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ORIGINAL ARTICLE

Frictional systems subjected to oscillating loads

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Abstract If a discrete elastic system with frictional interfaces is subjected to periodic loading, the eventual steady-state response may depend on the initial condition or an initial transient phase of the loading history. In cases where shakedown is possible, it is known that it will occur for all initial conditions if there is no coupling between slip displacements and normal contact tractions, but that when coupling is present, counter-examples can be developed where the steady-state depends on the initial conditions. In this paper, we explore the conjecture that this is a special case of a more general theorem that the time-varying terms in the steady-state solution for an uncoupled system are always independent of initial conditions. In such problems, the 'memory' of previous events can only be stored in the slip displacements at nodes that are presently strictly within the friction cone. If all the nodes slip at some point in the cycle, this memory must be continually exchanged between nodes, with a consequent degradation, or loss of memory, resulting in an asymptotic approach to a unique steady state. This behaviour is illustrated using a simple two-node example. When there exists a set of 'permanently stuck nodes', these constitute a repository for the system memory, but in uncoupled problems the displacements at these nodes have no effect on the normal tractions at the slipping nodes and hence on the time-varying terms in the solution These arguments are illustrated in the context of two examples: a random distribution of frictional microcracks in a block loaded in plane strain and a generalized Hertzian contact problem with friction.

Keywords Coulomb friction · Shakedown · Damping · Uniqueness

1 Introduction

Engineering systems typically comprise numerous separate components and the interfaces between these components are usually frictional. The resulting frictional contact problems are often of considerable technological importance. For example, frictional interactions are central to the modeling of the constitutive behaviour of granular materials and soils [19, 36, 40] and to the slip behaviour of tectonic plates [35]; they determine the rebound behaviour of bodies in oblique impact[37] with applications ranging from powder technology and fluidized beds [14, 26] to sports [12, 15]; they are determining factors in the performance of gripping robots [38] and of friction dampers [25, 34], and microslip at bolted joints influences the dynamic behaviour of machine tools [6, 28]. More generic surface interactions involving microslip include the contact of bodies with rough surfaces [7, 39] and the development of appropriate interaction laws for use in the analysis of masonry structures [1] and for the discrete element method [13, 27].

Many of these applications involve periodic (cyclic) loading, due to vibration or to repetitive operations and the energy dissipated in friction under these circumstances is an important measure of system performance. It has been estimated that frictional hysteresis in assembled structures accounts for more energy dissipation than internal material damping, but the effective damping in such cases is notoriously difficult to quantify. Also, the energy dissipated in

The author dedicates this paper to Michel Jean on the occasion of his 70th birthday.

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microslip regions produces a favourable environment for the generation of fretting fatigue cracks [29, 33], which are a primary mode of failure in many nominally static contacting systems subject to vibration, including notably the blade root contact in aero engines [32].

In this paper, we shall discuss aspects of the response of frictional-elastic systems to periodic loading with particular reference to uniqueness of solution and the amount of frictional dissipation.

2 The Coulomb friction law

The Coulomb law has been criticized extensively by tribologists, but it is sufficiently close to the experimentally observed behaviour of many macroscopic systems to justify use in engineering design, because of its simplicity. Of course, this simplicity is something of an illusion. The governing equations are linear (in two-dimensions), but the associated inequalities lead to quite complex behaviour, including non-uniqueness, non-existence [22], jumps in the quasi-static incremental response [2], instability [9] and wedging [5] when the coefficient of friction is 'sufficiently large'. However, in this paper, we focus on cases where the coefficient of friction is below this critical value, noting that Klarbring [23] has defined a criterion for determining the critical coefficient for discrete systems. It then follows that the incremental problem is well-posed and has a unique solution.

To fix ideas, consider the problem of a two-dimensional elastic body that makes frictional contact with one or more rigid obstacles in some contact area Γ_c. We define local normal and tangential tractions at each point in Γ_c as p, q respectively and the corresponding normal and tangential displacements as w, v. We adopt a sign convention such that p represents a compressive contact traction and w represents a positive gap between the elastic body and the obstacle as shown schematically in Fig. 1. With this notation, the incremental Coulomb friction law states that each point must be in one of four states defined as

$$\begin{array}{lllll} \text{Stick} & w=0; & \dot{v}=0; & p\geq 0; & |q|\leq fp \\ \text{Separation} & w>0; & p=0; & q=0 & \\ \text{Forward slip} & w=0; & \dot{v}>0; & p\geq 0; & q=-fp \\ \text{Backward slip} & w=0; & \dot{v}<0; & p\geq 0; & q=fp\ , \end{array} \tag{1}$$

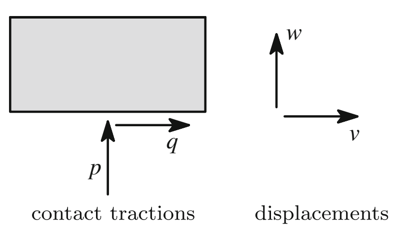

Fig. 1 Notation and sign convention for contact tractions and displacements

where f is the coefficient of friction and the dot denotes the derivative with respect to time t.

2.1 Multinode discrete system

Consider a discrete two-dimensional system with N contact nodes $i \in (1, N)$. We can then define N-vectors p_i, q_i, w_i, v_i and if the system is linearly elastic, these quantities must be linearly related through the equations

$$\left\{ \begin{array}{c} q_i \\ p_i \end{array} \right\} = \left\{ \begin{array}{c} q_i^w \\ p_i^w \end{array} \right\} + \begin{bmatrix} A_{ij} & B_{ij} \\ B_{ji} & C_{ij} \end{bmatrix} \left\{ \begin{array}{c} v_j \\ w_j \end{array} \right\}, \tag{2}$$

where p_i^w, q_i^w are the contact tractions that would be produced if the nodes were all welded in contact with $v_i = w_i = 0$. Notice that

$$\boldsymbol{K} = \begin{bmatrix} A_{ij} & B_{ij} \\ B_{ji} & C_{ij} \end{bmatrix} \tag{3}$$

is the reduced stiffness matrix for the system and hence must be symmetric and positive definite. It follows that $\boldsymbol{A}, \boldsymbol{C}$ are also symmetric and positive definite, but no such restriction applies to the matrix $\boldsymbol{B}$ which defines the coupling between normal and tangential effects. In fact, we shall find that the behaviour of the system is critically dependent on this coupling matrix.

2.2 History-dependence and memory

A critical feature of the frictional contact problem is that the instantaneous state generally depends on the loading history as well as on the instantaneous values of the applied external loads. This arises because the stick condition in (1) includes the slip velocity $\dot{v}$ which is a time derivative. It is easy to demonstrate that if the external loads applied to a single node discrete system permit a solution that is strictly within the limiting friction condition (i.e. $|q| < fp$), then the system can also support a range of distinct equilibrium states without violating the stick condition. Which state will be realized in practice depends on the initial conditions of the system and/or on the loading path from that condition to the present state. More general frictional systems share this feature. In particular, it suggests the possibility that if the system is subjected to a periodic external load, the final state may depend on the initial condition or on the transient path from an initial unloaded state to the periodic cycle.

History dependence requires that the system should in some sense posess *'memory'*. Furthermore, it is clear that the system memory must reside in the slip displacements at nodes that are not slipping (at the present instant), and that lie strictly within the boundaries defined by the frictional inequalities. After all, if at some instant all nodes were either slipping or separated, the instantaneous displacements would be completely defined in terms of the instantaneous external loads by the set of N Eqs. (1, 2) and hence would be independent of the previous history.

The slip displacements are contained in the vector v_i and the trajectory in v_i-space is therefore a good way of tracking the evolution of the system. If the applied loads are periodic in time, we anticipate eventually reaching a steady state which will define a closed loop in this space. We might reach this state after a finite number of load cycles, or alternatively, we might approach it asymptotically. However, because of the history-dependence, there is the possibility that the steady state may depend on the initial condition, or on the initial loading path.

2.2.1 *Problems involving no separation*

To explore this history dependence in more detail, we consider the subset of problems in which no nodes leave contact ($w_i = 0$ for all i), in which case Eq. (2) reduce to

$$q_i = q_i^w + A_{ij} v_j$$
$$p_i = p_i^w + B_{ji} v_j \ .$$

The instantaneous state of the system is then defined by a point $P(v_1, v_2, v_3, \ldots, v_N)$ in v_i-space and this point must satisfy the frictional constraints

$$\begin{aligned} (A_{ji} - fB_{ji}) v_i(t) &\le f p_j^w(t) - q_j^w(t) \\ (A_{ji} + fB_{ji}) v_i(t) &\ge -f p_j^w(t) - q_j^w(t) \ , \end{aligned} \tag{4}$$

which define a set of $2N$ directional hyperplanes. These hyperplanes advance and recede as the load varies periodically, but retain the same slope (which is determined by $\boldsymbol{A}, \boldsymbol{B}$ and f. During periods of slip, one or more of the advancing hyperplanes 'push' the point P around the space.

For the assumption of complete contact to be satisfied, it is necessary that at all times there exist at least one point that is not excluded by the constraint set (4). If this condition is violated, it is easy to show that the full system will involve one or more nodes in separation.

2.2.2 *The two-node system*

This process is most easily illustrated for the simple two-node system ($N = 2$), for which there are just four such constraints

$$\begin{aligned} (A_{11} - fB_{11}) v_1 + (A_{12} - fB_{12}) v_2 &\le f p_1^w - q_1^w \quad \mathrm{I} \\ (A_{11} + fB_{11}) v_1 + (A_{12} + fB_{12}) v_2 &\ge -f p_1^w - q_1^w \quad \mathrm{II} \\ (A_{21} - fB_{21}) v_1 + (A_{22} - fB_{22}) v_2 &\le f p_2^w - q_2^w \quad \mathrm{III} \\ (A_{21} + fB_{21}) v_1 + (A_{22} + fB_{22}) v_2 &\ge -f p_2^w - q_2^w \quad \mathrm{IV} \end{aligned}$$

which control the motions I: $\dot{v}_1 < 0$;II: $\dot{v}_1 > 0$; III: $\dot{v}_2 < 0$; IV: $\dot{v}_2 > 0$.

The process is illustrated in Fig. 2, where each constraint is represented by a straight line, with the disallowed side of this line shaded. For example, if the external loads change in such a way that $-fp_2^w - q_2^w$ increases, constraint IV will advance, pushing P in the direction $\dot{v}_2 > 0$ (forward slip at node 2).

During periodic loading, the constraints in Fig. 2 will advance and recede in a periodic sequence and we anticipate eventually reaching a steady state in which P traces out a unique path in v_i-space.

2.3 Shakedown

A special case of some interest is that in which the system 'shakes down', meaning that there is no frictional slip in the steady state, all nodes remaining permanently stuck. In other words, slip that may have occurred early in the loading process is sufficient to prevent further slip in the steady state. A necessary condition for shakedown is the existence of a *safe shakedown state*—i.e. a position of the point P that is never excluded by any of the constraints. Suppose we identify the extreme positions (maximum advance) of the constraints, which we denote by I^{E}, II^{E} etc., as shown in Fig. 3. The set $\mathcal{H}$ of all safe shakedown states P will then comprise the *safe shakedown region* that is not excluded by any of these extreme constraints.

Tribologists (e.g. [10]) speculated for many years that a variant of Melan's theorem [30] should apply to frictional systems. In other words, that the existence of a non-null safe shakedown region is also a *sufficient* condition for shakedown to occur. However, Klarbring et al. [24] have since shown that such a theorem applies if and only if there is no coupling between the tangential and normal contact problems. For the discrete system of equation (2), this is

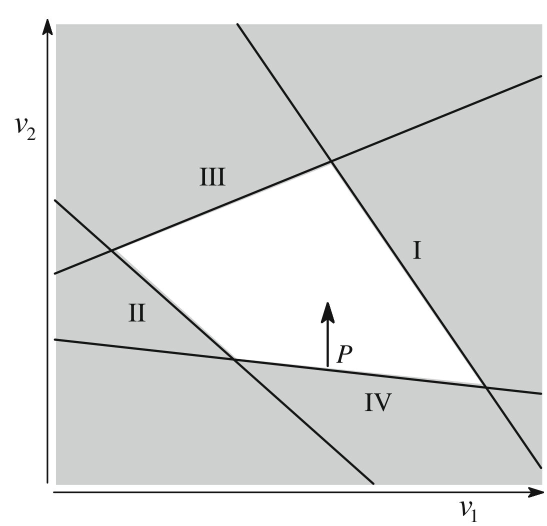

Fig. 2 Motion of the instantaneous state P due to motion of constraint IV

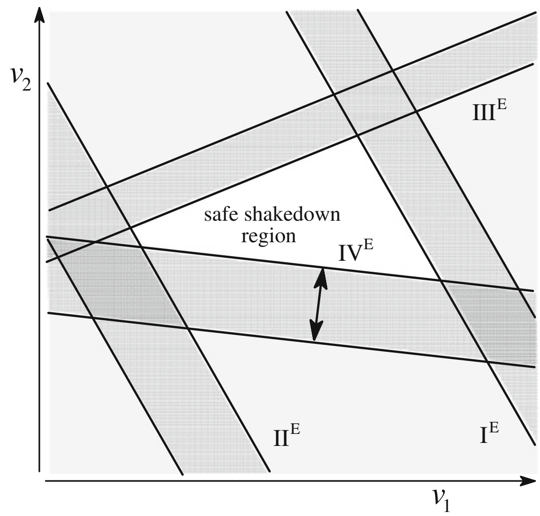

Fig. 3 Each constraint cycles between two extreme positions (whilst retaining the same slope). The safe shakedown region is the set of points that is never excluded by any constraint

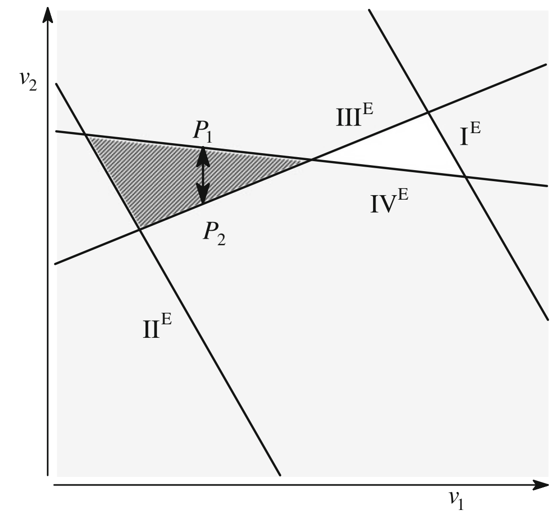

Fig. 4 Extreme constraint II^E does not abut the (unshaded) safe shakedown region, so cyclic slip can occur between P_1 and P_2

equivalent to the condition that the matrix $\boldsymbol{B} = \boldsymbol{0}$. The constraints (4) then reduce to

$$A_{ji} v_i(t) \leq f p_j^w(t) - q_j^w(t)$$
$$A_{ji} v_i(t) \geq -f p_j^w(t) - q_j^w(t)$$

implying that the two hyperplanes corresponding to a given node j are parallel. In particular, for the two-node system, the safe shakedown region of Fig. 3 will then be a parallelogram.

For the more general coupled case where $\boldsymbol{B} \neq \boldsymbol{0}$, Ahn et al. [3] have shown that P always reaches $\mathcal{H}$ if this is a quadrilateral, but not always if it is triangular, as shown in Fig. 4.

For example, with an appropriate initial condition, the system can end up oscillating between the points P_1 and P_2, since the constraint II never advances far enough to push P into the safe shakedown region defined by the 'active' extreme constraints I^E, III^E, IV^E in Fig. 4. This criterion can be generalized to the N-node discrete system to state that the system will always shake down if all of the $2N$ extreme constraints are active in defining $\mathcal{H}$—i.e. if all the extreme constraints abut this region in at least one point. When one or more of the extreme constraints are inactive, we can always devise initial conditions such that the system remains in a state of cyclic slip, despite the fact that $\mathcal{H}$ is non-null.

Consider a system of external loads defined by

$$\boldsymbol{p}^w(t) = \boldsymbol{p}_0 + \lambda \boldsymbol{p}_1(t) \; ; \; \boldsymbol{q}^w(t) = \boldsymbol{q}_0 + \lambda \boldsymbol{q}_1(t), \tag{5}$$

where λ is a scalar load factor characterizing the magnitude of the periodic (time-varying) components. If λ is increased, the extreme constraints will exclude more of the space and we can identify lower and upper bounds λ_1, λ_2 such that for $\lambda < \lambda_1$, $\mathcal{H}$ is defined by all the $2N$ constraints and for $\lambda > \lambda_2$ it is null. Thus, dependence of shakedown on the initial condition occurs only in the range $\lambda_1 < \lambda < \lambda_2$. By contrast, if $\boldsymbol{B} = \boldsymbol{0}$ and the constraints are parallel, they will all remain active until $\mathcal{H}$ becomes null in the condition where one pair of parallel constraints conspire to exclude the entire space. In the context of the simple two-node system, this is equivalent to the statement that a parallelogram is *a fortiori* a quadrilateral, so Ahn's criterion confirms that Melan's theorem applies to systems with $\boldsymbol{B} = \boldsymbol{0}$. For the N-node system and $\boldsymbol{B} \neq \boldsymbol{0}$, the number of facets in $\mathcal{H}$ will decrease monotonically with increasing λ, generally becoming a simplex of v_i-space somewhat below the limiting value λ_2. In fact, the values of λ at which the number of facets changes can be determined from the condition that $N + 1$ of the $2N$ constraints intersect in a point.

3 Loading above the shakedown limit: dissipation

Suppose we apply a system of external loads such that $\lambda > \lambda_2$ and the safe shakedown region $\mathcal{H}$ is null. We must then anticipate that the steady state will involve cyclic slip and as a result some energy will be dissipated in friction. In view of the above discussion, an immediate question is whether this steady state is unique, or whether it depends on the initial conditions.

We shall explore the following conjecture:-

Conjecture That for uncoupled systems in the steady-state, the tractions at the slipping nodes, the time-varying terms in the remaining tractions and displacements, and hence the energy dissipation per cycle are independent of initial conditions.

If this conjecture were true, the frictional Melan's theorem would be a special case—that in which the steady-state dissipation is zero. By contrast, for coupled systems we anticipate that all features of the steady state including the energy dissipation per cycle will sometimes depend of initial conditions.

There is some anecdotal evidence for this conjecture.

(1) Fretting fatigue tests are very consistent for smooth 'Hertzian-like' contact geometries, but much more erratic when a flat indenter is pressed against a plane surface. The former geometry is reasonably approximated by two half planes, which involves no coupling, but the latter involves significant normal-tangential coupling.
(2) Experimental measurements of the effective damping in bolted joints shows that the results are very erratic. Apparently identical systems give different results and even the same system, if disassembled and then reassembled, can give very different results. Depending on the assembly protocol employed, this might be equivalent to a change in initial conditions.

For uncoupled systems, the normal nodal forces are unique, since they are equal to p_j^w and are defined by the periodic loading. It follows that *during sliding* (in a given direction) the tangential tractions are also unique. To fix ideas, we consider some scenarios for the uncoupled two-node system. In Fig. 5, the extreme positions of III^E, IV^E leave a 'safe' space, but I^E, II^E overlap, forcing cyclic slip at node 1. We assume that these extreme positions are reached at different times in the periodic cycle and that there is no time at which the instantaneous constraints overlap, since this would imply periods of separation at node 1. For example, consider the scenario in which constraint I advances to I^E and then recedes, after which II advances to II^E and then recedes.

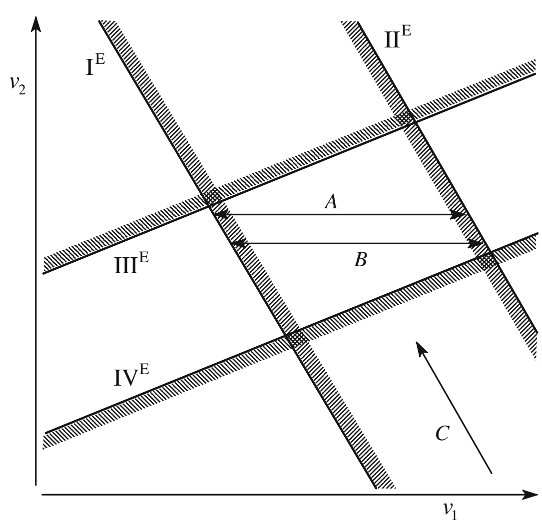

Fig. 5 Steady-state scenario with node 2 permanently stuck

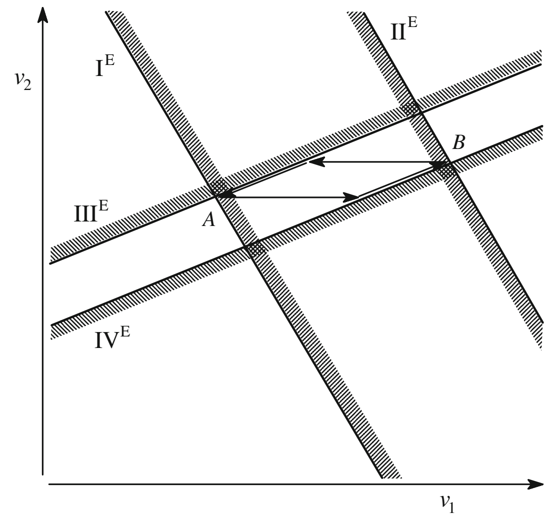

Fig. 6 Both nodes slip and the steady state is unique

The lines A and B define two possible steady states consistent with this scenario. In both cases, node 2 is permanently stuck during the steady state, but with different values for the 'locked-in' displacements v_2.

The existence of a safe region between III^E and IV^E is not sufficient to guarantee a steady-state with node 2 stuck. Figure 6 shows a scenario similar to Fig. 5, but with this space reduced. In this case, the steady state involves cyclic slip at both nodes and it is unique.

More generally, we can see that non-uniqueness of the steady state is associated with the locked-in displacements at the set $\mathcal{S}_0$ of *permanently stuck nodes*, which constitute the memory of the system. This conclusion also applies to coupled systems.

3.1 An alternative representation for uncoupled systems

In Fig. 5, the trajectories A and B differ only by a translation in direction C and the time-derivatives of the displacements (and hence of all the tractions) are the same for all such scenarios. In fact, we can make the steady state appear unique if we look in the direction C, which is equivalent to projecting the motion on a line perpendicular to C.

To develop a similar projection for the N-node uncoupled system, we need to change the basis of the slip-displacement N-vector $\boldsymbol{v}$. We define a new N-vector $\boldsymbol{\tau}$ through the linear operation

$$\boldsymbol{\tau} \equiv \boldsymbol{A}\boldsymbol{v} \quad \text{or} \quad \tau_j = A_{ji} v_i \; .$$

The constraints at node j now take the form

$$\tau_j \le f p_j^w - q_j^w$$
$$\tau_j \ge -f p_j^w - q_j^w \, .$$

Physically, τ_j is the *change* in the tangential reaction q_j due to the slip displacements v_i.

Figure 7 shows the two-node trajectories A, B from Fig. 5 when projected into τ-space. The constraints are now perpendicular to the axes, but slip at one node now corresponds to an inclined trajectory. Projection of A or B onto a line perpendicular to τ_2 generates a unique one-dimensional trajectory in the reduced space τ_1.

For the N-node system, suppose that in the steady state, M nodes slip at least once during each cycle and $(N - M)$ nodes never slip (they comprise the set $\mathcal{S}_0$ of permanently stuck nodes). We shall obtain a unique steady-state trajectory if we project the actual trajectory on the M-dimensional space $\mathcal{S}_1 \equiv \tau_j, j \notin \mathcal{S}_0$.

It then follows that the tractions at the slipping nodes are

$$q_j = q_j^w + \tau_j$$

and hence they are unique, since they depend only on quantities defined in $\mathcal{S}_1$. Notice that in this reduced space, each constraint will be active at some time during each cycle.

3.2 Systems in which $\mathcal{S}_0$ is null

As a first step towards a possible proof of the conjecture, we consider the restricted class of problems in which all nodes slip at some time during each period and hence the set $\mathcal{S}_0$ is null. We propose the following lemma:-

Lemma *That for a system in which $\mathcal{S}_0$ is null, the system must approach a unique steady state, even if only asymptotically.*

A heuristic argument in favour of this lemma is that the system memory is stored at any instant in the displacements v_i at nodes which are instantaneously stuck. However, since no node is permanently stuck, the memory must be exchanged between nodes during every cycle and this exchange can only involve a degradation of memory. To illustrate this concept, we consider the coupled two-node system of Fig. 8, in which the extreme positions of the four constraints are reached in the sequence $\mathrm{I^E}$, $\mathrm{III^E}$, $\mathrm{II^E}$, $\mathrm{IV^E}$ and each constraint recedes before the next one advances, so only one node is slipping at any given time.

Two initial conditions are considered and in each case the cycle tends asymptotically towards a unique steady state. In fact, we can solve for the steady state in this example, since each corner of the rectangle must lie on one of the four extreme constraint lines and each pair of adjacent corners shares the value of one of the displacements v_1,v_2. This provides eight linear equations for the eight co-ordinates defining the corner points and the solution is therefore unique, which establishes the lemma for loading scenarios of this form.

For the N-node system, things are rather more complex, but it remains reasonable that the system memory must degrade with each nodal exchange. The algebraic proof that the steady state is unique when $\mathcal{S}_0$ is null can also be extended to N nodes, but only under the rather severe restriction that only one node slips at a time.

It is worth remarking that for the N-node system, if there is any time at which all N nodes are slipping, the steady state must be reached immediately, since the system then loses any memory of previous conditions. For a formal proof of this result, suppose that all the nodes are slipping at time $t = t_0$, in which case one of the constraints (4) must be replaced by an equality for each of the nodes $j = (1, N)$. This provides N linear equations for the N unknowns $v_j(t_0)$

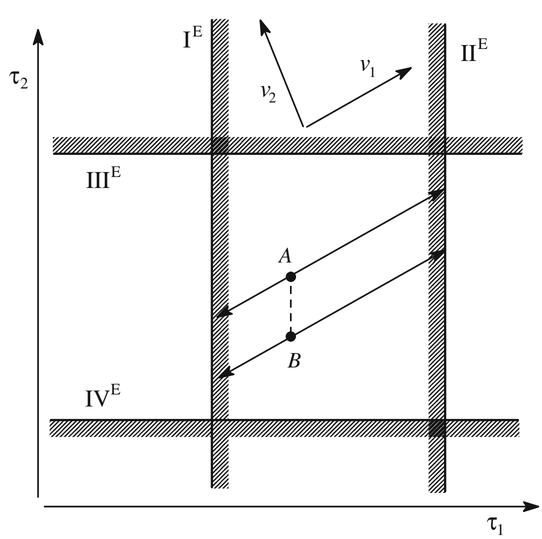

Fig. 7 The trajectories of Fig. 5 presented in τ-space

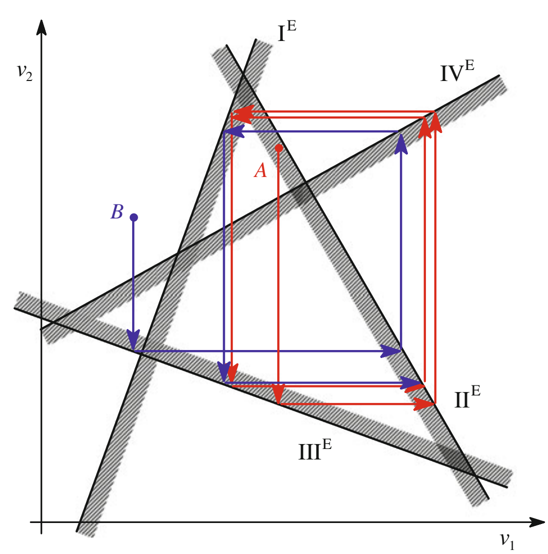

Fig. 8 Convergence of a coupled system on a unique steady state

and hence these instantaneous nodal displacements are uniquely determined in terms of the instantaneous loadings $p_j^w(t_0)$, $q_j^w(t_0)$ and are independent of the initial conditions $v_j(0)$. The subsequent periodic problem could then be redefined as an evolutionary problem starting from this unique initial condition at $t = t_0$ and is therefore henceforth independent of $v_j(0)$.

3.3 Consequences for systems with $\mathcal{S}_0 \neq 0$

Suppose now that we have a system in which there exists a non-null set $\mathcal{S}_0$ of permanently stuck nodes (or strictly of nodes that cease to slip after a finite number of cycles). Once these nodes have all slipped for the last time, their slip displacements v_j will be time-invariant and we can focus our attention on the evolution of the reduced system comprising the nodes that continue to slip. Of course, the slip displacements at the nodes in $\mathcal{S}_0$ will contribute to the tractions at the remaining nodes, but this contribution will also be time-invariant, so the evolution of the reduced system will be that corresponding to different vectors $\boldsymbol{p}_0, \boldsymbol{q}_0$ in Eq. 5. Thus, if the above lemma is regarded as proven, the steady-state will depend only on the slip displacements in $\mathcal{S}_0$.

Consider now the case where the system is uncoupled, so the slip displacements in $\mathcal{S}_0$ make no contribution to $\boldsymbol{p}_0$ in the reduced system. There will remain a contribution to $\boldsymbol{q}_0$, but this can be relaxed out without affecting the time-varying terms in the steady state. To explain this, imagine a scenario where only $\boldsymbol{p}_0, \boldsymbol{q}_0$ are applied to the reduced system and for the moment we set the coefficient of friction at the nodes in this system (i.e. the slipping nodes) to zero. The loads $\boldsymbol{q}_0$ will then produce tangential nodal displacements such as to reduce the tangential reactions to zero, without affecting the normal reactions. In other words, the system will relax to an equilibrium state. If we now reinstate the friction coefficient and resume the periodic loading pattern, the time-varying terms in the response will be identical with those for different slip displacements in $\mathcal{S}_0$. The only difference lies in the datum about which these time-varying terms oscillate. If instead we retain the friction coefficient throughout, the lemma shows that we shall tend asymptotically to this same state.

We conclude that a rigorous proof of the conjecture would be established if (i) we could provide a mathematical proof of the 'memory-loss' lemma and (ii) establish that the set $\mathcal{S}_0$ is unique for a given steady-state loading scenario.

4 Two examples

To illustrate the ideas presented in this paper, we briefly discuss results for two specific systems. One is a system of plane cracks in an otherwise homogeneous elastic material and the other is the generalized Hertzian contact problem for two bodies of similar materials.

4.1 A distribution of plane cracks

The elasticity solution for a single plane crack under a uniform far-field stress shows that the crack will either open completely, close completely and remain stuck everywhere, or close completely and slip everywhere. There are no conditions under which partial closure or partial slip can occur. It follows that a sufficiently sparse distribution of N plane cracks acts (mathematically) like a completely uncoupled discrete N-node frictional system. All the off-diagonal components of the stiffness matrix (3), including the whole matrix $\boldsymbol{B}$ are zero.

Ideally, we would like to be able to extend this argument to cases where the cracks are close enough for there to be some interaction. A representative randomly generated pattern of cracks is shown in Fig. 9. Kachanov [21] has shown that a good approximation to this interaction can be obtained by situating each crack in a uniform stress field comprising (i) the externally applied stress and (ii) a perturbation representing the *average* stress along the crack face due to the stress perturbations at all the other cracks. This average traction is easily calculated using the complex-variable solution to the isolated crack problem (crack i) to determine the forces transmitted across the line of crack j in the geometry of Fig. 10.

It then follows that the behaviour of the system of N cracks mimics that of an N-node discrete elastic system of which the stiffness matrix (3) is determined from the solution for a single crack in a uniform stress field [18]. The resulting discrete system is coupled ($\boldsymbol{B} \neq \boldsymbol{0}$).

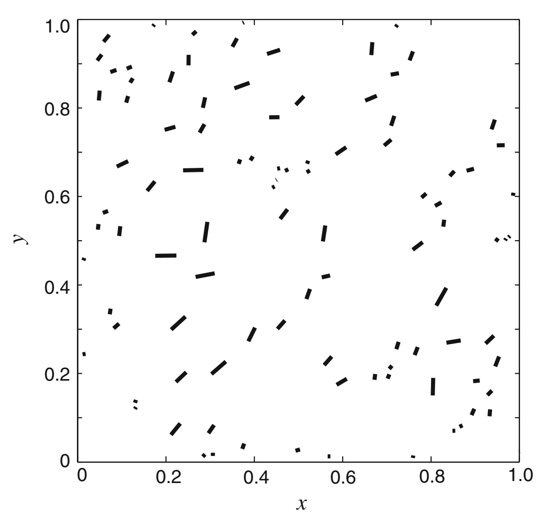

Fig. 9 A random distribution of microcracks

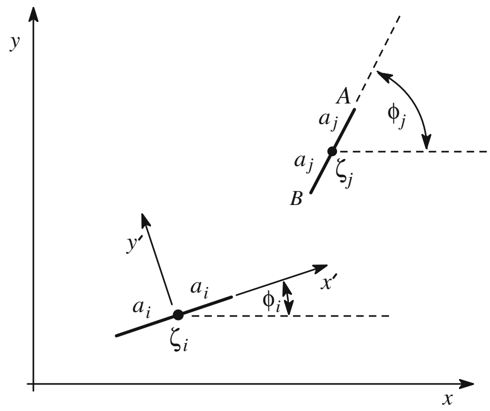

Fig. 10 A pair of interacting cracks

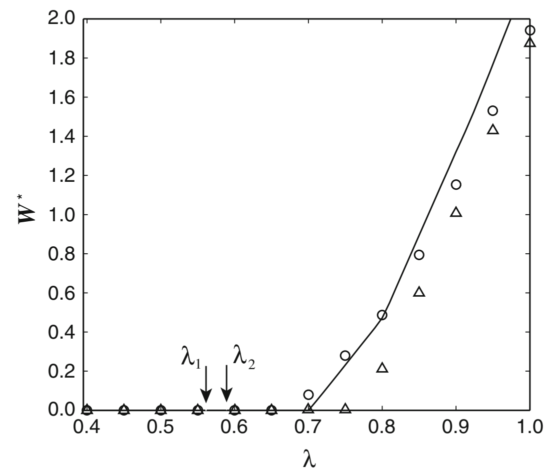

Fig. 11 Dimensionless dissipation rate W^* as a function of the scalar load factor λ

The interaction coefficients (stiffness matrix) were calculated for a random distribution of 100 cracks and the system was subjected to a far-field loading comprising a constant hydrostatic compressive stress and a periodic shear stress and defined by

$$\boldsymbol{S} = \boldsymbol{S}_0 + \lambda \boldsymbol{S}_1 \cos(\omega t)\ ,$$

where $\boldsymbol{S} \equiv \{\sigma_{xx}, \sigma_{xy}, \sigma_{yy}\}$ and

$$\boldsymbol{S}_0 = p_0\{-1, 0, -1\}; \quad \boldsymbol{S}_1 = p_0\{0, 1, 0\}.$$

Numerous runs were performed with different initial conditions at each value of λ and the resulting steady-state energy dissipation per cycle W was normalized as

$$W^* \equiv \frac{W\mu}{4(1-\nu)p_0^2\bar{a}^2},$$

where μ, ν are the modulus of rigidity and Poisson's ratio respectively and $\bar{a}$ is the mean crack semi-length. Figure 11 shows the maximum (◯) and minimum (Δ) levels of dissipation obtained as a function of λ. Ahn's algorithm [3] was used to determine the lower and upper bounds λ_1,λ_2 for shakedown, so that for $\lambda < \lambda_2$, the minimum dissipation is zero and for $\lambda < \lambda_1$, the *maximum* dissipation is also zero. The numerical results confirmed this conclusion and also showed that for $\lambda > \lambda_2$ there is a significant difference between the dissipation (and hence for the effective hysteretic damping in the material) depending on initial conditions. However, further increase in λ led to a decrease in history-dependence and at sufficiently large λ, the dissipation becomes unique. We recall that our lemma proposes that even for coupled systems, the steady-state cycle is unique if the periodic component in the load is sufficient to ensure that all nodes experience slip during every cycle and this explanation was confirmed in the present case by monitoring the proportion of cracks that experience at least one period of slip and/or opening per cycle in the steady state. When λ is sufficiently large for this to reach 100%, the dissipation was found to be unique.

4.2 A generalized Hertzian contact problem

As a second example, we consider the problem illustrated in Fig. 12, where two non-conforming bodies of similar materials are pressed together by a force P and sheared by a force Q, both of which can vary in an arbitrary but periodic way.

This is a continuum problem, which could be brought under the scope of the present discrete algorithm by a suitable finite element discretization. However, we shall show that it is possible to establish a continuum version of the above conjecture without recourse to discretization.

We suppose (i) that the linear dimensions of the bodies are very large compared with the extent of the contact region, (ii) that the loads P, Q are the resultants of tractions applied to the distant boundaries of these bodies and (iii) that the slope of the undeformed surfaces is everywhere small compared with unity. Under these conditions, the bodies can be approximated as half-planes in estimating the local deformations in the contact region due to the distributions of contact tractions. This approximation of course underlies the classical Hertzian theory of elastic contact and it has been applied in numerous continuum solutions of

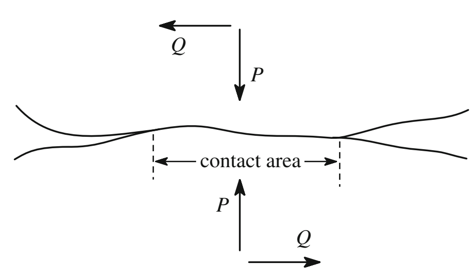

Fig. 12 A generalized Hertzian contact problem

problems in contact mechanics, including problems involving Coulomb friction [16, 20].

Notice that as long as the materials of the two bodies are similar, there will be no coupling between the tangential and normal contact problems. In fact, the uncoupled nature of the problem is retained as long as Dundurs' bimaterial parameter β is equal to zero [20]. Also, the uncoupled condition will be satisfied (including in discrete contact problems for finite bodies) whenever the contact occurs on a plane of symmetry between symmetrically loaded bodies of similar materials.

We remark here that it is an open question as to the conditions under which the general continuum frictional contact problem is well posed. However, as long as the above conditions are satisfied, the contact tractions and displacements are related by Cauchy singular integral equations [16] and the evolutionary solution can be shown to exist and be unique using the properties of the solutions of these integrals.

We consider the case where the bodies are subject to some form of vibration, so that the forces P, Q vary sinusoidally in time, but not necessarily in phase. As a result, the loading trajectory will trace an ellipse in PQ-space, as shown in Fig. 13. As in the discrete case, some initial transient is needed to reach this periodic state, represented in the figure by the line segment OA. We restrict attention to the case where $|Q| < f\,P$ at all times, so that there is no gross slip (sliding).

Cattaneo [8] and Mindlin [31] considered the case in which the contacting bodies have quadratic profiles (the classical Hertzian case) and are first loaded by a normal force P and then by an increasing tangential force Q. The resulting tangential tractions can be written as the superposition of the 'full slip' tractions over the entire contact area and a similar distribution over a smaller central stick area. In fact, this 'corrective' distribution is

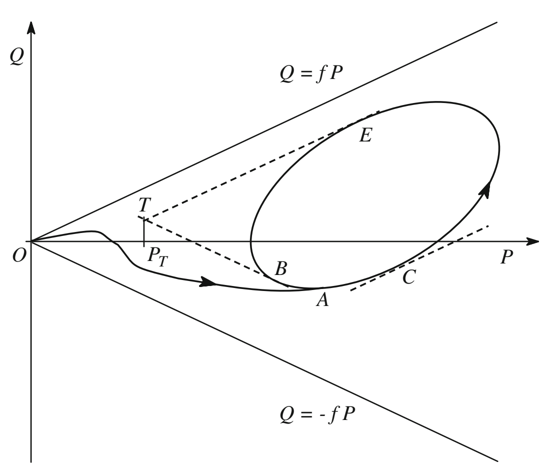

Fig. 13 The loading cycle in PQ-space

proportional to the normal tractions at a smaller value of the normal load P at which the contact area would be equal to what is now the stick area. Ciavarella [11] and Jäger [17] showed that this form of superposition is not restricted to Hertzian contact, but applies exactly to all uncoupled two-dimensional contact problems provided the bodies can be approximated as half planes. For general loading scenarios, it can be shown that there will be no (incremental) slip as long as

$$\frac{dP}{dt} > 0 \quad \text{and} \quad \frac{\partial |Q|}{\partial P} < f \tag{6}$$

and that in all other cases the incremental tangential tractions have the form of a Ciavarella-Jäger superposition. Combining these results, we were able to conclude that [4]

(1) No slip occurs anywhere in the contact area between B and C in Fig. 13, where the conditions (6) are satisfied.
(2) A slip region develops in the region CE surrounding a stick region that reaches its minimum extent at the point E, where the local tangent has slope f.
(3) The entire contact area sticks instantaneously at E, after which a region of reverse slip grows from the outer edge in the segment EB.

When the point E is reached on the second cycle, the tractions are identical to those at this point in the first cycle, so the steady state is established from this point onwards. Also [4], after the point E is passed for the second time, no further slip occurs in a region $\mathcal{A}_T$ which can be determined by (i) constructing tangents to the loading curve at the points E, B (where the slopes are $\pm f$), (ii) finding their intersection point T with coordinates (P_T, Q_T) and (iii) determining the contact area $\mathcal{A}_T$ for a normal load P_T. This construction is independent of the initial loading phase OA, showing that for this fairly general continuum problem, the permanent stick zone $\mathcal{S}_0$ is unique. Notice that the slip displacements locked into $\mathcal{A}_T$ do depend on OA as do the tractions in this region, but the differences between different solutions are self-equilibrated and the tractions in parts of the contact area outside $\mathcal{A}_T$ are unique.

Notice also that in this example, the steady state is reached after a finite period of loading, in contrast to (for example) Fig. 8, where it is approached asymptotically. The reason for this distinction is that in the half plane problem there is a point in the cycle (just before E) at which all points not in $\mathcal{S}_0$ are slipping simultaneously, whereas in Fig. 8, although both nodes slip, they never do so simultaneously and there is always a repository for system memory. Notice that the uncoupled scenario in Fig. 6 also involves both nodes slipping simultaneously and here also the steady state is reached in a finite number of cycles.

5 Conclusions

In this paper, we advance the conjecture that for frictional elastic contact problems subjected to periodic loading, important features of the steady state will be independent of the initial conditions if the system is 'uncoupled', meaning that slip displacements at the contact interface have no effect on the normal tractions. In particular, there will be a unique permanent stick zone within which tangential tractions may be non-unique, but the tractions outside this zone and the time-varying terms in the displacements will be unique. The previously proven 'Melan' theorem for frictional systems is a special case of this conjecture.

A heuristic proof is offered, based on the concept that dependence on initial conditions demands a system 'memory' that can reside only in regions that are instantaneously stuck. Memory from the regions outside the permanent stick zone must be exchanged between nodes during each cycle and will therefore degrade as the process continues. Memory stored in the permanent stick zone has no effect on the normal tractions at slipping nodes.

The conclusion is supported by examples using low-order discrete systems and also by the results of two practical examples.

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