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A shrink-fit shaft subject to torsion

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Abstract

The problem studied is an elastic, circular shaft, fitted into a cavity normal to the free surface of a half-space. The cavity is smaller than the shaft, so that there is a residual radial stress. A torque is applied to the shaft, giving rise to a region of slip between the shaft and the socket. Its extent is determined by forming an integral equation whose kernel is given by a circular ring dislocation, which has a Burgers vector whose magnitude is constant, oriented in the tangential direction. The problem has direct application to the study of shrink fitted shafts in wheels, whose diameter is large compared with the shaft. © 2002 Éditions scientifiques et médicales Elsevier SAS. All rights reserved.

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1. Introduction

The solution is sought to the problem of the penetration of slip for a shaft shrink-fitted onto an elastically similar half-space, and subjected to remote torsion, shown in Fig. 1(a). It is assumed that the interface between the shaft and socket is subject to Coulomb friction, applied on a pointwise basis, and where the coefficient of friction is f. This problem is of direct relevance to the design of shrink-fitted shafts in wheels which carry a torque, although the outer boundary of the wheel is implicitly neglected, so that the solution is most appropriate when the radius of the wheel is large compared with the radius of the shaft, a. The problem is to be formulated as an integral equation representing the effect of slip, whose kernel is an axi-symmetric ring dislocation, representing an element of twist displacement. Attention is focused on the behaviour of the interface between the half-space and the socket, and we wish to avoid a detailed consideration of how the length of the shaft projecting from the half-space will modify the nominal state of stress, or bilateral solution, in the assembly when all slip is prevented. Solution of the problem will therefore involve three stages, as follows:

1. Determine the bilateral solution for the assembly, assuming complete adhesion. As mentioned, this formally will require a numerical solution, but useful bounding solutions may be obtained in the following way. Assume, for the time being, that the shaft is not homogeneous, but has a different stiffness external to the half space from that part which is embedded. If the external stiffness is low the state of stress transferred into the embedded part is prescribed by traction conditions, and following elementary torsion theory, this means that $\sigma_{z\theta} \sim Ar$. On the other hand, if the external part of the shaft is extremely rigid, the load is transferred as an imposed displacement condition, and $u_\theta \sim Cr$. Thus, the stress state induced in a half space by each of these surface boundary conditions, imposed separately over the disk $r \leqslant a$ is needed, together with $\sigma_{z\theta} = 0$, $r > a$ and $\sigma_{zz} = 0$ everywhere.

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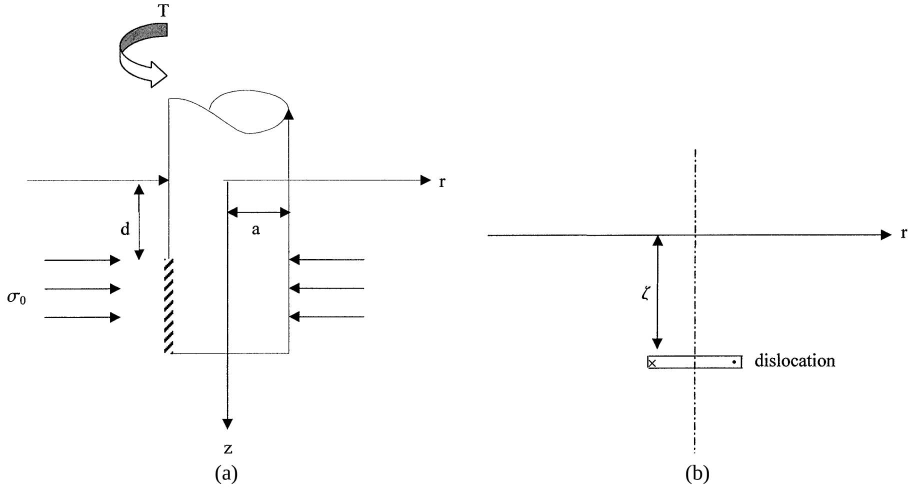


Fig. 1. A shrink-shaft subject to torsion: (a) general arrangement; (b) dislocation used.

2. Determine the stress state induced by a Somigliana dislocation, representing a 'twist' discontinuity, located at a depth d, and of radius a. Specialise the solution to the traction component, $\sigma_{r\theta}$, present on the cylinder $r = a$.
3. Form an integral equation describing the slip region, and solve.

2. Dislocation solution

The second phase of the solution process presents the greatest element of novelty, and that will therefore be described first, in detail. This phase will be further split into two parts: the stress state induced in an *infinite* space by the θ-dislocation will be found first, and then the free surface introduced by the method of images. In order to facilitate its use as a kernel of the integral equation the stress state on the line $r = a$ will be considered in some detail, and, in particular, it will be shown that the stress component $\sigma_{r\theta}$ varies like $1/|z|$ as the dislocation core is approached.

2.1. Formulation

The problem to be solved is equivalent to the rigid body rotation of a thin disk, of radius a, whose periphery describes the line of the dislocation. The magnitude of the Burgers vector of the dislocation is the displacement discontinuity present at the edge of the disk, and is taken to be Ba. We will initially take the origin of coordinates to be the centre of the disk. The solution required is equivalent to a rigid body relative rotation of the two disks $0 < r < a$ in the half-space $z \leqslant 0$ and $0 < r < a$ in the half-space $z \geqslant 0$. In particular, the stress state induced in the half-space $z \geqslant 0$ may be found by employing a half-space formulation with the surface boundary conditions

$$\begin{aligned} &u_\theta = \frac{1}{2}Br, \quad 0 < r < a,\\ &u_\theta = 0, \quad r > a,\\ &u_r = 0, \quad \text{all } r,\\ &\sigma_{zz} = 0, \quad \text{all } r. \end{aligned}$$

The internal state of stress within the half-space may then be found either from Green and Zerna (1968), solution E, or Barber (1992), table 16.1. The potential, ψ, is connected to the internal displacements by the following

$$\mu u_r = 0, \qquad \mu u_\theta = -\frac{\partial \psi}{\partial r}, \qquad \mu u_z = 0, \tag{1}$$

where μ is the modulus of rigidity, and, from the corresponding strains and Hooke's law we see that the only surviving components of stress are

$$\sigma_{\theta z} = -\frac{\partial^2 \psi}{\partial r \partial z}, \qquad \sigma_{r\theta} = \frac{1}{r}\frac{\partial \psi}{\partial r} - \frac{\partial^2 \psi}{\partial r^2}, \tag{2}$$

where $\psi(r,z)$ is harmonic, and hence a solution of

$$\nabla^2\psi \equiv \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} = 0.$$

This solution automatically satisfies the conditions $\sigma_{zz} = 0$ and $\sigma_{rr} = 0$ on the surface $z = 0$ so that the remaining boundary value problem is of the form

$$\mu u_\theta = -\frac{\partial\psi}{\partial r} = \begin{cases} -f(r); & 0 < r < a,\ z = 0, \\ 0; & r > a,\ z = 0. \end{cases} \tag{3}$$

We take the following form as a representation for ψ:

$$\psi = \operatorname{Re}\int_0^a g(\xi)\ln(z_1 + R_1)\,\mathrm{d}\xi, \tag{4}$$

where $g(t)$ is a real function of the real parameter t and

$$z_1 = z + i\xi, \qquad R_1 = \sqrt{r^2 + z_1^2}$$

which is equivalent to the Green and Zerna (1968) approach for even $g(\xi)$ and permits the generation of an appropriate Abel integral equation for $f(r)$ in terms of $g(\xi)$.

The surface values of $\partial\psi/\partial r$ are required. Now, in general, from (4)

$$\begin{aligned} \frac{\partial\psi}{\partial r} &= \operatorname{Re}\int_0^a \frac{rg(\xi)}{R_1(z_1 + R_1)}\,\mathrm{d}\xi \\ &= \frac{1}{r}\operatorname{Re}\int_0^a g(\xi)\left(1 - \frac{z_1}{R_1}\right)\mathrm{d}\xi \end{aligned}$$

but

$$R_1 = \begin{cases} \sqrt{r^2 - \xi^2}; & \xi \leqslant r,\ z = 0, \\ i\sqrt{\xi^2 - r^2}; & r \leqslant \xi,\ z = 0 \end{cases}$$

so that, on $z = 0$, $0 < r < a$ we have

$$\left(\frac{\partial\psi}{\partial r}\right)_{z=0} = \frac{1}{r}\int_0^a g(\xi)\,\mathrm{d}\xi - \frac{1}{r}\int_0^a \frac{\xi g(\xi)}{\sqrt{\xi^2 - r^2}}\,\mathrm{d}\xi$$

whilst on $z = 0$, $r > a$ we have

$$\left(\frac{\partial\psi}{\partial r}\right)_{z=0} = \frac{1}{r}\int_0^a g(\xi)\,\mathrm{d}\xi$$

so, in summary

$$\frac{\partial\psi}{\partial r} = \begin{cases} \frac{1}{r}\int_0^a g(\xi)\,\mathrm{d}t - \frac{1}{r}\int_r^a \frac{\xi g(\xi)}{\sqrt{\xi^2 - r^2}}\,\mathrm{d}\xi; & 0 < r < a,\ z = 0, \\ \frac{1}{r}\int_0^a g(\xi)\,\mathrm{d}\xi; & r > a,\ z = 0. \end{cases}$$

In order to satisfy the boundary values we must have, therefore,

$$\int_r^a \frac{\xi g(\xi)}{\sqrt{\xi^2 - r^2}}\,\mathrm{d}t = -rf(r) \tag{5}$$

and

$$\int_0^a g(\xi)\,\mathrm{d}\xi = 0. \tag{6}$$

The inversion of Abel equation (5) to find $g(\xi)$ is given by

$$g(\xi) = \frac{1}{\xi}\frac{2}{\pi}\frac{\mathrm{d}}{\mathrm{d}\xi}\int_t^a \frac{r^2 f(r)}{\sqrt{r^2-\xi^2}}\,\mathrm{d}r. \tag{7}$$

Substituting this into (7) gives the following restriction on those functions $f(r)$ which satisfy the boundary value problem:

$$\lim_{r\to 0}\bigl(rf(r)\bigr) = 0. \tag{8}$$

With the surface displacement given, which has no poles at $r = 0$, so that (8) is satisfied, we find

$$g(\xi) = -\frac{\mu B}{\pi}\frac{a^2-2\xi^2}{\sqrt{a^2-\xi^2}} \tag{9}$$

and the resultant stress state may be found by differentiation alone, from equation (4).

2.2. *Stress state in full space*

In order to represent slip around a circular shaft, the stress component $\sigma_{r\theta}$ on the surface $r = a$ is needed and hence the following three derivatives: $(\partial\psi/\partial r)_{r=a}$, $(\partial^2\psi/\partial r^2)_{r=a}$, $(\partial^2\psi/\partial z\partial r)_{r=a}$. We will therefore specialise to this, and adopt normalised coordinates by the following substitutions

$$s = \frac{t}{a}, \qquad \omega = \frac{z}{a}.$$

First

$$\frac{\partial\psi}{\partial r} = \mathrm{Re}\, e\int_0^a g(\xi)\frac{1}{z_1+R_1}\frac{r}{R_1}\,\mathrm{d}\xi$$

from which

$$\frac{2\pi}{\mu Ba}\left(\frac{\partial\psi}{\partial r}\right)_{r=a} = \int_{-1}^{1}\frac{1-2s^2}{\sqrt{1-s^2}}\frac{\omega+is}{\sqrt{1+(\omega+is)^2}}\,\mathrm{d}s.$$

In order to present this and subsequent integrals in a form which may subsequently be reduced to standard elliptic integrals, we introduce the notation of the celebrated Lipschitz–Hankel integral

$$I(m,n,\lambda) = \int_0^\infty J_m(x)J_n(x)\mathrm{e}^{-\omega x}x^\lambda\,\mathrm{d}x,$$

where $J_n(x)$ is the Bessel function of the first kind of order n. Although the Lipschitz–Hankel integral does depend on ω, the value of ω never varies throughout manipulations of the integrals. This explains the absence of ω as a variable in the notation. This integral has been extensively studied by Eason, Noble and Sneddon (1955). Hence, after a very considerable amount of algebra, we find that

$$\left(\frac{\partial\psi}{\partial r}\right)_{r=a} = -\frac{\mu Ba}{2}I(1,2,0). \tag{10}$$

Turning, now, to the evaluation of $\partial^2\psi/\partial r^2$, we note that

$$\frac{\partial\psi}{\partial r} = \frac{\mu B}{2\pi r}\int_{-a}^{a}\frac{a^2-2t^2}{\sqrt{a^2-t^2}}\frac{z_1}{R_1}\,\mathrm{d}t$$

and, again after considerable manipulation, it is found that

$$\left(\frac{\partial^2\psi}{\partial r^2}\right)_{r=a} = \frac{\mu B}{2}\bigl[I(1,2,0)-I(0,2,1)\bigr]. \tag{11}$$

The evaluation of $(\partial^2\psi/\partial z\partial r)_{r=a}$ is quite straightforward. It is clearly an even function of z, because $\partial\psi/\partial r$ is odd, and we write

$$\left(\frac{\partial^2\psi}{\partial z\partial r}\right)_{r=a} = \frac{\partial}{\partial z}\left(\frac{\partial\psi}{\partial r}\right)_{r=a} = \frac{\mu B}{2} I(1,2,1) \tag{12}$$

using the result

$$\frac{\partial}{\partial\omega} I(m,n,\lambda) = -I(m,n,\lambda+1).$$

2.3. Results: infinite space solution

The expressions for the derivatives of the potentials derived in the above section may be collected to arrive at the following explicit results for the only two non-zero components of stress induced.

$$(\sigma_{r\theta})_{r=a} = \frac{\mu B}{2}\bigl[I(0,2,1) - 2I(1,2,0)\bigr] \tag{13}$$

and

$$(\sigma_{z\theta})_{r=a} = -\frac{\mu B}{2} I(1,2,1). \tag{14}$$

It is useful to note that, using the result $\frac{2}{x}J_1(x) = J_0(x) + J_2(x)$, we can re-write this result as

$$(\sigma_{r\theta})_{r=a} = -\frac{\mu B}{2} I(2,2,1).$$

Finally, using the extensive tables of relationships between the Lipschitz–Hankel integrals and standard elliptic integrals, compiled by Eason, Noble and Sneddon (1955), we may re-write these results in terms of simpler functions as

$$(\sigma_{r\theta})_{r=a} = \frac{\mu B}{4\pi k^2}\left[8k'\bigl(2-k^2\bigr)K(k) - \frac{k^4+16k'^2}{k'}E(k)\right], \tag{15}$$

$$(\sigma_{z\theta})_{r=a} = \frac{\mu B}{4\pi k}\bigl[\bigl(5k^2-8\bigr)K(k) + \bigl(8-k^2\bigr)E(k)\bigr], \tag{16}$$

where $K(\cdot)$, $E(\cdot)$ are complete elliptic integrals of the first and second kinds, respectively, and

$$k^2 = \frac{4}{4+\omega^2},$$
$$k'^2 = 1-k^2.$$

The behaviour of the stress component, $(\sigma_{r\theta})_{r=a}$, as $|z|\to 0$, is of particular interest, as this will form the dominant term in an integral representation of the effects of slip, as described in the Introduction. Note that $\omega = 2k'/k$ so that, as $z\to 0$, $\omega\to 0$, $k\to 1$ and $k'\to 0$. Also

$$\lim_{k\to 1} k'K(k) = 0,$$
$$\lim_{k\to 1} E(k) = 1.$$

Hence:

$$\begin{aligned}(\sigma_{r\theta})_{r=a} &= -\frac{\mu B}{2\pi}\frac{1}{\omega} + \cdots \\ &= -\frac{\mu Ba}{2\pi}\frac{1}{z} + \cdots,\end{aligned} \tag{17}$$

where … represents terms which are not singular in z.

2.4. The half-space

We turn, now, to a consideration of the stress state when the dislocation loop lies parallel with the free surface, at a depth d, as shown in Fig. 1(b). In the infinite space solution, described above, the stresses σ_{zz} and σ_{rz}, which form two of the tractions on z = constant planes, are zero everywhere, and it therefore remains only to annul the traction $\sigma_{z\theta}$. It may be seen from the form of the solution for a dislocation in an infinite space that $\sigma_{z\theta}$ is an even function of z. Consider, then, placing two dislocations, of the same radius, but with their centres located at points $z = \pm d$. The one placed at $z = +d$ is of strength Ba, whilst the one placed at $z = -d$ is of strength $-Ba$. The resultant stress state may be found by superposition, and, if we denote the stress state produced by a dislocation at the origin, in an inifinite space, by $\sigma_{ij}(r, z; Ba)$ the resultant stress state produced by the shifted dislocation and its image, s_{ij}, is given by

$$s_{ij}(r, z; Ba) = \sigma_{ij}(r, z - d; Ba) + \sigma_{ij}(r, z + d; -Ba)$$

and the traction $\sigma_{z\theta}$ on $z = 0$ clearly vanishes.

3. Bilateral solutions

The bilateral solutions needed are the stress states within a half-space, due to the given displacment or traction boundary conditions, present on the contact disk $r \leqslant a$. The solutions needed are closely related to that already described: in each case, external to the disk $r = a$ the shearing traction must vanish, and the two other components of traction must vanish everywhere, so that

$$\begin{aligned} \sigma_{z\theta} &= 0, \quad r > a, \\ \sigma_{zz} &= \sigma_{rz} = 0 \quad \text{everywhere} \end{aligned} \tag{18}$$

and again solution E of Green and Zerna is appropriate. We therefore take, as a possible form for ψ

$$\psi = \operatorname{Im} \int_0^a h(\xi) \ln(z_1 + R_1)\, \mathrm{d}\xi, \tag{19}$$

where z_1, R_1 have been defined in the last section. This is connected to the internal stresses and displacements by

$$\mu u_r = 0, \qquad \mu u_\theta = -\frac{\partial \psi}{\partial r} \equiv f(r), \qquad \mu u_z = 0, \tag{20}$$

and the only surviving components of stress are

$$\sigma_{\theta z} = -\frac{\partial^2 \psi}{\partial r \partial z}, \qquad \sigma_{r\theta} = \frac{1}{r}\frac{\partial \psi}{\partial r} - \frac{\partial^2 \psi}{\partial r^2}, \tag{21}$$

where $\psi(r, z)$ is harmonic. Differentiating equation (19), we find that, on the surface

$$\frac{\partial \psi}{\partial z} = \operatorname{Im} \int_0^a \frac{h(\xi)}{\sqrt{r^2 - \xi^2}}\, \mathrm{d}\xi, \quad z = 0.$$

Now this integral has no imaginary part when $r > a$, and therefore $\partial^2\psi/\partial r \partial z \equiv 0$, $r > a$, $z = 0$. Hence $\sigma_{z\theta} = 0$ on the surface, exterior to the disk, thereby satisfying equation (18) automatically. This much of the formulation is common to both boundary value problems. Brief notes on the evaluation of each will now be presented.

3.1. Reissner–Sagoci problem

First, the displacement-controlled problem will be examined. It has the following boundary conditions, and its solution is associated with the names Reissner and Sagoci (Gladwell, 1980). In addition to boundary conditions (18), we have

$$u_\theta = Ar, \quad r \leqslant a. \tag{22}$$

The surface value of $\partial\psi/\partial r$ within the contact disk is

$$f(r) \equiv \frac{\partial \psi}{\partial r} = -\frac{1}{r} \int_0^r \frac{\xi h(\xi)}{\sqrt{r^2 - \xi^2}}\, \mathrm{d}\xi, \quad 0 \leqslant r < a,\ z = 0. \tag{23}$$

Inverting gives

$$h(\xi)=\frac{2}{\pi\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\int_0^{\xi}\frac{r\{-rf(r)\}}{\sqrt{\xi^2-r^2}}\,\mathrm{d}r. \tag{24}$$

Now substitute in for $f(r)$ and evaluate the integral, with the result

$$h(\xi)=\frac{4\mu A}{\pi}\xi. \tag{25}$$

The full displacement and stress fields may now be constructed from a knowledge of $\partial\psi/\partial r$, which is given by

$$\frac{\partial\psi}{\partial r}=\frac{2\mu A}{\pi i}\int_{-a}^{a}\frac{\xi}{z_1+R_1}\frac{r}{R_1}\,\mathrm{d}\xi=-\frac{2\mu A}{\pi i r}\int_{-a}^{a}\frac{\xi z_1}{R_1}\,\mathrm{d}\xi. \tag{26}$$

This expression may easily be evaluated. Make the substitutions

$$\begin{aligned} r\cos\omega &= iz-\xi=i(z+i\xi)=iz_1,\\ r\sin\omega &= \sqrt{r^2+z_1^2}=R_1 \end{aligned}$$

so that

$$\begin{aligned}\int_{-a}^{a}\frac{\xi z_1}{R_1}\,\mathrm{d}\xi &= \int\left(rz\cos\omega+ir^2\cos^2\omega\right)\mathrm{d}\omega\\ &= zR_1-\frac{1}{2}z_1R_1+\frac{1}{2}ir^2\tan^{-1}\left(\frac{R_1}{iz_1}\right)+\text{constant}.\end{aligned}$$

An elegant solution may be constructed using oblate spheroidal coordinates and Love's connection (Love, 1927), but for our purposes it is easier to continue with cylindrical coordinates r, z and use the oblate coordinates as naturally occurring parameters as follows. Write

$$z=ast,\qquad r=a\sqrt{1+s^2}\sqrt{1-t^2}, \tag{27}$$

s and t are oblate spheroidal coordinates and $s=0$ is the disc $z=0,\ 0\leqslant r\leqslant a$, whilst $t=0$ is the anti-disk $z=0,\ a\leqslant r<\infty$. Then

$$\sqrt{r^2+(z+ia)^2}=a(s+it)$$

and

$$\left[\tan^{-1}\left(\frac{R_1}{iz_1}\right)\right]_{-a}^{a}=\tan^{-1}\left(\frac{2s}{s^2-1}\right)=2\tan^{-1}\left(\frac{1}{s}\right).$$

Hence

$$\frac{\partial\psi}{\partial r}=-\frac{2\mu A}{\pi i r}\int_{-a}^{a}\frac{\xi z_1}{R_1}\,\mathrm{d}\xi=-\frac{2\mu Aa^2}{\pi r}\left(1-t^2\right)\left\{\left(1+s^2\right)\tan^{-1}\left(\frac{1}{s}\right)-s\right\},$$

from which

$$\frac{1}{r}\frac{\partial\psi}{\partial r}=-\frac{2\mu A}{\pi}\left\{\tan^{-1}\left(\frac{1}{s}\right)-\frac{s}{1+s^2}\right\}. \tag{28}$$

Thus

$$\begin{aligned}\sigma_{r\theta} &= -\frac{4\mu Ar^2}{\pi a}\frac{s}{(s^2+t^2)(1+s^2)^2},\\ \sigma_{z\theta} &= -\frac{4\mu Ar}{\pi}\frac{t}{(s^2+t^2)(1+s^2)}.\end{aligned} \tag{29}$$

The surface shear traction distribution is therefore given by

$$\sigma_{z\theta}(r)=\frac{4\mu Ar}{\pi\sqrt{a^2-r^2}}$$

and equilibrium with the applied torque, T^0, gives

$$T^0=\frac{16\mu Aa^3}{3},$$

i.e.

$$\frac{a^3\sigma_{z\theta}(r)}{T^0}=\frac{3}{4\pi}\frac{r}{\sqrt{a^2-r^2}}. \tag{30}$$

3.2. Linearly varying traction

The second bilateral solution corresponds to the shear traction distribution within a simple shaft. From elementary torsion theory, it is of the form

$$\frac{a^3\sigma_{z\theta}(r)}{T^0}=\frac{2r}{\pi a} \tag{31}$$

or

$$\sigma_{z\theta}=Cr,$$

where

$$C=\frac{2T^0}{\pi a^4}.$$

From equation (21) this boundary value problem is of the form

$$\left(\frac{\partial^2\psi}{\partial r\partial z}\right)_{z=0}=\begin{cases}-Cr; & 0<r\leqslant a,\ z=0,\\ 0; & r>a,\ z=0.\end{cases} \tag{32}$$

From equation (19)

$$\frac{\partial\psi}{\partial z}=\begin{cases}-\int_r^a\frac{h(\xi)}{\sqrt{\xi^2-r^2}}\,\mathrm{d}\xi; & 0<r<a,\ z=0,\\ 0; & r>a,\ z=0.\end{cases} \tag{33}$$

This latter result is because $(\partial\psi/\partial z)_{z=0}$ has no imaginary part for $r>a$ and is the reason for this choice of ψ. It means, of course, that the condition of zero traction outside the circle $r=a$ will be automatically satisfied. Thus

$$Cr=-\left(\frac{\partial^2\psi}{\partial r\partial z}\right)_{z=0}=\frac{\mathrm{d}}{\mathrm{d}r}\int_r^a\frac{h(\xi)}{\sqrt{\xi^2-r^2}}\,\mathrm{d}\xi,\quad \text{on } r\leqslant a. \tag{34}$$

This Abel integral equation is easily inverted to give:

$$\frac{h(\xi)}{\xi}=-\int_t^a\frac{2Cr/\pi}{\sqrt{r^2-\xi^2}}\,\mathrm{d}r,$$

i.e.

$$h(\xi)=-\frac{2C\xi}{\pi}\sqrt{a^2-\xi^2}. \tag{35}$$

The subsequent evaluation of the stress components, in principle requiring differentation alone, is, in practice, a lengthy process, and only the results on $r=a$ will be summarised here.

$$\begin{aligned}\sigma_{\theta z}(a,z)&=-\frac{\partial^2\psi}{\partial r\partial z}=aC\big[J(1,0,0)-\omega J(1,1,0)\big], \qquad (36)\\ \sigma_{r\theta}(a,z)&=\frac{1}{r}\frac{\partial\psi}{\partial r}-\frac{\partial^2\psi}{\partial r^2}=aC\left[J(0,0,0)-\frac{4\omega}{3}J(1,1,-1)-\frac{4}{3}J(1,0,-1)\right],\end{aligned}$$

where

$$J(m,n,\lambda)=\int_0^\infty \mathrm{e}^{-\omega x} J_m(x) J_n(\rho x) x^\lambda \,\mathrm{d}x$$

and

$$\omega=\frac{z}{a}.$$

Not that $J(m,n,\lambda)$ reduces to $I(m,n,\lambda)$ when $\rho=1$. These J integrals are derived explicitly using results in (Eason et al., 1955) to give

$$\begin{aligned}\sigma_{\theta z}(a,z) &= aC\left[\frac{1}{2}-\frac{\omega}{2\pi k}\big(4-k^2\big)K(k)+\frac{2\omega}{k\pi}E(k)\right],\\ \sigma_{r\theta}(a,z) &= \frac{4aC}{3k\pi}\left[\left(\omega^2+\frac{3}{4}k^2\right)K(k)-\big(\omega^2+2\big)E(k)\right].\end{aligned} \tag{37}$$

4. Integral equation formulation

The bilateral solutions give the shear traction distribution present along the surface of the shaft. An abitrary shrink-fit residual stress, $\sigma_{rr}=\sigma_0$ (<0) is also present. The normalised forms of the bilateral stresses will denoted by $(a^3\widetilde{\sigma}_{r\theta}(z)/T,\ a^3\sigma_0/T)$ and the variation of the shear traction with depth is shown in Fig. 2. The ratio $(\widetilde{\sigma}_{r\theta}(z)/\sigma_0)$ becomes infinite as $z\to 0$, regardless of the ratio $T/a^3\sigma_o$, which is the key dimensionless ratio representing the relative effects of torque and residual compression. It therefore follows that, regardless of the value of the coefficient of interfacial friction, there will always be a slip zone, which starts at the surface and extends into the solid with increasing torque.

Suppose that the depth of penetration of slip is d. The shear traction present, $S(z)$, has contributions $\widetilde{\sigma}_{r\theta}$ from the bilateral solution, together with the effect of an unknown distribution of dislocations, $B_\theta(\zeta)$, i.e.

$$S(z)=\widetilde{\sigma}_{r\theta}(z)+T^0\int_0^d B_\theta(\zeta)K(z,\zeta)\,\mathrm{d}\zeta.$$

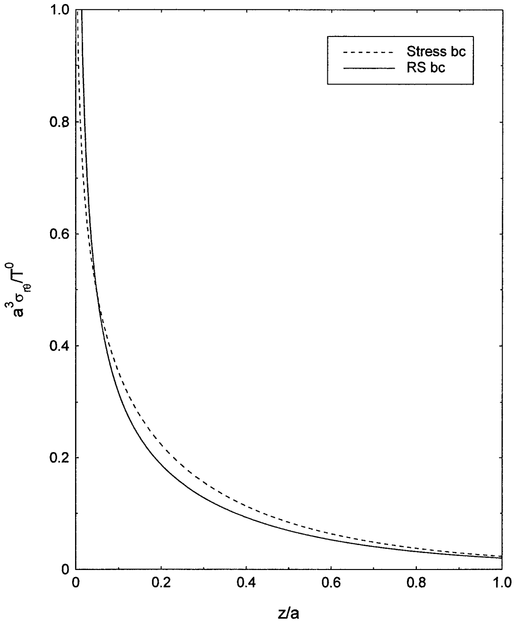


Fig. 2. The two bounding bilateral solutions (normalised w.r.t. T^0/a^3) shown on $r=a$ as a function of z/a.

If there is limiting friction over the interval $0 \leqslant z \leqslant d$, this may be expressed by putting

$$S(z) = \pm f\sigma_0.$$

After normalization, therefore, we arrive at the integral equation under a monotonically increasing torque:

$$f\frac{a^3\sigma_0}{T^0} = \frac{a^3\widetilde{\sigma}_{r\theta}(z)}{T^0} + \int_0^d B_\theta(\zeta)K(z,\zeta)\,\mathrm{d}\zeta, \quad 0 \leqslant z \leqslant d, \tag{38}$$

where the dislocation density, $B_\theta(\zeta)$, is defined by

$$B_\theta(\zeta) = \frac{\mathrm{d}b_\theta}{\mathrm{d}z}$$

and the kernel, $K(z,\zeta)$, is defined by equation (15)

$$\sigma_{r\theta}(z) = K(z,\zeta)b_\theta(\zeta).$$

It has already been noted that this kernel has a Cauchy singularity. The equation may therefore be solved in the usual way (Hills et al., 1996), giving the following numerical representation of the quadrature. The dislocation density must remain bounded at $r = a, z = d$ so Erdogan's method of extending the range of definition of the integral to $z = \pm d$ (Erdogan and Gupta, 1973) is used. Equation (38) is first normalised over the range ± 1 by the substitutions:

$$z = ds, \qquad \zeta = dr.$$

We then let:

$$B_\theta(r) = \frac{4\sigma_0}{\mu}\phi(r)\sqrt{1-r^2}.$$

The integral equation may now be discretised using a Gauss–Jacobi quadrature, which takes the form

$$\sum_{i=1}^{N}\frac{1-r_i^2}{2(N+1)}\phi(r_i)K(s_k,r_i) = f - \widetilde{\sigma}_{r\theta}, \quad k = 1,2,\ldots,N+1, \tag{39}$$

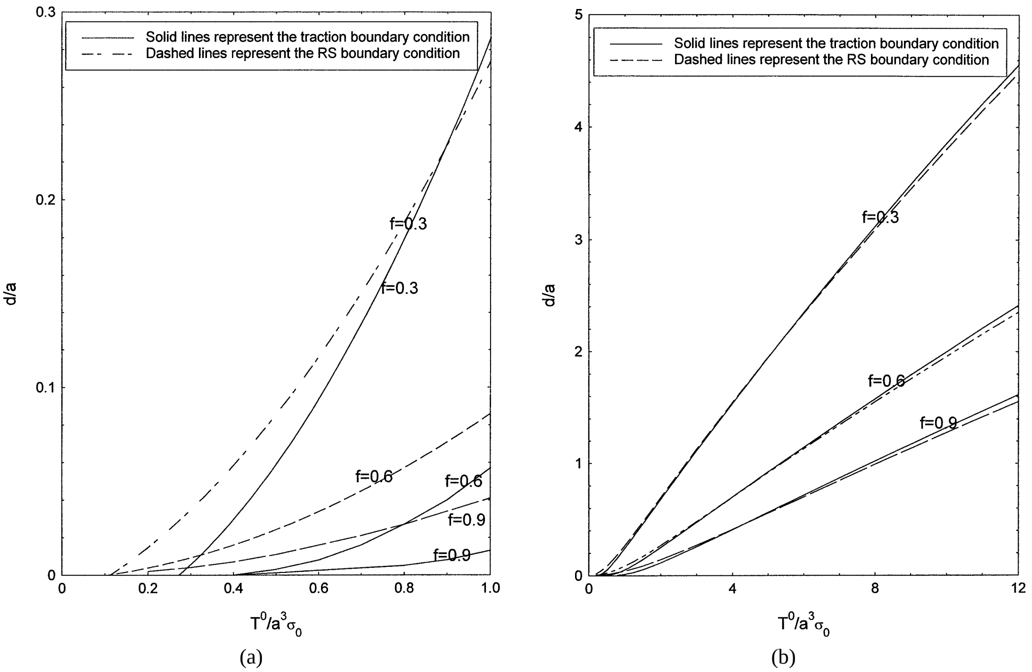
$$r_i = \cos\left(\frac{i\pi}{2(N+1)}\right), \qquad s_k = \cos\left(\frac{(2k-1)\pi}{4(N+1)}\right).$$

It will be recognised that there is one more equation than there are unknown coefficients $\phi(N)$, and this is to be anticipated, as a bounded solution to equation (38) has been sought, so that a side condition arises, which is encapsulated in the additional equation. In practice, a value of d/a is guessed, and one equation in (39) is discarded. The remaining $N \times N$ matrix is inverted, giving the values of $\phi(r_i)$, and the discarded equation is then multiplied out. This will not be satisfied exactly, and hence the value of d/a is adjusted and the process repeated until a satisfactory agreement is attained.

5. Results

First, the two 'bounding' bilateral solutions, Fig. 2, show a remarkably close agreement over a wide range of depths. In this and in subsequent figures we employ the legends 'stress BC' to denote the linearly varying traction boundary condition, and 'RS BC' to denote the Reissner–Sagoci or linearly varying displacement boundary condition. As the two surface traction distributions are statically equivalent it would be reasonable to expect them to converge on the same value for large z/a, from St. Venant's principle, but this figure shows that they are very nearly the same at values as small as unity. We would therefore expect the full solutions to the problem to be very similar.

The primary unknown is the depth of penetration of the slip zone, d/a, and this is shown as a function of the magnitude of applied torque, non-dimensionalised with respect to the compressive pre-stress ($T^0/a^3\sigma_0$) in Fig. 3, (a) and (b). These display results for a large range of torques, and show that the results are close, but do not coincide. Further results, in which the depth of penetration is examined as far as $d/a = 20$ show that the bounding solutions do tend to the same value, as we would expect from a consideration of the bilateral solution. It is clear, however, that the convergence of the slip penetration solution is very much slower than the underlying stress field itself. As expected, the depth of slip penetration is greatest for the lowest coefficient of friction, and vice versa. The corrected shear traction (normalised with respect to σ_0) is also plotted in Fig. 4 for a value of $d = 0.2$.



(a) (b)

Fig. 3. The penetration of slip, d/a, as a function of normalised torque, $T^0/a^3\sigma_0$. In this figure and in Figs. 4 and 5 results are shown for the two bonding solutions and $f = 0.3, 0.6, 0.9$: (a) shallow slip $0 \leqslant T^0/a^3\sigma_0 \leqslant 1.0$; (b) deep penetration $0 \leqslant T^0/a^3\sigma_0 \leqslant 12$.

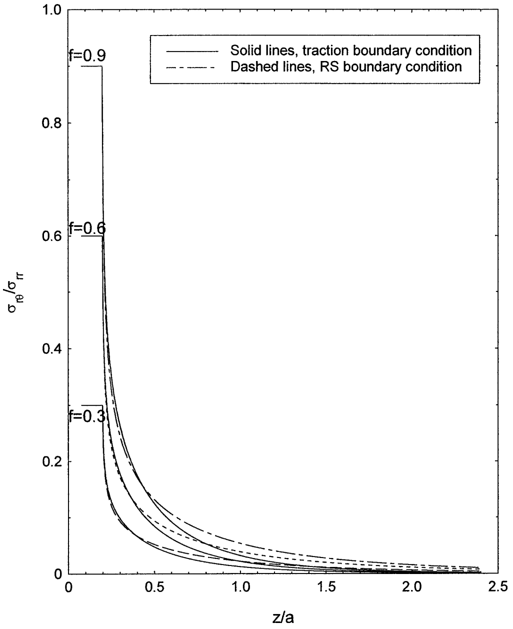


Fig. 4. Shear tractions distributions for $d/a = 0.2$.

Additional torsional compliance is introduced into the problem by the presence of interfacial slip. This may easily be calculated from the dislocation density by evaluating the relative twist across the interface, $\Delta\theta$, from the equation

$$\Delta\theta(z) = \int_z^d B_\theta(\zeta)\,\mathrm{d}\zeta. \tag{40}$$

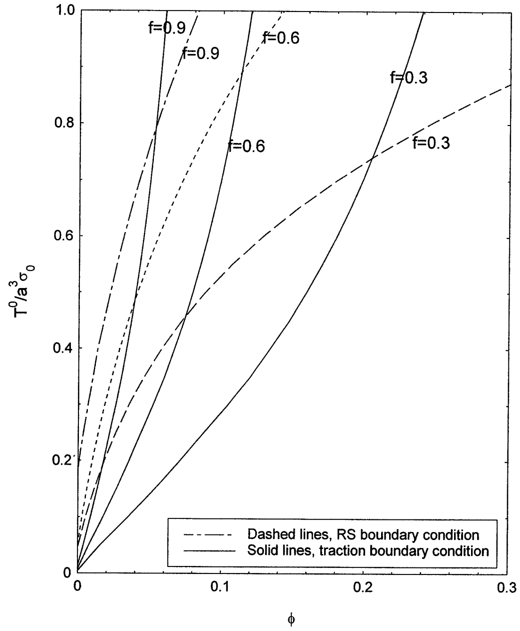


Fig. 5. The torsional compliance associated with slip.

The surface relative slip displacement, $\Delta\theta(0)$, is plotted in Fig. 5 for the two bounding solutions.

6. Conclusion

The paper describes in detail the solution for the slip penetration of a shaft embedded in an elastic half-space. The solution has been formed from two bilateral solutions which, in some sense, described bounding behaviour for the internal stress state within the half-space under adhesive conditions. The formulation for a novel dislocation is also given, all three solutions being formed using Green and Zerna's formulation. The use of an integral equation to solve the frictional contact problem is described in detail.

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