

## INDENTATION OF THE SEMI-INFINITE ELASTIC SOLID BY A HOT SPHERE

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**Summary**—An exact solution is obtained for the problem of the semi-infinite elastic solid, indented by a hot rigid sphere. The relation between the contact circle radius, the compressive load and the temperature of the sphere is expressed in closed form, whilst series and integral formulations are given for the contact pressure distribution.

### NOTATION

$r, \theta, z$	cylindrical polar co-ordinates
$a$	radius of contact circle
$E$	Young's modulus
$k$	thermal conductivity
$p_{zz}$	direct stress in the $z$ direction (tensile stress positive)
$P$	normal compressive load
$q_z$	heat flow per unit area in the $z$ direction
$R$	radius of the sphere
$s$	parametric equivalent of $r$
$T$	temperature
$T_0$	temperature of the heated sphere
$u_z$	displacement in the $z$ direction
$\alpha$	coefficient of linear thermal expansion
$\nu$	Poisson's ratio
$\chi_2(x)$	Legendre's chi-function

### INTRODUCTION

IF A hot, rigid sphere is maintained at a uniform temperature, whilst being pressed into the plane surface of a semi-infinite elastic solid, the contact area will be a circle, the radius of which will be affected by thermal distortion. An approximate solution for this problem has already been obtained.<sup>1</sup> In this paper, an exact solution is given for the radius of the contact circle and the distribution of contact stress as functions of the temperature difference, the applied load and geometrical and physical properties.

### STATEMENT OF THE PROBLEM

We assume that the elastic solid is homogeneous and isotropic with Young's modulus  $E$ , Poisson's ratio  $\nu$ , thermal conductivity  $k$  and coefficient of thermal expansion  $\alpha$ . Frictionless, contact conditions are assumed and heat flow between the solids is only permitted to take place by conduction through the contact area. Following normal practice, the effect of elastic deformation on the heat conduction problem is ignored and the thermal boundary conditions are therefore referred to the solid in its undeformed state. In

cylindrical polar co-ordinates  $(r, \theta, z)$ , they are

$$\begin{aligned} \text{heat flow} \quad q_z &= 0, \quad z = 0, r > a; \\ \text{temperature} \quad T &= T_0, \quad z = 0, r \leq a; \\ T &\rightarrow 0, \quad z \rightarrow \infty, \end{aligned}$$

where the positive  $z$  axis is directed into the semi-infinite solid.

The mechanical boundary conditions are

$$\begin{aligned} \text{normal contact stress} \quad & p_{zz} = 0, \quad z = 0, r > a; \\ \text{shear stresses} \quad & p_{rz} = p_{\theta z} = 0, \quad z = 0, \text{ for all } r; \\ \text{normal displacement } u_z \text{ continuous at} \quad & z = 0, \quad r \leq a; \\ \text{applied compressive load} \quad & P = - \int_0^a \int_0^{2\pi} p_{zz}(z = 0) r \, d\theta \, dr. \end{aligned}$$

Although the contact radius ( $a$ ) is an unknown dependent variable related to the compressive load ( $P$ ), it is convenient to consider it as independent, i.e. to find the compressive load necessary to maintain a given contact area in the presence of heat flow. This relation can then easily be inverted.

### SOLUTION

The method of solution is that described in ref. (2). The heat conduction problem is first solved for the distribution of heat flow across the contact area and the latter is then used to find the surface displacement which would occur on the plane in the absence of surface stresses, using properties of the point heat source. The actual contact stresses are those which would be needed to establish the required contact area between the rigid sphere and an isothermal, elastic solid with the calculated distorted profile. This isothermal problem is solved by a method due to Segedin.<sup>3</sup>

The first stages have been adequately treated in the approximate solution<sup>1</sup> and need only be briefly mentioned here.

In the steady state, the heat flow through the contact area will be

$$q_z = \frac{2kT_0}{\pi(a^2 - r^2)^{\frac{1}{2}}} \quad (1)$$

per unit area, where  $q_z$  is taken as positive in the direction of the  $z$  axis. The derivation of the corresponding result for the conduction of electricity into a semi-infinite solid from a disk on the surface maintained at constant potential is given by Maxwell,<sup>4</sup> Art. 308.

The surface displacement which would be produced by this heat input in the absence of surface stress can be found by integrating the known solution for a thin annular heat source,<sup>2</sup> which is

$$u_z = \begin{cases} \frac{Q\alpha(1+\nu)}{2\pi k} \ln\left(\frac{r}{s}\right) + A, & r > s, \\ A, & r \leq s, \end{cases} \quad (2)$$

where  $Q$  is the strength of the source,  $s$  is its radius and  $A$  is an arbitrary rigid body displacement which will be zero if we take the origin as a reference point.  $u_z$  is defined as positive in the direction of the  $z$  axis.

Using this result and neglecting the rigid body displacement we have

$$\begin{aligned} u_z &= \frac{2\alpha T_0(1+\nu)}{\pi} \int_0^r \frac{s \ln(r/s) \, ds}{(a^2 - s^2)^{\frac{1}{2}}} \\ &= \frac{2\alpha T_0(1+\nu)a}{\pi} \left[ \ln\left\{ \frac{1}{2} \left[ 1 + \left(1 - \frac{r^2}{a^2}\right)^{\frac{1}{2}} \right] \right\} + 1 - \left(1 - \frac{r^2}{a^2}\right)^{\frac{1}{2}} \right], \quad r \leq a, \end{aligned} \quad (3)$$

for the normal surface displacement produced by the heat input of equation (1) in the absence of surface stress.

THE EQUIVALENT ISOTHERMAL PROBLEM

We now seek a distribution of contact stress,  $p_{zz}$ , which will be just sufficient to cause the spherical punch to make contact over the circle  $r \leq a$  with a half-space distorted by the displacements of equation (3). This is the sum of two parts: (i) the contact stress between the sphere and an undistorted half-space; (ii) the contact stress equal and opposite to that needed to produce the displacement defined by equation (3).

The method used for the solution of both problems is that developed by Segedin.<sup>3</sup> He showed that a stress distribution

$$p_{zz} = \begin{cases} \frac{-E}{\pi(1-\nu^2)} \int_r^a \left(\frac{s}{a}\right)^{2m-1} \frac{ds}{(s^2-r^2)^{\frac{1}{2}}}, & r \leq a, \\ 0, & r > a, \end{cases} \tag{4}$$

$$p_{rz} = p_{\theta z} = 0 \quad \text{for all } r,$$

acting on the plane surface,  $z = 0$ , of a semi-infinite solid, produces a normal surface displacement

$$u_z = \frac{a}{2m} - \frac{(2m-1)!}{2^{2m} m! m!} \frac{r^{2m}}{a^{2m-1}} \tag{5}$$

within the circle  $r \leq a$  and satisfies the condition of continuity at  $r = a$ .

This result is easily verified by a suitable integration of the well-known result for a point force on the surface of a semi-infinite solid. The first term of equation (5) represents a rigid body displacement which will not affect the local stress distribution in a contact problem.

The stress distribution of equation (4) corresponds to a total compressive load

$$P = \frac{2Ea^2}{(2m+1)(1-\nu^2)}. \tag{6}$$

Hence, if the required displacement profile can be expressed as a power series in  $(r^2/a^2)$ , we can build up the corresponding contact stress distribution as a series of terms of the form of equation (4).

Segedin<sup>3</sup> applies his method to the problem of the isothermal indentation of a semi-infinite solid by a rigid sphere, which is the first part of the solution we require (problem (i), above). In particular, he shows that the radius of the contact area ( $a$ ), the radius of the sphere ( $R$ ) and the applied compressive load ( $P_1$ ) are related by the equation

$$P_1 = \frac{ER^2}{1-\nu^2} \left[ \left(1 + \frac{a^2}{R^2}\right) \tan^{-1} \left(\frac{a}{R}\right) - \frac{a}{R} \right]. \tag{7}$$

To solve problem (ii), we first express the thermoelastic surface displacements (equation (3)) as a power series in  $(r^2/a^2)$  obtaining

$$u_z = \frac{2\alpha T_0(1+\nu)a}{\pi} \sum_{m=1}^{\infty} \frac{(2m-2)!}{2^{2m} m! m!} \left(\frac{r^2}{a^2}\right)^m. \tag{8}$$

The distribution of normal stress over the circle  $r \leq a$  which would be just sufficient to produce the same profile within the circle is then found from equations (4), (5) and (8) to be

$$p_{zz} = \frac{2\alpha T_0(1+\nu)a}{\pi} \sum_{m=1}^{\infty} \frac{(2m-2)!}{2^{2m} m! m!} \frac{2^{2m} m! m!}{(2m-1)!} \frac{E}{\pi(1-\nu^2)a} \int_r^a \left(\frac{s}{a}\right)^{2m-1} \frac{ds}{(s^2-r^2)^{\frac{1}{2}}} \tag{9}$$

$$= \frac{2E\alpha T_0}{\pi^2(1-\nu)} \int_r^a \sum_{m=1}^{\infty} \frac{(s/a)^{2m-1} ds}{(2m-1)(s^2-r^2)^{\frac{1}{2}}} \tag{9}$$

$$= \frac{E\alpha T_0}{\pi^2(1-\nu)} \int_r^a \ln \left(\frac{1+s/a}{1-s/a}\right) \frac{ds}{\sqrt{(s^2-r^2)}}. \tag{10}$$

This integral can be evaluated (see Appendix) to give

$$p_{zz} = \frac{2E\alpha T_0}{\pi^2(1-\nu)} \left[ \frac{\pi^2}{8} - \chi_2 \left( \frac{a - \sqrt{(a^2 - r^2)}}{a + \sqrt{(a^2 - r^2)}} \right) \right], \quad (11)$$

where  $\chi_2$  is Legendre's chi-function, which is defined by the series

$$\chi_2(x) = \sum_{m=1}^{\infty} \frac{x^{2m-1}}{(2m-1)^2} \quad (12)$$

and is discussed and tabulated by Lewin.<sup>5</sup>

The total load associated with this stress distribution could be found by integrating equation (11) over the contact circle, but it is more convenient to sum the series of integrals of the form of equation (6).

Thus, the total load

$$\begin{aligned} P_2 &= -\frac{2\alpha T_0(1+\nu)}{\pi} \sum_{m=1}^{\infty} \frac{1}{(2m-1)} \frac{2Ea^2}{(2m+1)(1-\nu^2)} \\ &= -\frac{2E\alpha T_0 a^2}{\pi(1-\nu)}. \end{aligned} \quad (13)$$

#### RELATION BETWEEN CONTACT RADIUS AND APPLIED LOAD

The total applied load ( $P_0$ ) is the difference between the two partial values ( $P_1, P_2$ ) given by equations (7) and (13). Thus

$$P_0 = \frac{2E\alpha T_0 a^2}{\pi(1-\nu)} + \frac{ER^2}{(1-\nu^2)} \left[ \left( 1 + \frac{a^2}{R^2} \right) \tan^{-1} \left( \frac{a}{R} \right) - \frac{a}{R} \right] \quad (14)$$

or in non-dimensional form

$$\frac{(1-\nu^2)P_0}{ER^2} = \frac{2\alpha T_0(1+\nu)}{\pi} \left( \frac{a}{R} \right)^2 + \left( 1 + \frac{a^2}{R^2} \right) \tan^{-1} \left( \frac{a}{R} \right) - \frac{a}{R}. \quad (15)$$

#### DISTRIBUTION OF CONTACT STRESS

The contact stress distribution can be obtained by subtracting equation (11) from the contact stress for an isothermal spherical indentation. Segedin<sup>3</sup> does not give an explicit expression for the latter, but it is easily shown from his paper to be

$$p_{zz} = -\frac{E}{\pi(1-\nu^2)} \int_r^a \sum_{m=1}^{\infty} \frac{2m}{(2m-1)} \left( \frac{s}{R} \right)^{2m-1} \frac{ds}{\sqrt{(s^2 - r^2)}} \quad (16)$$

$$= -\frac{E}{\pi(1-\nu^2)} \int_r^a \left[ \frac{1}{2} \ln \left( \frac{R+s}{R-s} \right) + \frac{Rs}{R^2 - s^2} \right] \frac{ds}{\sqrt{(s^2 - r^2)}}. \quad (17)$$

This integral is less tractable than that of equation (10). However, in most situations  $a \ll R$  and it is sufficient to take the first few terms of equation (16), i.e.

$$p_{zz} = -\frac{E\sqrt{(a^2 - r^2)}}{\pi(1-\nu^2)R} \left[ 2 + \frac{2.4}{1.3} \frac{a^2}{3R^2} \left( \frac{r^2}{a^2} + \frac{1}{2} \right) + \frac{2.4.6}{1.3.5} \frac{a^4}{5R^4} \left( \frac{r^4}{a^4} + \frac{1}{2} \frac{r^2}{a^2} + \frac{1.3}{2.4} \right) + \dots \right]. \quad (18)$$

The stress distribution under combined heating and loading is thus

$$\begin{aligned} p_{zz} &= -\frac{2E\alpha T_0}{\pi^2(1-\nu)} \left[ \frac{\pi^2}{8} - \chi_2 \left( \frac{a - \sqrt{(a^2 - r^2)}}{a + \sqrt{(a^2 - r^2)}} \right) \right] - \frac{E\sqrt{(a^2 - r^2)}}{\pi(1-\nu^2)R} \left[ 2 + \frac{2.4}{1.3} \frac{a^2}{3R^2} \left( \frac{r^2}{a^2} + \frac{1}{2} \right) \right. \\ &\quad \left. + \frac{2.4.6}{1.3.5} \frac{a^4}{5R^4} \left( \frac{r^4}{a^4} + \frac{1}{2} \frac{r^2}{a^2} + \frac{1.3}{2.4} \right) + \dots \right]. \end{aligned} \quad (19)$$

In general, the heat flow tends to increase the stress near the edges of the contact area at the expense of the centre, though the contact stress is always greatest at the centre.

## INDENTATION BY A COLD PUNCH

Tensile contact stresses are inadmissible in an indentation problem and hence the stress distribution defined by equation (19) is the solution to our problem if and only if  $p_{zz} \leq 0$ ;  $z = 0$ , for all  $r$ . This condition is satisfied provided that  $\alpha T_0 \geq 0$ , since the function  $\chi_2(x) \leq \pi^2/8$  for  $0 \leq x \leq 1$  (see ref. 5).

However, if  $\alpha T_0 < 0$ , i.e. if the solid is indented by a cooled sphere, we find that equation (19) always defines a tensile stress in the vicinity of  $r = a$ . Writing equation (19) in the integral form

$$p_{zz} = -\frac{E}{\pi^2(1-\nu^2)} \int_r^a \left\{ \alpha T_0(1+\nu) \ln \left( \frac{1+s/a}{1-s/a} \right) + \frac{\pi}{2} \left[ \ln \left( \frac{1+s/R}{1-s/R} \right) + \frac{2Rs}{R^2-s^2} \right] \right\} \frac{ds}{(s^2-r^2)^{3/2}} \quad (20)$$

(from equations (10) and (17)), we note that the term  $\ln[(1+s/a)/(1-s/a)]$  increases without limit as  $s$  approaches  $a$ . For particular values of  $a$ ,  $\nu$ ,  $R$  and a negative value of  $\alpha T_0$ , there must therefore be some value of  $r$  such that the integrand in equation (20) is negative for  $r \leq s \leq a$ . Hence, equation (20) will define tensile stresses in this region. In physical terms, the distorted plane surface defined by equation (3) is of such a shape that when a sphere is pressed into it, contact is established outside the circle  $r = a$ , before this circle is itself completely in contact.

This suggests that there may be one or more concentric annular contact regions as well as or instead of a central circular area. It has been shown<sup>1</sup> that the normal surface displacement  $u_z$  due to a point heat source on the surface of a semi-infinite solid satisfies the equation

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u_z = 0 \quad (21)$$

except at the source itself. ( $x, y$  are any two orthogonal directions in the plane of the surface.)

Furthermore, a point compressive load produces displacements which satisfy

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u_z > 0 \quad (22)$$

except at its point of application. Hence, if the surface of a semi-infinite solid is subjected to a distributed heat source (and/or sink) and a compressive normal stress, the inequality (22) will be satisfied wherever  $q_z = 0$  and  $p_{zz} = 0$ ; i.e. in all regions which are not in contact.

Now suppose there is a region on the surface of the solid which is in contact with the sphere, but which is totally surrounded by regions of contact. In such a region there must be a point at which the distance between the sphere and the solid surface is a local maximum. The sphere is everywhere convex, and hence at such a point we must have

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u_z < 0,$$

i.e. the deformed surface must be concave upwards.

But we have already established that the incompatible inequality (22) must be satisfied throughout a non-contact region and hence our hypothesis that there are totally enclosed non-contact regions must be false. In other words, the contact region must be simply connected. A similar argument would apply if the punch had any arbitrary convex profile.

We must therefore conclude that there is no solution for the indentation of an elastic solid by a cooled punch, which satisfies boundary conditions of the form assumed. This type of result has already been remarked for different geometries and possible interpretations are discussed elsewhere.<sup>1</sup>

## REFERENCES

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## APPENDIX

We wish to evaluate the integral

$$I = \int_r^a \frac{1}{2} \ln \left( \frac{1+s/a}{1-s/a} \right) \frac{ds}{\sqrt{(s^2-r^2)}}. \quad (23)$$

The following solution is somewhat devious, but the author has been unable to discover a more direct method.

We make the substitution  $s = r/\cos \theta$  and use the result

$$\frac{1}{2} \ln \left( \frac{1+y}{1+y} \right) = \int_0^y \frac{dz}{(1-z^2)} \quad (24)$$

to obtain

$$I = \int_0^\psi \int_0^{\cos \psi / \cos \theta} \frac{z \, dz \, d\theta}{(1-z^2) z \cos \theta}, \quad (25)$$

where

$$\cos \psi = r/a.$$

If we treat  $(z, \theta)$  as a set of polar co-ordinates we have

$$I = \iint_A \frac{dA}{(1-z^2) z \cos \theta}, \quad (26)$$

where the area of integration ( $A$ ) is the triangle shown in Fig. 1.

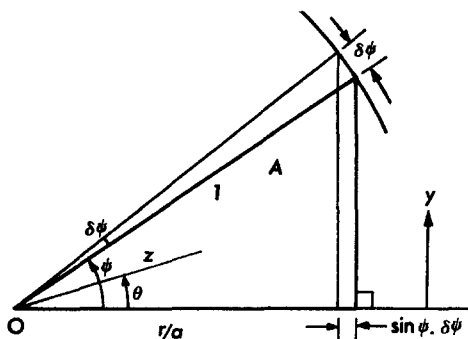


FIG. 1.

As we change the independent variable  $r$ , the vertex furthest from the origin moves on the unit circle as shown and hence

$$\frac{dI}{d\psi} = \int_0^1 \frac{z \, dz}{(1-z^2) z \cos \psi} - \int_0^{\sin \psi} \frac{\sin \psi \, dy}{\cos \psi (\sin^2 \psi - y^2)}. \quad (27)$$

Unfortunately, these integrals are not bounded because of the behaviour of the integrand near to the unit circle, but their difference is bounded and can be found by replacing the radius of the latter by  $c$  ( $< 1$ ) and taking the limit as  $c \rightarrow 1$ . This procedure gives

$$\frac{dI}{d\psi} = -\frac{\ln(\sin \psi)}{\cos \psi}. \quad (28)$$

When  $\psi = 0$ ,  $r = a$  and  $I = 0$ . Hence,

$$I = - \int_0^{\arccos r/a} \ln(\sin \psi) \frac{d\psi}{\cos \psi}, \quad (29)$$

$$= - \int_0^{\sqrt{1-r^2/a^2}} \frac{\ln(x) dx}{(1-x^2)} \quad (30)$$

$$= \frac{\pi^2}{8} - \chi_2 \left( \frac{1 - \sqrt{1-r^2/a^2}}{1 + \sqrt{1-r^2/a^2}} \right) \quad (31)$$

(see Lewin;<sup>5</sup> p. 17).