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THEORIES OF THERMOELASTIC INSTABILITY DURING SOLIDIFICATION BASED ON THE NEUMANN TEMPERATURE SOLUTION

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The evolution of contact pressure at the metal/mold interface in an idealized, directional solidification process is examined using a thermohypoelastic stress function and the Neumann temperature solution. Nonuniform thermal distortion of the casting is established through the application of an oscillatory temperature profile at the mold interface. This is an attempt to study the effects of the often highly irregular cooling profiles in typical industrial casting processes (albeit in an idealized fashion). In many of these processes, it is virtually impossible to cool the ingot uniformly due to a variety of process conditions. Nonuniform thermal distortion causes the contact pressure at the mold interface as well as the macroscopic freezing front morphology to oscillate. If the contact pressure exceeds the hydrostatic pressure of the residual molten metal to the point where its net value is zero, then an air gap is nucleated. This represents a condition in which the casting and the mold surfaces locally begin to separate. If air gaps nucleate beneath the thinnest sections of the undulating freezing front, then a condition of growth instability exists since further heat extraction beneath these regions is significantly reduced. The thickest regions of the casting continue to grow due to improved contact and hence the freezing front nonuniformities become significantly exaggerated with time.

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NOMENCLATURE

a	coefficient associated with residual stress for Part 1
$\bar{a}_0(\tau)$	dimensionless coefficient defined by Eq. (46)
$a_i(\tau)$	coefficients of Eq. (30)
$b_i(\tau)$	coefficients of Eq. (50)
$\bar{b}_1(\tau)$	dimensionless coefficient defined by Eq. (69*b*)
c	specific heat
d	constant of integration
$erf(x)$	error function
g	acceleration due to gravity
i	zero or an integer
k	thermal diffusivity
m	wave number
n	summation index
p	cooling parameter defined by Eq. (18)
$s(x,t)$	casting thickness
$s_0(t)$	location of mean melt line
$s_1(t)$	evolution of spatial perturbation in casting thickness
t	time
$t^*(x, z)$	layer formation time
$\vec{u}$	displacement field
$\dot{u}_{z_1}$	total first order normal velocity
$\dot{u}_{z_1}^{H}$	normal velocity associated with homogeneous solution
$\dot{u}_{Z_1}^{P}$	normal velocity associated with particular solution
w	$\tanh(ms_0(t))$
x	lateral spatial variable
y	longitudinal spatial variable
z	spatial variable through the casting thickness
A_n, B_n, C_n, D_n	coefficients defined by Eqs. (19) and (28)
$\bar{C}_n$, $\bar{D}_n$	dimensionless coefficients defined by Eqs. (48)
E	Young's modulus
F	residual stress function for Part 2
$\bar{F}$	dimensionless form of F defined by Eq. (64)
G	$E/2(1 + \nu)$
H	height of the fluid above mold surface
K	thermal conductivity
L	latent heat
$P(x, t)$	contact pressure
$\bar{P}(x, \tau)$	dimensionless contact pressure defined by Eq. (47)
$P_0(t)$	zeroth-order contact pressure

$P_1(t)$	evolution of spatial perturbation in contact pressure
$S_0(\tau)$	dimensionless location of mean melt line
$S_1(\tau)$	dimensionless evolution of spatial perturbation in casting thickness
ΔT_0	$T_m - T_0^*$
$T(x, z, t)$	temperature distribution in casting
T_m	fusion temperature
T_0^*	uniform mold surface temperature
$T_0(z, t)$	zeroth-order temperature distribution
$\bar{T}_1^*(\tau)$	arbitrary perturbation in mold surface temperature
T_1^*	uniform temperature perturbation
$T_1(z, t)$	first-order temperature distribution
$W(a, b)$	function defined by Eq. (19*g*)
Z	dimensionless thickness variable defined by Eq. (17*b*)
α	coefficient of thermal expansion
θ	dimensionless spatial variable defined by Eq. (63)
ν	Poisson's ratio
ρ	density
ϕ	Airy stress function
λ	parameter defined in Neumann's solution for Parts 1 and 2
$\sigma_{ij}(x, z, t)$	total stress tensor
σ_{ij_0}	zeroth-order stress tensor
σ_{ij_1}	first-order stress tensor
τ	dimensionless time defined by Eq. (17*a*)
$\hat{\tau}$	dimensionless time defined by Eq. (65*b*)
ψ	displacement potential
$\underline{\Delta}_i(\lambda, \tau)$	functions defined by Eqs. (58)
$\bar{\Delta}_i(\lambda, \tau)$	functions defined by Eqs. (66)
Φ	confluent hypergeometric function
Λ_n	function defined by Eqs. (19*e*) and (19*f*)
P	particular solution
H	homogeneous solution

INTRODUCTION

During the solidification of metals in metal molds, the resulting properties of the cast product are largely controlled by the physics of the mold/casting interface. Initially, the contact pressure between the solidifying shell and mold is that due to the hydrostatic pressure in the molten fluid. As solidification proceeds, the temperature gradient through the metal shell produces a thermal distortion of the shell that influences the contact pressure. If the molten fluid pressure is insufficiently high, thermal distortion of the shell may overcome the pressure of the molten fluid leading to air gap nucleation at the metal/mold interface. Air gap nucleation leads to a drop in the thermal conductance at the metal/mold interface.

The reduction in heat extraction at the metal/mold interface due to air gap nucleation can result in a condition known as growth instability in which the freezing front exhibits large cellular undulations. These undulations, which are on a scale that greatly exceeds the dendrite arm spacing, are generally not observed in most industrial casting operations unless the casting process is interrupted and the residual fluid decanted. Nevertheless, freezing front growth instability can lead to catastrophic failure of a casting operation known as breakout, in which the lateral strength of localized regions of the metal shell does not sufficiently resist the pressure from the residual molten fluid. Growth instability can also enhance both surface and internal cracking of the cast product through nonuniform thermal distortions, and this can lead to an undesirable metallurgical structure within the cast product.

In the present article, we reconsider the thermoelastic model of growth instability of Li and Barber [1] in which the casting is subjected to a spatially periodic temperature profile along the mold interface. This sets up nonuniform thermal distortion of the casting and hence a spatially periodic freezing front morphology. If the contact pressure along the mold interface initially falls to zero beneath the thinnest sections of the casting, which is air gap nucleation, while simultaneously increasing beneath the thickest sections, then unstable growth of the casting results. Rather than use the simple polynomial temperature solution considered by Li and Barber [1], we apply the perturbation solution of the Neumann problem due to Li and Barber [2]. In doing so, we remove the small Stefan number constraint in the original theory. We address the thermal stress problem using the same temperature distribution but with two different stress functions. In the first part of this article, the stress field in the casting is derived from a stress function that in truncated series form, approximates the exact stress function in an asymptotic sense. An additional logarithmic term in the stress function is necessary in order to avoid singularities in the algebraic expressions that result when imposing the mechanical boundary conditions. In the second part, we consider the full stress function, which includes a time-independent term from which is derived the residual stress in the casting. Unlike the contact pressure derived in the first part, the contact pressure predicted in the second part can only be calculated numerically since it is necessary to evaluate a differential equation for the residual stress function. Comparisons of the evolution of the contact pressure predicted by each model are made for selected values of Stefan number.

PART 1. THEORY BASED ON TRUNCATED STRESS FUNCTION

THE THERMAL STRESS PROBLEM

The thermal stress problem for a solidifying metal is necessarily complicated since instantaneous growth of the casting is accompanied by lateral thermal distortion due to the temperature gradient through the shell thickness. As the casting grows, newly solidified layers of metal are continuously nucleated from the melt. Each layer is immediately subjected to instantaneous thermal stress due to the temperature gradient through the thickness of the solid; however, each layer of the casting is subject to

a fluid pressure from the residual melt and may also be prevented from lateral contraction at its boundaries due to sticking. This gives rise to an uncertainty in the stress field that ultimately manifests itself as residual stress. The residual stress is by definition the stress state of the solid after it has been cooled to a uniform temperature and is subject to no boundary tractions. Hence, the residual stress, which is time-independent, will vary through the thickness of the solid and depend on the history of the solidification process.

If the solidified material behaves elastically, the total stress distribution can be expressed as a linear combination of:

1. A particular solution σ^P corresponding to the temperature field.
2. An isothermal solution σ^H that is allowed to vary in time so as to satisfy the time-varying terms in the boundary conditions.
3. A residual stress σ^R that is neither time-varying nor required to satisfy compatibility. In general, σ^R may be subsumed under σ^H.

The particular solution can be represented in terms of a thermoelastic displacement potential ψ (see Westergaard [3]) that is derived from the displacement field $\vec{u}$ through

$$\vec{\nabla}\psi = \frac{E\vec{u}}{1+\nu} \tag{1}$$

The corresponding equilibrium relation is

$$\nabla^2\psi = \frac{E\alpha T}{1-\nu} \tag{2}$$

The stress and displacement components are then expressed in terms of ψ with the following relations:

$$\sigma^P_{xx} = -\frac{\partial^2\psi}{\partial z^2} \qquad \sigma^P_{zz} = -\frac{\partial^2\psi}{\partial x^2} \qquad \sigma^P_{xz} = \frac{\partial^2\psi}{\partial x\partial z} \tag{3a}$$

$$2Gu^P_z = \frac{\partial\psi}{\partial z} \tag{3b}$$

where $G = E/2(1+\nu)$. The homogeneous and residual stress fields may be derived from the Airy stress function ϕ that must satisfy the following compatibility relation:

$$\frac{\partial}{\partial t}\nabla^4\phi = 0 \tag{4}$$

Equation (4) states that the incompatible strains must be independent of time. The general solution to Eq. (4) may be decomposed into two parts, i.e.,

$$\phi = \phi_1 + \phi_2 \tag{5a}$$

where

$$\frac{\partial \phi_1}{\partial t} = 0 \qquad \nabla^4 \phi_2 = 0 \tag{5b}$$

A physical interpretation of Eqs. (5*b*) is that the residual stress field, or the stress that remains when the fully solidified casting has been cooled to a uniform temperature and has traction-free boundaries, is derived from ϕ_1. The hydrostatic condition on the stress state at the freezing front is used to determine the function ϕ_1. The stress field due to the instantaneous compatible response of the solid to the changing temperature field is derived from ϕ_2. The function ϕ_2 is derived by imposing the appropriate mechanical boundary conditions on stress. The stress field may be derived from the Airy stress function using

$$\sigma^H_{xx} = \frac{\partial^2 \phi}{\partial z^2} \qquad \sigma^H_{zz} = \frac{\partial^2 \phi}{\partial x^2} \qquad \sigma^H_{xz} = -\frac{\partial^2 \phi}{\partial x \partial z} \tag{6}$$

This formulation ensures that the resulting stress field satisfies equilibrium.

HEAT CONDUCTION SOLUTION

Figure 1 shows the system that will be modeled; it consists of an infinite number of cells of solidified metal growing nonuniformly due to spatially periodic heat extraction. Each cell is assumed to be of finite width and infinitely long in the y-direction. The governing equation for heat transfer in this two-dimensional case is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{k}\frac{\partial T}{\partial t} \qquad t > 0 \qquad \forall x, z \tag{7}$$

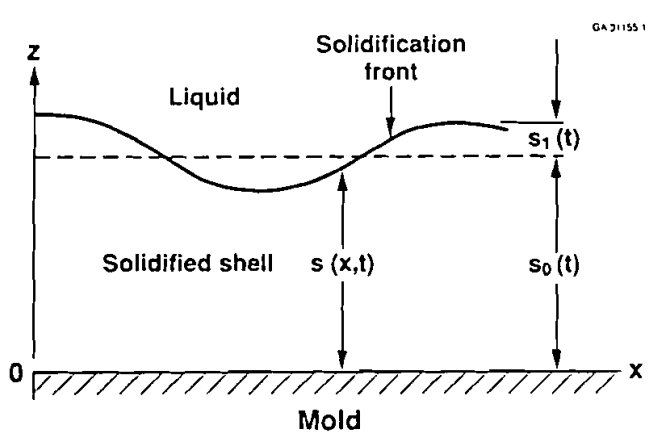

Fig. 1 Geometry of the system.

subject to the following conditions:

$$T = T_0^* + \tilde{T}_1^*(t) \cos (mx) \qquad z = 0 \tag{8}$$

$$T = T_m \qquad z = s \tag{9}$$

$$\frac{\partial T}{\partial z} = \frac{\rho L}{K} \frac{\partial s}{\partial t} \qquad z = s \tag{10}$$

$$s = 0 \qquad t = 0 \tag{11}$$

The temperature distribution in the casting is denoted by $T = T(x, z, t)$, the thermal diffusivity by k, the thermal conductivity by K, the density by ρ, the fusion temperature of the metal by T_m, the latent heat of fusion by L, and the casting thickness by $s = s(x, t)$. A spatial perturbation in the mold surface temperature, given by $\tilde{T}_1^* \cos (mx)$, is superposed onto a uniform temperature T_0^* in Eq. (8). The addition of this idealized term is an attempt to account for the fact that it is virtually impossible to effect uniform cooling in most industrial casting processes; it is quite likely that such spatial variations are due to small stochastic variations in the mold surface temperature arising from a variety of process conditions. The constant m is the wave number, which is proportional to the reciprocal of the wavelength of the spatial perturbation in the mold surface temperature. Note that the amplitude of the temperature perturbation is small compared to the difference between the molten metal temperature and the mold surface temperature, i.e., $T_1^* \ll \Delta T_0 = T_m - T_0^*$.

Equation (7) is the transient heat conduction equation. The mold/casting interface condition, given by Eq. (9), asserts that the freezing front is isothermal at the fusion temperature. Equation (10) is the energy balance at the freezing front that governs the release of latent heat at the freezing front (see Patel [4]). Equation (11) states that there is no solid at initial time.

The solution to the temperature problem given by Eqs. (7)–(11) is that due to Li and Barber [2]. The resulting temperature field and casting thickness may be expressed by

$$s = s_0(t) + s_1(t) \cos (mx) \qquad s_0(t) \gg s_1(t) \tag{12}$$

$$T = T_0(z, t) + T_1(z, t) \cos (mx) \qquad T_0(z, t) - T_0^* \gg T_1(z, t) \tag{13}$$

The zeroth-order solution for the casting temperature and thickness is that of the Neumann problem

$$T_0(z, t) = T_0^* + \frac{\Delta T_0}{erf(\lambda)} erf\left(\frac{z}{\sqrt{4kt}}\right) \qquad \Delta T_0 = T_m - T_0^* \tag{14}$$

$$s_0(x, t) = \lambda\sqrt{4kt} \tag{15}$$

where $erf(x)$ is the error function and λ is a dimensionless constant that is determined from

$$\frac{c\Delta T_0}{L} = \sqrt{\pi}\,\lambda e^{\lambda^2}\, erf(\lambda) \tag{16}$$

Note that Eq. (16) defines the Stefan number of the material.

For convenience in the analysis, the following dimensionless variables are introduced:

$$\tau = m^2 kt \qquad \text{dimensionless time} \tag{17a}$$

$$Z = \frac{z}{\sqrt{4kt}} = \frac{mz}{2\sqrt{\tau}} \qquad \text{dimensionless thickness variable} \tag{17b}$$

$$S(\tau, mx) = ms(x, t) = S_0(\tau) + S_1(\tau)\cos(mx) \qquad \text{dimensionless casting thickness} \tag{17c}$$

The perturbed temperature at the mold interface is assumed to vary exponentially in time according to

$$\tilde{T}_1^*(\tau) = T_1^* e^{p\tau} \tag{18}$$

where T_1^* is a constant with dimension of temperature and p is a dimensionless cooling parameter. The solution to the first-order temperature problem is expressed in terms of confluent hypergeometric functions as follows (see Abramowitz and Stegun [5]):

$$T_1(Z, \tau) = e^{-(\tau+Z^2)} \sum_{n=0}^{\infty} \tau^n \left[A_n \Phi\left(n + \frac{1}{2}, \frac{1}{2}; Z^2\right) + B_n Z \Phi\left(n + 1, \frac{3}{2}; Z^2\right) \right] \tag{19a}$$

$$S_1(\tau) = -\frac{c\sqrt{\tau} e^{-(\tau+\lambda^2)}}{\lambda L} \sum_{n=0}^{\infty} \tau^n \left[A_n \Phi\left(n + \frac{1}{2}, \frac{1}{2}; \lambda^2\right) + B_n \lambda \Phi\left(n + 1, \frac{3}{2}; \lambda^2\right) \right] \tag{19b}$$

where the coefficients in Eqs. (19a) and (19b) are given by

$$A_n = \frac{T_1^*(1 + p)^n}{n!} \tag{19c}$$

$$B_n = \frac{T_1^*(1 + p)^n}{n!} \Lambda_n \tag{19d}$$

and

$$\Lambda_n = \frac{2nW(n - 1/2, 1/2) + 2n\lambda W(n, 3/2)\Lambda_{n-1} - (1 + p)(2n + 1)W(n + 3/2, 1/2)}{2(1 + p)(n + 1)\lambda W\left(n + 2, \frac{3}{2}\right)} \tag{19e}$$

$$\Lambda_0 = -\frac{(2\lambda + 1/\lambda)}{e^{-\lambda^2} + (\sqrt{\pi}/2)(2\lambda + 1/\lambda)\, erf(\lambda)} \tag{19f}$$

$$W(a, b) = \Phi(a, b, \lambda^2) \tag{19g}$$

Equations (17*c*) and (19*b*) place the thinnest regions of the casting above points along the mold surface that correspond to $mx = \pm 2i\pi$ where i is either zero or an integer.

DETERMINATION OF THE STRESS FIELD

The mechanical boundary conditions at the mold interface are

$$\sigma_{xz} = 0 \qquad z = 0 \qquad \forall\tau \tag{20a}$$

$$\dot{u}_z = 0 \qquad z = 0 \qquad \forall\tau \tag{20b}$$

where the dot denotes differentiation with respect to τ. Equations (20) define a condition of frictionless contact at the mold interface. Note that the boundary condition on displacement can only be stated in terms of a time derivative since there is no reference state for displacement. We also assume that the material is initially solidified in a state of hydrostatic compression, i.e.,

$$\sigma_{xx} = \sigma_{zz} = -\rho g(H - s) \qquad \sigma_{xz} = 0 \qquad z = s \tag{21}$$

(see Richmond [6]).

In view of the form of the temperature field Eq. (13), the resulting stress field may be written as

$$\sigma(x, z, t) = \sigma_0(z, t) + \sigma_1(x, z, t) \qquad \sigma_0(z, t) \gg \sigma_1(x, z, t) \tag{22}$$

where $\sigma_1(x, z, t)$ is a small perturbation on the stress field $\sigma_0(z, t)$ of the zeroth-order problem. The contact pressure at the mold interface, which is defined by $P(x, t) = -\sigma_{zz}(x, 0, t)$, will take the form

$$P(x, t) = P_0(t) + P_1(t)\cos(mx) \qquad P_0(t) \gg P_1(t) \tag{23a}$$

where

$$P_0(t) = -\sigma_{zz_0}(0, t) = \rho g(H - s_0) \tag{23b}$$

$$P_1(t) \cos(mx) = -\sigma_{zz_1}(x, 0, t) \tag{23c}$$

The Zeroth Order Solution

The zeroth order particular stress field is determined by choosing an appropriate displacement potential satisfying Eq. (2) and subsequently applying Eqs. (3*a*) and (14). This gives

$$\sigma^{P}_{xx_0} = -\frac{E\alpha}{1-\nu}\left\{\Delta T_0 \frac{erf(Z)}{erf(\lambda)} + T_0^*\right\} \tag{24a}$$

$$\sigma^{P}_{zz_0} = 0 \qquad \sigma^{P}_{xz_0} = 0 \tag{24b}$$

which is the stress field corresponding to the temperature $T_0(z, t)$. To complete this solution, it is necessary to superpose a homogeneous stress field in order to satisfy the mechanical boundary conditions Eqs. (20) and (21). This is achieved with the uniform biaxial stress field

$$\sigma^{H}_{xx_0} = \frac{E\alpha T_m}{1-\nu} - \rho g(H - s_0) \qquad \sigma^{H}_{zz_0} = -\rho g(H - s_0) \qquad \sigma^{H}_{xz_0} = 0 \tag{25}$$

The complete zeroth-order thermal stress field, therefore, is

$$\sigma_{xx_0}(z, t) = -\rho g(H - s_0) + \frac{E\alpha\Delta T_0}{1-\nu}\left\{1 - \frac{erf(Z)}{erf(\lambda)}\right\} \tag{26a}$$

$$\sigma_{zz_0}(z, t) = -\rho g(H - s_0) \qquad \sigma_{xz_0}(z, t) = 0 \tag{26b}$$

The First-Order Solution

The thermoelastic problem corresponding to the temperature field $T_1(Z, \tau) \cos(mx)$ is now considered. A suitable particular solution that satisfies Eq. (2) can be obtained with a displacement potential of the form

$$\psi = \sum_{n=1}^{\infty} [C_n\psi_{n_1} + D_n\psi_{n_2}] \tag{27a}$$

where

$$\psi_{n_1} = \tau^n e^{-(\tau+Z^2)}\Phi\left(n + \frac{1}{2}, \frac{1}{2}; Z^2\right) \tag{27b}$$

$$\psi_{n_2} = \tau^n e^{-(\tau+Z^2)} Z\Phi\left(n + 1, \frac{3}{2}; Z^2\right) \tag{27c}$$

are taken from Eq. (19*a*). Using Eqs. (2), (19*a*), and (27), we can determine the coefficients C_n and D_n, with the results expressed in recurrence relation form for $n \neq 1$ as

$$C_{n+1} = \frac{1}{n+1}\left[\frac{E\alpha}{m^2(1-\nu)}A_n + C_n\right] \tag{28a}$$

$$D_{n+1} = \frac{1}{n+1/2}\left[\frac{E\alpha}{m^2(1-\nu)}B_n + D_n\right] \tag{28b}$$

while for $n = 1$

$$C_1 = \frac{E\alpha}{m^2(1-\nu)}A_0 \tag{28c}$$

$$D_1 = \frac{2E\alpha}{m^2(1-\nu)}B_0 \tag{28d}$$

where A_n and B_n are defined in Eqs. (19*c*) and (19*d*). A derivation of Eqs. (28) is given in the appendix.

From Eqs. (3*a*) and (27), the first-order stress components of the particular solution are

$$\sigma^{P}_{xx_1}(x, z, \tau) = -m^2 e^{-(\tau+Z^2)}\sum_{n=1}^{\infty}\tau^{n-1}\left[nC_n\Phi\left(n-\frac{1}{2}, \frac{1}{2}; Z^2\right) + \left(n-\frac{1}{2}\right)D_n Z\Phi\left(n, \frac{3}{2}; Z^2\right)\right]\cos(mx) \tag{29a}$$

$$\sigma^{P}_{zz_1}(x, z, \tau) = m^2 e^{-(\tau+Z^2)}\sum_{n=1}^{\infty}\tau^{n}\left[C_n\Phi\left(n+\frac{1}{2}, \frac{1}{2}; Z^2\right) + D_n Z\Phi\left(n+1, \frac{3}{2}; Z^2\right)\right]\cos(mx) \tag{29b}$$

$$\sigma^{P}_{xz_1}(x, z, \tau) = -\frac{1}{2}m^2 e^{-(\tau+Z^2)}\sum_{n=1}^{\infty}\tau^{n-1/2}\left[4nC_n Z\Phi\left(n+\frac{1}{2}, \frac{3}{2}; Z^2\right) + D_n\Phi\left(n, \frac{1}{2}; Z^2\right)\right]\sin(mx) \tag{29c}$$

We represent the homogeneous solution of the first-order perturbation problem in terms of an Airy stress function ϕ. This can be made to satisfy Eq. (4) by writing it as the sum of four linearly independent biharmonic terms with arbitrarily time-varying coefficients and an arbitrary time-independent function, which does not have to be biharmonic. However, since we are interested in the initial stages of solidifi-

cation, when the layer thickness s is small, it is natural to think in terms of a series approximation to ϕ in the form

$$\phi(x, z, \tau) = \sum_{i=0}^{N} a_i(\tau) z^i \cos(mx) \tag{30}$$

If $N > 3$, substitution of Eq. (30) into Eq. (4) yields a set of recurrence relations that defines the time derivatives $\dot{a}_i(\tau)$ for $i > 3$ in terms of the first four coefficients. Thus the linearly independent terms in Eq. (30) comprise the first four functions $\dot{a}_i(\tau)$, $i = 0, 3$ and a time-independent arbitrary constant in each of the coefficients for $i > 3$.

A further complication arises when this strategy is applied to the present problem. When the boundary conditions are imposed using the truncated series and equating coefficients of like powers, the resulting system of algebraic equations is found to be singular; in other words, two of the equations are not linearly independent. This can also be seen as an indication that the equation governing the function ϕ has a repeated root, which therefore requires a special solution of different form. Procedures for determining such special solutions are discussed by Barber [7]. A more detailed analysis in the present case shows that the special solution required is of the form $az^2 \log(z)$, where a is a time-independent arbitrary constant. With this addition, the assumed form of ϕ is

$$\phi(x, z, \tau) = \left\{ \sum_{i=0}^{N} a_i(\tau) z^i + az^2 \log(z) \right\} \cos(mx) \tag{31}$$

Li and Barber [1] showed that a four-term polynomial series is exactly equivalent to the beam theory of Richmond et al. [8] for the related problem of a mold with a prescribed perturbation in heat flux at zero Stefan number. In this part of the paper, therefore, we shall construct an approximate "beam-like" theory by truncating the series in Eq. (31) at $N = 3$. Notice, however, that in contrast to a true beam theory, the present method is developed in the sense of an asymptotic approximation to the exact result and hence permits nonpolynomial terms in the distributions where appropriate. As explained above, the present problem requires a logarithmic term in a consistent asymptotic expansion that could not be represented in a nonuniform beam theory model.†

Using Eqs. (3a), (6), (29), and (31), the complete stress field for the first-order problem is written as

$$\sigma_{xx_1}(x, z, \tau) = \left\{ \begin{array}{l} 2a_2(\tau) + 6a_3(\tau)z + a(2\log(z) + 3) \\ -m^2 e^{-(\tau+Z^2)} \displaystyle\sum_{n=1}^{\infty} \tau^{n-1} \left[nC_n \Phi\left(n - \frac{1}{2}, \frac{1}{2}; Z^2\right) \right. \\ \left. + \left(n - \frac{1}{2}\right) D_n Z \Phi\left(n, \frac{3}{2}; Z^2\right) \right] \end{array} \right\} \cos(mx) \tag{32a}$$

†In fact, earlier attempts to develop a direct beam theory solution to the present problem gave results that were poor approximations to the exact result even at very small values of time.

$$\sigma_{zz_1}(x, z, \tau) = -m^2 \left\{ \begin{array}{l} a_0(\tau) + a_1(\tau)z + a_2(\tau)z^2 + a_3(\tau)z^3 \\ + az^2 \log(z) \\ -e^{-(\tau+Z^2)} \sum_{n=1}^{\infty} \tau^n \left[C_n \Phi\left(n + \frac{1}{2}, \frac{1}{2}; Z^2\right) \right. \\ \left. + D_n Z \Phi\left(n + 1, \frac{3}{2}; Z^2\right) \right] \end{array} \right\} \cos(mx) \quad (32b)$$

$$\sigma_{xz_1}(x, z, \tau) = m \left\{ \begin{array}{l} a_1(\tau) + 2a_2(\tau)z + 3a_3(\tau)z^2 \\ + az(2 \log(z) + 1) \\ -\frac{m}{2} e^{-(\tau+Z^2)} \sum_{n=1}^{\infty} \tau^{n-1/2} \left[4nC_n Z \Phi\left(n + \frac{1}{2}, \frac{3}{2}; Z^2\right) \right. \\ \left. + D_n \Phi\left(n, \frac{1}{2}; Z^2\right) \right] \end{array} \right\} \sin(mx) \quad (32c)$$

The strains due to these stress components are not required to be compatible since there is no initial reference state for strain. However, the strain rates are compatible and, therefore, can be expressed as displacement derivatives and subsequently integrated to give the velocity components. The velocity component through the thickness of the casting may be obtained from $\dot{u}_{z_1} = \dot{u}^{\mathrm{H}}_{z_1} + \dot{u}^{\mathrm{P}}_{z_1}$ where

$$\dot{u}^{\mathrm{H}}_{z_1} = \frac{m^2 k}{2G} \left[(1 - \nu) \int \dot{\sigma}^{\mathrm{H}}_{zz}\, dz - \nu \int \dot{\sigma}^{\mathrm{H}}_{xx}\, dz \right] \quad (33a)$$

and

$$\dot{u}^{\mathrm{P}}_{Z_1} = \frac{m^3 k}{4G} \frac{\partial}{\partial \tau} \left(\frac{1}{\sqrt{\tau}} \frac{\partial \psi}{\partial Z} \right) \quad (33b)$$

Equation (33*a*), which is the elastic constitutive relation for plane strain, neglects associated rigid body displacements. Equation (33*b*) results from Eqs. (3*b*), (17*a*), and (17*b*).

Linear combination of Eqs. (33) gives

$$\dot{u}_{z_1} = \frac{1}{2G} \left\{ \begin{array}{l} [-m^2(1 - \nu)]\dot{a}_0(\tau)z + \left[-m^2(1 - \nu)\frac{z^2}{2} + \nu - 2 \right] \dot{a}_1(\tau) \\ + \left[-m^2(1 - \nu)\frac{z^2}{3} - 2\nu \right] \dot{a}_2(\tau)z \\ + \left[-m^2(1 - \nu)\frac{z^4}{4} - 3\nu z^2 + 6\frac{(1 - \nu)}{m^2} \right] \dot{a}_3(\tau) \\ + \frac{m}{2} e^{-(\tau+Z^2)} \sum_{n=1}^{\infty} \left[\left(n - \frac{1}{2}\right)\tau^{n-3/2} - \tau^{n-1/2} \right] \\ \cdot \left[4nC_n Z \Phi\left(n + \frac{1}{2}, \frac{3}{2}; Z^2\right) + D_n \Phi\left(n, \frac{1}{2}; Z^2\right) \right] \end{array} \right\} \cos(mx) \quad (34)$$

where we have used Eqs. (6), (27), and (31).

We now consider the boundary conditions corresponding to the first-order problem. Since the perturbation is small, we can expand the stress field in the vicinity of the mean freezing front, $z = s_0(t)$, in a Taylor series. For example, the first boundary condition given by Eqs. (21) becomes

$$-\rho g(H - s_0) = \sigma_{xx_0}(s_0, t) + \frac{\partial \sigma_{xx_0}(s_0, t)}{\partial z} s_1(t) \cos(mx)$$

$$+ \frac{\partial^2 \sigma_{xx_0}(s_0, t)}{\partial z^2} \frac{s_1^2(t) \cos^2(mx)}{2!}$$

$$+ \cdots + \left[\sigma_{xx_1}(x, s_0, t) + \frac{\partial \sigma_{xx_1}(x, s_0, t)}{\partial z} s_1(t) \cos(mx) + \cdots \right] \quad (35)$$

Separating periodic and uniform terms, dropping second- and higher order product terms in the small quantities, σ_{xx_1}, s_1, and using Eq. (26*a*), we obtain the boundary condition for σ_{xx_1} at $z = s_0(t)$ or $Z = \lambda$, i.e.,

$$\sigma_{xx_1}(x, s_0, \tau) = -\frac{E\alpha}{1 - \nu} T_1(\lambda, \tau) \cos(mx) \quad (36a)$$

where T_1 is defined in Eq. (19*a*) and we have used the following expression from Li and Barber [2]:

$$S_1(\tau) = -\frac{c\sqrt{\tau}}{\lambda L} T_1(\lambda, \tau) \quad (36b)$$

Applying the same procedure to the remaining boundary conditions in Eqs. (20), we obtain

$$\sigma_{zz_1}(x, s_0, \tau) = 0 \qquad \sigma_{xz_1}(x, s_0, \tau) = 0 \quad (37)$$

Also, from Eqs. (21), we have

$$\sigma_{xz_1}(x, 0, \tau) = 0 \qquad \dot{u}_{z_1}(x, 0, \tau) = 0 \quad (38)$$

Substituting the shear stress and normal displacement components from Eqs. (32*c*) and (34) into Eqs. (38), we find that the time-dependent coefficients $a_1(\tau)$ and $a_3(\tau)$ can be written as

$$a_1(\tau) = \frac{m}{2} e^{-\tau} \sum_{n=1}^{\infty} \tau^{n-1/2} D_n \qquad a_3(\tau) = \frac{m^3}{12} e^{-\tau} \sum_{n=1}^{\infty} \tau^{n-1/2} D_n + d \quad (39)$$

where d is an arbitrary constant of integration.

Substituting for the stress components from Eqs. (32) into the remaining boundary conditions, Eqs. (36) and (37), we obtain the following relations for the remaining unknown terms $a_0(\tau)$, $a_2(\tau)$, and a:

$$
\begin{aligned}
2a_2(\tau) + a(2\log(s_0) + 3) = &-\frac{m^3 s_0}{2} e^{-\tau} \sum_{n=1}^{\infty} \tau^{n-1/2} D_n - 6\, ds_0 \\
&- \frac{E\alpha}{1-\nu} e^{-(\tau+\lambda^2)} \sum_{n=0}^{\infty} \tau^n \left[A_n \Phi\left(n + \frac{1}{2}, \frac{1}{2}; \lambda^2\right) \right. \\
&\left. + B_n \lambda \Phi\left(n + 1, \frac{3}{2}; \lambda^2\right) \right] \\
&+ m^2 e^{-(\tau+\lambda^2)} \sum_{n=1}^{\infty} \tau^{n-1} \left[nC_n \Phi\left(n - \frac{1}{2}, \frac{1}{2}; \lambda^2\right) \right. \\
&\left. + \left(n - \frac{1}{2}\right) D_n \lambda \Phi\left(n, \frac{3}{2}; \lambda^2\right) \right]
\end{aligned} \tag{40a}
$$

$$
\begin{aligned}
a_0(\tau) + a_2(\tau)s_0^2 + as_0^2 \log(s_0) = &-\left(1 + \frac{m^2 s_0^2}{6}\right) \frac{m s_0}{2} e^{-\tau} \sum_{n=1}^{\infty} \tau^{n-1/2} D_n - ds_0^3 \\
&+ e^{-(\tau+\lambda^2)} \sum_{n=1}^{\infty} \tau^n \left[C_n \Phi\left(n + \frac{1}{2}, \frac{1}{2}; \lambda^2\right) \right. \\
&\left. + D_n \lambda \Phi\left(n + 1, \frac{3}{2}; \lambda^2\right) \right]
\end{aligned} \tag{40b}
$$

$$
\begin{aligned}
2a_2(\tau)s_0 + as_0(2\log(s_0) + 1) = &-\left(1 + \frac{m^2 s_0^2}{2}\right) \frac{m}{2} e^{-\tau} \sum_{n=1}^{\infty} \tau^{n-1/2} D_n - 3\, ds_0^2 \\
&+ \frac{m}{2} e^{-(\tau+\lambda^2)} \sum_{n=1}^{\infty} \tau^{n-1/2} \left[4nC_n \lambda \Phi\left(n + \frac{1}{2}, \frac{3}{2}; \lambda^2\right) \right. \\
&\left. + D_n \Phi\left(n, \frac{1}{2}; \lambda^2\right) \right]
\end{aligned} \tag{40c}
$$

Using Eqs. (40a) and (40c) to eliminate $a_2(\tau)$ and replacing $s_0(t)$ by $2\lambda\sqrt{\tau}/m$ using Eqs. (15) and (17a), we find

$$
\begin{aligned}
a = &(1 - 2\lambda^2\tau) \frac{m^2}{8\lambda} e^{-\tau} \sum_{n=1}^{\infty} \tau^{n-1} D_n - 3d\lambda \frac{\sqrt{\tau}}{m} \\
&- \frac{E\alpha}{2(1-\nu)} e^{-(\tau+\lambda^2)} \sum_{n=0}^{\infty} \tau^n \left[A_n \Phi\left(n + \frac{1}{2}, \frac{1}{2}; \lambda^2\right) + B_n \lambda \Phi\left(n + 1, \frac{3}{2}; \lambda^2\right) \right]
\end{aligned}
$$

$$+\frac{m^2}{2}e^{-(\tau+\lambda^2)}\sum_{n=1}^{\infty}\tau^{n-1}\left[nC_n\Phi\left(n-\frac{1}{2},\frac{1}{2};\lambda^2\right)+\left(n-\frac{1}{2}\right)D_n\lambda\Phi\left(n,\frac{3}{2};\lambda^2\right)\right]$$

$$-\frac{m^2}{8\lambda}e^{-(\tau+\lambda^2)}\sum_{n=1}^{\infty}\tau^{n-1}\left[4nC_n\lambda\Phi\left(n+\frac{1}{2},\frac{3}{2};\lambda^2\right)+D_n\Phi\left(n,\frac{1}{2};\lambda^2\right)\right] \tag{41}$$

Equation (41) cannot be satisfied for all values of times τ, since a must be time-independent in order that Eq. (31) should satisfy the compatibility equation (4). However, Eq. (41) can be expanded as a power series in τ except for the term involving d. Thus, by setting $d = 0$ and choosing a to equal the constant term in the right-hand side of Eq. (41), we can satisfy this equation up to the order of $\tau^{1/2}$. This leads to

$$a = -\frac{E\alpha}{2(1-\nu)}\left[A_0 + B_0\frac{\sqrt{\pi}}{2}erf(\lambda)\right] + \frac{m^2}{8\lambda}[1-e^{-\lambda^2}]D_1 \tag{42}$$

or, using Eqs. (28*c*) and (28*d*),

$$a = -\frac{m^2}{2}\left[C_1 + \frac{1}{4}\left(\sqrt{\pi}\,erf(\lambda) - \frac{1-e^{-\lambda^2}}{\lambda}\right)D_1\right] \tag{43}$$

We note that Eq. (41) can be satisfied to a higher order in τ by including terms of higher order in the stress function ϕ of Eq. (31). For example, if we include the next term $a_4(\tau)z^4$, the coefficient $a_4(\tau)$ will be determined by Eq. (4) except for an arbitrary (time-independent) constant, which will then appear in Eq. (41) multiplying a term of order τ. This constant can then be chosen to satisfy Eq. (41) to the order of τ^1. Inclusion of further terms in Eq. (31) allows the procedure to be extended to higher order. Once a is determined, the remaining coefficients, $a_0(\tau)$ and $a_2(\tau)$, may be determined from Eqs. (40*a*), (40*b*), and (43).

The perturbation in the contact pressure is $P_1(\tau)\cos(mx)$ where

$$P_1(\tau) = m^2\left[a_0(\tau) - e^{-\tau}\sum_{n=1}^{\infty}\tau^n C_n\right] \tag{44}$$

results from Eqs. (23*c*) and (29*b*), C_n is determined from Eqs. (32*b*), and

$$a_0(\tau) = \lambda^2\tau(2e^{-\tau}-3)\left[C_1 + \frac{\sqrt{\pi}}{4}D_1\,erf(\lambda)\right] + \frac{3}{4}\lambda\tau(1-e^{-\lambda^2})D_1$$

$$+\frac{2E\alpha\lambda^2}{m^2(1-\nu)}e^{-(\tau+\lambda^2)}\sum_{n=1}^{\infty}\tau^{n+1}\left[A_n\Phi\left(n+\frac{1}{2},\frac{1}{2};\lambda^2\right)+B_n\lambda\Phi\left(n+1,\frac{3}{2};\lambda^2\right)\right]$$

$$+ \lambda\left(\frac{4}{3}\lambda^2\tau - 1\right)e^{-\tau}\sum_{n=1}^{\infty}\tau^n D_n - 2\lambda^2 e^{-(\tau+\lambda^2)}\sum_{n=1}^{\infty}\tau^n\left[nC_n\Phi\left(n-\frac{1}{2},\frac{1}{2};\lambda^2\right)\right.$$

$$\left.+\left(n-\frac{1}{2}\right)D_n\lambda\Phi\left(n,\frac{3}{2};\lambda^2\right)\right] + e^{-(\tau+\lambda^2)}\sum_{n=1}^{\infty}\tau^n\left[C_n\Phi\left(n+\frac{1}{2},\frac{1}{2};\lambda^2\right)\right.$$

$$\left.+ D_n\lambda\Phi\left(n+1,\frac{3}{2};\lambda^2\right)\right] \tag{45}$$

DIMENSIONLESS FORMULATION

It is convenient to recast Eq. (44) in dimensionless form by defining the following dimensionless variables:

$$\bar{a}_0(\tau) = \frac{m^2(1-\nu)}{E\alpha\Delta T_0}a_0(\tau) \tag{46}$$

$$\bar{P}(x,\tau) = \frac{(1-\nu)}{E\alpha\Delta T_0}P(x,t) \tag{47}$$

The dimensionless recurrence relations for C_n and D_n for $n \neq 1$ are

$$\bar{C}_{n+1} = \frac{m^2(1-\nu)}{E\alpha\Delta T_0}C_{n+1} = \frac{1}{n+1}\left[\frac{T_1^*}{\Delta T_0}\frac{(1+p)^n}{n!} + \bar{C}_n\right] \tag{48a}$$

$$\bar{D}_{n+1} = \frac{m^2(1-\nu)}{E\alpha\Delta T_0}D_{n+1} = \frac{1}{n+1/2}\left[\frac{T_1^*}{\Delta T_0}\frac{(1+p)^n}{n!}\Lambda_n + \bar{D}_n\right] \tag{48b}$$

while for $n = 1$

$$\bar{C}_1 = \frac{m^2(1-\nu)}{E\alpha\Delta T_0}C_1 = \frac{T_1^*}{\Delta T_0} \tag{48c}$$

$$\bar{D}_1 = \frac{m^2(1-\nu)}{E\alpha\Delta T_0}D_1 = \frac{2T_1^*}{\Delta T_0}\Lambda_0 \tag{48d}$$

The dimensionless perturbed contact pressure becomes

$$\bar{P}_1(\tau) = \bar{a}_0(\tau) - e^{-\tau}\sum_{n=1}^{\infty}\tau^n\bar{C}_n \tag{49}$$

PART 2. THEORY BASED ON FULL STRESS FUNCTION

DETERMINATION OF THE STRESS FIELD

The method of Part 1 can be extended to arbitrary order in τ by including a sufficient number of terms in the stress function series given by Eq. (31). In practice, however, the algebra becomes unmanageable beyond the first few terms, so the principal value of this approach is to provide an asymptotic solution to the problem at small values of time. To extend the solution to larger values of τ, we replace Eq. (31) with the most general stress function satisfying Eq. (4), which is

$$\phi = \{[b_1(\tau) + b_2(\tau)z] \cosh(mz) + [b_3(\tau) + b_4(\tau)z] \sinh(mz) + F(z)\} \cos(mx) \quad (50)$$

where $F(z)$ is an arbitrary, time-independent function, related to the residual stress in the casting. The first-order boundary conditions, given by Eqs. (36*a*), (37), and (38), are to be used to determine $F(z)$ and the time-dependent coefficients $b_i(\tau)$. The total stress distribution corresponding to the first-order problem may be written in terms of the Airy stress function ϕ, defined in Eq. (50), and the displacement potential ψ (see Eq. (27*a*)) as

$$\sigma_{xx_1} = \frac{\partial^2\phi}{\partial z^2} - \frac{m^2}{4\tau}\frac{\partial^2\psi}{\partial Z^2} \quad (51a)$$

$$\sigma_{zz_1} = \frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\psi}{\partial x^2} \quad (51b)$$

$$\sigma_{xz_1} = -\frac{\partial^2\phi}{\partial x\partial z} + \frac{m}{2\sqrt{\tau}}\frac{\partial^2\psi}{\partial x\partial Z} \quad (51c)$$

Substitution of Eqs. (27) and (50) into Eqs. (51) leads to

$$\sigma_{xx_1} = m^2 \left\{ \begin{aligned} &\left[b_1(\tau) + b_2(\tau)z + \frac{2b_4(\tau)}{m}\right]\cosh(mz) \\ &+ \left[b_3(\tau) + b_4(\tau)z + \frac{2b_2(\tau)}{m}\right]\sinh(mz) + \frac{1}{m^2}\frac{d^2F(z)}{dz^2} \\ &- e^{-(\tau+Z^2)}\sum_{n=1}^{\infty}\tau^{n-1}\left[nC_n\Phi\left(-\frac{1}{2}+n, \frac{1}{2}; Z^2\right)\right. \\ &\left. + \left(n-\frac{1}{2}\right)D_nZ\Phi\left((n, \frac{3}{2}; Z^2\right)\right] \end{aligned} \right\} \cos(mx) \quad (52a)$$

$$\sigma_{zz_1} = m^2 \left\{ \begin{array}{l} -[b_1(\tau) + b_2(\tau)z] \cosh(mz) + [b_3(\tau) \\ + b_4(\tau)z] \sinh(mz) + F(z) \\ + e^{-(\tau+Z^2)} \sum_{n=1}^{\infty} \tau^n \left[C_n \Phi\left(\frac{1}{2} + n, \frac{1}{2}; Z^2\right) \right. \\ \left. + D_n Z \Phi\left(1 + n, \frac{3}{2}; Z^2\right) \right] \end{array} \right\} \cos(mx) \tag{52b}$$

$$\sigma_{xz_1} = m^2 \left\{ \begin{array}{l} \left[b_1(\tau) + b_2(\tau)z + \frac{b_4(\tau)}{m} \right] \sinh(mz) \\ + \left[b_3(\tau) + b_4(\tau)z + \frac{b_2(\tau)}{m} \right] \cosh(mz) + \frac{1}{m} \frac{dF(z)}{dz} \\ - \frac{e^{-(\tau+Z^2)}}{2} \sum_{n=1}^{\infty} \tau^{n-1/2} \left[4nC_n Z \Phi\left(\frac{1}{2} + n, \frac{3}{2}; Z^2\right) \right. \\ \left. + D_n \Phi\left(n, \frac{1}{2}; Z^2\right) \right] \end{array} \right\} \sin(mx) \tag{52c}$$

We first proceed to derive the conditions on the stress field that are necessary to satisfy the frictionless contact conditions along the mold interface. Using Eqs. (33) to derive the normal velocity $\dot{u}_{z_1}$, and then applying Eq. (20*b*), we find

$$-\frac{(1-2\nu)}{m} \dot{b}_2(\tau) + \dot{b}_3(\tau) = \frac{e^{-\tau}}{2} \sum_{n=1}^{\infty} \left[\left(n - \frac{1}{2}\right) \tau^{n-3/2} - \tau^{n-1/2} \right] D_n \tag{53}$$

Substitution of Eq. (52*c*) into Eq. (20*a*) gives

$$\frac{1}{m} \dot{b}_2(\tau) + \dot{b}_3(\tau) = \frac{e^{-\tau}}{2} \sum_{n=1}^{\infty} \left[\left(n - \frac{1}{2}\right) \tau^{n-3/2} - \tau^{n-1/2} \right] D_n \tag{54}$$

Simultaneous solution of Eqs. (53) and (54) gives

$$b_2(\tau) = 0 \tag{55}$$

$$b_3(\tau) = \frac{e^{-\tau}}{2} \sum_{n=1}^{\infty} \tau^{n-1/2} D_n \tag{56}$$

where we have set the constant of integration equal to zero without loss of generality since it can be subsumed in the time-independent term F. Application of Eqs. (21), expanded according to Eq. (35), to Eqs. (52), yields, respectively,

$$b_1(\tau)\cosh(ms_0(t)) + b_4(\tau)\left[\frac{2}{m}\cosh(ms_0(t)) + s_0(t)\sinh(ms_0(t))\right]$$

$$= -\frac{1}{m^2}\frac{d^2F}{dz^2} - \frac{e^{-\tau}}{2}\sinh(ms_0(t))\sum_{n=1}^{\infty}\tau^{n-1/2}D_n + \Delta_1(\lambda,\tau) \tag{57a}$$

$$b_1(\tau)\cosh(ms_0(t)) + b_4(\tau)s_0(t)\sinh(ms_0(t))$$

$$= -F - \frac{e^{-\tau}}{2}\sinh(ms_0(t))\sum_{n=1}^{\infty}\tau^{n-1/2}D_n + \Delta_2(\lambda,\tau) \tag{57b}$$

$$b_1(\tau)\sinh(ms_0(t)) + b_4(\tau)\left[\frac{\sinh(ms_0(t))}{m} + s_0(t)\cosh(ms_0(t))\right]$$

$$= -\frac{1}{m}\frac{dF}{dz} - \frac{e^{-\tau}}{2}\cosh(ms_0(t))\sum_{n=1}^{\infty}\tau^{n-1/2}D_n + \Delta_3(\lambda,\tau) \tag{57c}$$

where

$$F = F(s_0(t)) \tag{58a}$$

$$\Delta_1(\lambda,\tau) = -e^{-(\tau+\lambda^2)}\left\{\begin{array}{l} C_1 + D_1\dfrac{\sqrt{\pi}}{4}erf(\lambda) + \dfrac{E\alpha}{m^2(1-\nu)}\displaystyle\sum_{n=1}^{\infty}\tau^n\left[A_n\Phi\left(n+\frac{1}{2},\frac{1}{2};\lambda^2\right)\right. \\ \left. + B_n\lambda\Phi\left(n+1,\dfrac{3}{2};\lambda^2\right)\right] \\ -\displaystyle\sum_{n=1}^{\infty}\tau^{n-1}\left[nC_n\Phi\left(n-\frac{1}{2},\frac{1}{2};\lambda^2\right) + \left(n-\frac{1}{2}\right)D_n\lambda\Phi\left(n,\frac{3}{2};\lambda^2\right)\right] \end{array}\right\} \tag{58b}$$

$$\Delta_2(\lambda,\tau) = e^{-(\tau+\lambda^2)}\sum_{n=1}^{\infty}\tau^n\left[C_n\Phi\left(n+\frac{1}{2},\frac{1}{2};\lambda^2\right) + D_n\lambda\Phi\left(n+1,\frac{3}{2};\lambda^2\right)\right] \tag{58c}$$

$$\Delta_3(\lambda,\tau) = \frac{e^{-(\tau+\lambda^2)}}{2}\sum_{n=1}^{\infty}\tau^{n-1/2}\left[4n\lambda C_n\Phi\left(n+\frac{1}{2},\frac{3}{2};\lambda^2\right) + D_n\Phi\left(n,\frac{1}{2};\lambda^2\right)\right] \tag{58d}$$

Equation (57*b*) may be subtracted from Eq. (57*a*) and the result solved to yield

$$b_4(\tau) = \frac{m}{2\cosh(ms_0(t))}\left[F - \frac{1}{m^2}\frac{d^2F}{dz^2} + \Delta_1(\lambda,\tau) - \Delta_2(\lambda,\tau)\right] \tag{59}$$

Substitution of Eq. (59) into Eq. (57*a*) with subsequent rearrangement gives the following expression for the remaining unknown coefficient $b_1(\tau)$:

$$b_1(\tau) = \frac{1}{\cosh(ms_0(t))} \left\{ \begin{array}{l} -\dfrac{ms_0(t)w}{2}\left[F - \dfrac{1}{m^2}\dfrac{d^2F}{dz^2}\right] \\ -F - \dfrac{e^{-\tau}}{2}\sinh(ms_0(t)) \displaystyle\sum_{n=1}^{\infty} \tau^{n-1/2}D_n \\ -\dfrac{ms_0(t)w}{2}\Delta_1(\lambda, \tau) + \left[1 + \dfrac{ms_0(t)w}{2}\right]\Delta_2(\lambda, \tau) \end{array} \right\} \tag{60}$$

where $w = \tanh(ms_0(t))$. We substitute Eqs. (59) and (60) into Eq. (57c) to generate the following ordinary differential equation for the residual stress function F:

$$\frac{1}{2m^2}[ms_0(t)w^2 - w - ms_0(t)]\frac{d^2F}{dz^2} + \frac{1}{m}\frac{dF}{dz} - \frac{1}{2}[ms_0(t)w^2 + w - ms_0(t)]F$$

$$= \frac{e^{-\tau}\cosh(ms_0(t))}{2}[w^2 - 1]\sum_{n=1}^{\infty}\tau^{n-1/2}D_n - \frac{1}{2}[ms_0(t) + w - ms_0(t)w^2]\Delta_1(\lambda, \tau)$$

$$-\frac{1}{2}[w + ms_0(t)w^2 - ms_0(t)]\Delta_2(\lambda, \tau) + \Delta_3(\lambda, \tau) \tag{61}$$

The left-hand side of Eq. (61) is identical to that of the differential equation for the residual stress function derived by Li and Barber [1]. We may rewrite Eq. (61) in the z-domain by defining $z = s_0(t)$, in which $t = t^*(s_0(t))$, where the layer formation time is

$$t^* = \frac{1}{k}\left(\frac{z}{2\lambda}\right)^2 \tag{62}$$

In addition, we define the dimensionless spatial variable θ where

$$\theta = mz \tag{63}$$

and

$$\bar{F}(\theta) = \frac{m^2(1 - \nu)}{E\alpha\Delta T_0}F(z) \tag{64}$$

Using the relations between A_n and C_n, B_n and D_n, defined in Eqs. (28), and the dimensionless quantities defined in Eqs. (48), we may thus write Eq. (61) as

$$\frac{1}{2}[w^2\theta - w - \theta]\frac{d^2\bar{F}}{d\theta^2} + \frac{d\bar{F}}{d\theta} - \frac{1}{2}[w^2\theta + w - \theta]\bar{F} = \frac{-e^{-\hat{\tau}}}{2\cosh(\theta)}\sum_{n=1}^{\infty}\hat{\tau}^{n-1/2}\bar{D}_n$$

$$-\frac{1}{2}[w^2\theta - w - \theta]\bar{\Delta}_1(\lambda, \hat{\tau}) - \frac{1}{2}[w + w^2\theta - \theta]\bar{\Delta}_2(\lambda, \hat{\tau}) + \bar{\Delta}_3(\lambda, \hat{\tau}) \tag{65a}$$

where

$$\hat{\tau} = \left(\frac{\theta}{2\lambda}\right)^2 \tag{65b}$$

and

$$\bar{\Delta}_1(\lambda, \hat{\tau}) = e^{-\hat{\tau}} \left\{ \begin{aligned} & \bar{C}_1 + \bar{D}_1 \frac{\sqrt{\pi}}{4} erf(\lambda) \\ & + \frac{T_1^* e^{-\lambda^2}}{\Delta T_0} \sum_{n=1}^{\infty} \hat{\tau}^n \left[\frac{(1+p)^n}{n!}\right] \left[\Phi\left(n + \frac{1}{2}, \frac{1}{2}; \lambda^2\right) \right. \\ & \left. + \lambda \Lambda_n \Phi\left(n+1, \frac{3}{2}; \lambda^2\right)\right] \\ & - e^{-\lambda^2} \sum_{n=1}^{\infty} \hat{\tau}^{n-1} \left[n\bar{C}_n \Phi\left(n - \frac{1}{2}, \frac{1}{2}; \lambda^2\right) + \left(n - \frac{1}{2}\right) \bar{D}_n \lambda \Phi\left(n, \frac{3}{2}; \lambda^2\right)\right] \end{aligned} \right\} \tag{66a}$$

$$\bar{\Delta}_2(\lambda, \hat{\tau}) = e^{-(\hat{\tau}+\lambda^2)} \sum_{n=1}^{\infty} \hat{\tau}^n \left[\bar{C}_n \Phi\left(n + \frac{1}{2}, \frac{1}{2}; \lambda^2\right) + \bar{D}_n \lambda \Phi\left(n+1, \frac{3}{2}; \lambda^2\right)\right] \tag{66b}$$

$$\bar{\Delta}_3(\lambda, \hat{\tau}) = \frac{e^{-(\hat{\tau}+\lambda^2)}}{2} \sum_{n=1}^{\infty} \hat{\tau}^{n-1/2} \left[4n\lambda \bar{C}_n \Phi\left(n + \frac{1}{2}, \frac{3}{2}; \lambda^2\right) + \bar{D}_n \Phi\left(n, \frac{1}{2}; \lambda^2\right)\right] \tag{66c}$$

Equation (65*a*) is an ordinary differential equation for $\bar{F}(\theta)$ that can be solved numerically by the Runge-Kutta method [9], given suitable initial conditions. The latter are in fact arbitrary, since the arbitrary constants implied by the general solution of Eq. (65*a*) simply determine the partition of an arbitrary, time-independent biharmonic function between the two functions ϕ_1 and ϕ_2 of Eq. (5*a*) and have no effect on the physical quantities predicted by the solution. We therefore use the following initial conditions:

$$\bar{F}(0) = 0 \qquad \frac{d\bar{F}(0)}{d\theta} = 0 \tag{67}$$

The time evolution of the perturbation in contact pressure may then be determined from

$$P_1(\tau) = m^2 \left[b_1(\tau) - e^{-\tau} \sum_{n=1}^{\infty} \tau^n C_n\right] \tag{68}$$

which comes from Eq. (52*b*). We may recast Eq. (68) in terms of the dimensionless variables defined by Eqs. (47) and (48) as follows:

$$\bar{P}_1(\tau) = \bar{b}_1(\tau) - e^{-\tau} \sum_{n=1}^{\infty} \tau^n \bar{C}_n \tag{69a}$$

where

$$\bar{b}_1(\tau) = \frac{m^2(1-\nu)}{E\alpha\Delta T_0} b_1(\tau) \tag{69b}$$

NUMERICAL SOLUTION PROCEDURE

Equation (65*a*), which is a second-order ordinary differential equation for $\bar{F}$ with variable coefficients, was solved numerically by the fourth-order Runge-Kutta method. The initial conditions for all cases considered are given by Eqs. (67) and the space step, $\Delta\theta$, was taken as 0.001. The cooling parameter p was set to zero for each calculation.

Following Li and Barber [2], it can be shown from the asymptotic properties of the hypergeometric function and Eqs. (19) that the series expressions in Eqs. (66) are absolutely convergent for all λ, p, and τ. To numerically evaluate these series, we set up the criterion that terms of value less than 10^{-13} are truncated and the iteration subsequently stopped. We also tested different step sizes to ensure the level of convergence would indeed fall within the range of 10^{-8} for all τ and λ except for the $\lambda = 0$; the temperature profile for this limiting case has a rather simple linear variation through the thickness at all times.

RESULTS AND DISCUSSION

The solution given in Part 2 can be made as accurate as desired up to any value of τ by using a suitably fine discretization of the differential equation (65*a*). By contrast, the asymptotic solution of Part 1 can only be expected to be accurate at small values of τ, but in this range it has the advantage of being expressed in analytical rather than numerical form. We therefore first compare the two solutions to determine in what range the more convenient asymptotic solution gives results of acceptable accuracy. This comparison also has the incidental advantage of giving us an independent check of the correctness of the numerical algorithm in the small time limit.

Figure 2 shows the short time behavior of the perturbation in contact pressure, $\bar{P}_1(\tau)$ predicted by each of the two models for $\lambda = 0.5$, which corresponds to a Stefan number of 0.59. Each theory predicts unstable growth of the casting since the thermal distortion acts to increase the contact pressure at the positions $mx = \pm(2i - 1)\pi$ that lie beneath thickness maxima on the freezing front (see Eqs. (17*c*) and (19*b*)). The results show that the two solutions agree closely in the range $0 < \tau < 0.01$, but that significant differences start to accumulate at larger values of τ. Thus, at a truncation of $N = 3$, the asymptotic solution is only usable at very small times.

To explore the evolution of the contact pressure perturbation at larger values of τ, we use the numerical solution of Part 2. Since the analysis shows that the perturbation is simply linearly proportional to $\Delta T_0/T_1^*$, we present the evolution of the contact pressure perturbation in terms of $\bar{P}_1(\tau)\ \Delta T_0/T_1^*$. Figure 3 shows the results

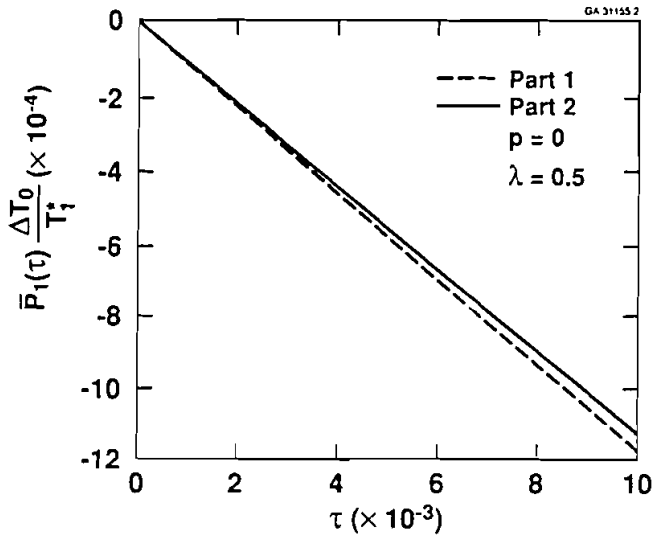

Fig. 2 Short time evolution of $\bar{P}_1(\tau)\ \Delta T_0/T_1^*$ from parts 1 and 2 for $p = 0$ and $\lambda = 0.5$.

obtained for $\lambda = 0.2$, 0.5, and 1.0, which correspond to Stefan numbers of 0.08, 0.59, and 4.0, respectively. In each case, the pressure perturbation increases approximately linearly with τ, the most rapid increase being associated with the largest value of λ. This seems to indicate that large values of λ, which correspond to large Stefan numbers, are conducive to instability in unidirectional solidification, but this is misleading, since the rate of solidification is also a function of λ, with a casting of given thickness solidifying more rapidly if λ is large. In fact, the thickness of the

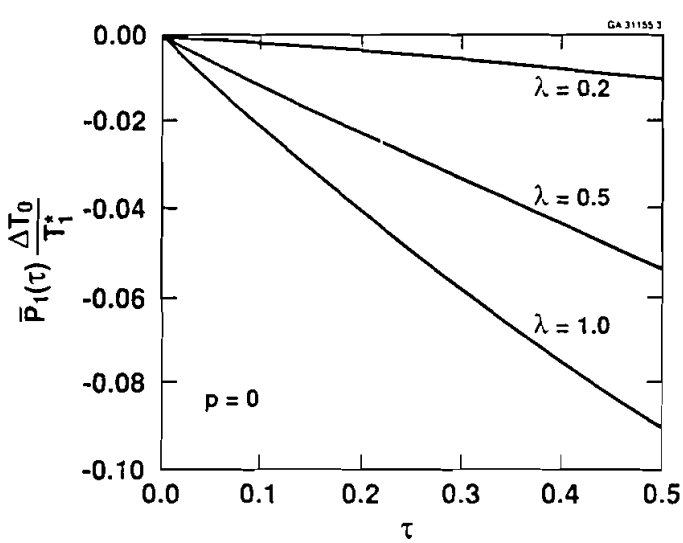

Fig. 3 Evolution of $\bar{P}_1(\tau)\ \Delta T_0/T_1^*$ from part 2 for $p = 0$ and $\lambda = 0.2$, 0.5, and 1.0.

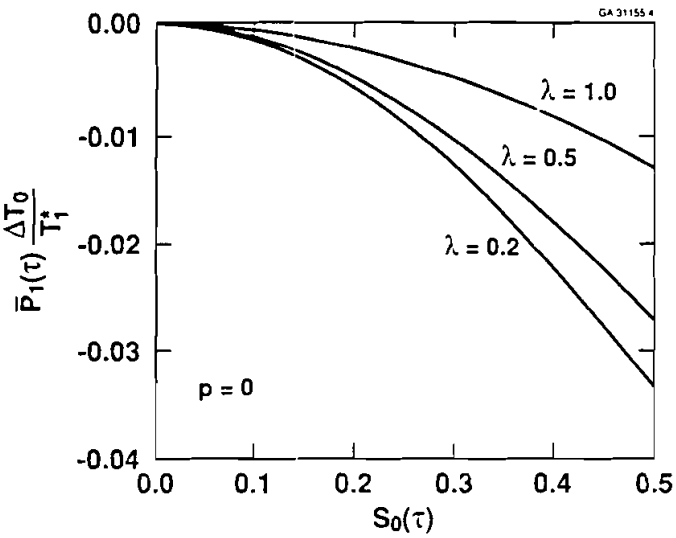

Fig. 4 Variation of $\bar{P}_1(\tau)\ \Delta T_0/T_1^*$ from part 2 with mean melt thickness $S_0(\tau)$ for $p = 0$ and $\lambda = 0.2$, 0.5, and 1.0.

solidified layer in the zeroth order solution is given by Eq. (15), which includes a λ multiplier.

Some measure of the effect of λ on the possibility of air gap nucleation during solidification of a casting of given thickness can be obtained by plotting the pressure perturbation as a function of the dimensionless, instantaneous, zeroth-order layer thickness S_0, as shown in Fig. 4. We find that the magnitude of the pressure perturbation at the time when the mean melt layer thickness has reached a given value is largest when λ is smallest, indicating that large Stefan numbers are actually less conducive to instability for a casting of given thickness.

CONCLUSIONS

Freezing front growth instability and associated air gap nucleation in an idealized casting process have been explored with a thermohypoelastic stress function approach. The first model employed a stress function written in series form that is appropriate for the earliest stages of the process. The second model employed the full stress function that is the exact solution to the compatibility relations but requires a numerical solution. Each model indicated that when the ingot is subjected to an oscillatory mold surface temperature, air gaps nucleate beneath the thinnest regions of the casting. This results in continued nonuniform growth of the freezing front, which has undesirable effects on the quality of the ingot. The two solutions agree closely at sufficiently small values of dimensionless time τ, but diverge significantly when $\tau > 0.01$. As the Stefan number increases, the time rate of change of the contact pressure increases, leading to shorter air gap nucleation times. The total solidification time, however, also decreases in such cases and we find that air gap

formation during solidification of a casting of given thickness is favored by low values of Stefan number.

APPENDIX

Derivation of Strain Potential ψ

The procedure for integration of Eq. (2) is outlined below. Each property of confluent hypergeometric functions has been taken from Abramowitz and Stegun [5]. Two linearly independent solutions of the hypergeometric differential equation

$$\beta''(x) + 2\mu\beta'(x) - \mu\beta(x) = 0 \tag{A.1}$$

are

$$\beta_1(\mu, x) = \Phi\left(-\frac{\mu}{4}, \frac{1}{2}; -x^2\right) \tag{A.2}$$

and

$$\beta_2(\mu, x) = x\Phi\left(\frac{1}{2} - \frac{\mu}{4}, \frac{3}{2}; -x^2\right) \tag{A.3}$$

Using the following property of hypergeometric functions:

$$\Phi'(c_1, c_2; z) \equiv \frac{d\Phi(c_1, c_2; z)}{dz} = \frac{c_1}{c_2}\Phi(c_1 + 1, c_2 + 1; z) \tag{A.4}$$

we may write

$$\beta_1'(\mu, x) \equiv \frac{dB_1(\mu, x)}{dx} = \mu x\Phi\left(1 - \frac{\mu}{4}, \frac{3}{2}; -x^2\right) \tag{A.5}$$

Let $\mu \to \mu - 2$ in Eq. (A.3) and equate the resulting expression with Eq. (A.5). This gives

$$\beta_1'(\mu, x) = \mu\beta_2(\mu - 2, x) \tag{A.6}$$

Let us apply Eq. (A.4) to Eq. (A.3). This gives

$$\beta_2'(\mu, x) \equiv \frac{d\beta_2(\mu, x)}{dx} = \Phi\left(\frac{1}{2} - \frac{\mu}{4}, \frac{3}{2}; -x^2\right) - \frac{x^2(2 - \mu)}{3}\Phi\left(\frac{3}{2} - \frac{\mu}{4}, \frac{5}{2}; -x^2\right) \tag{A.7}$$

Consider the following property of hypergeometric functions:

$$c_2\Phi(c_1, c_2; z) - c_2\Phi(c_1 - 1, c_2; z) - z\Phi(c_1, c_2 + 1; z) = 0 \tag{A.8}$$

Let $c_1 = \frac{3}{2} - \frac{\mu}{4}$, $c_2 = \frac{3}{2}$, and $z = -x^2$. Eq. (A.8) becomes

$$x^2\Phi\left(\frac{3}{2} - \frac{\mu}{4}, \frac{5}{2}; -x^2\right) = \frac{3}{2}\Phi\left(\frac{1}{2} - \frac{\mu}{4}, \frac{3}{2}; -x^2\right) - \frac{3}{2}\Phi\left(\frac{3}{2} - \frac{\mu}{4}, \frac{3}{2}; -x^2\right) \quad \text{(A.9)}$$

Equation (A.9) is substituted into Eq. (A.7) to yield

$$\beta_2'(\mu, x) = \frac{\mu}{2}\Phi\left(\frac{1}{2} - \frac{\mu}{4}, \frac{3}{2}; -x^2\right) + \left(1 - \frac{\mu}{2}\right)\Phi\left(\frac{3}{2} - \frac{\mu}{4}, \frac{3}{2}; -x^2\right) \quad \text{(A.10)}$$

Consider the following property of hypergeometric functions:

$$(1 + c_1 - c_2)\Phi(c_1, c_2; z) - c_1\Phi(c_1 + 1, c_2; z)$$
$$+ (c_2 - 1)\Phi(c_1, c_2 - 1; z) = 0 \quad \text{(A.11)}$$

Let $c_1 = \frac{1}{2} - \frac{\mu}{4}$, $c_2 = \frac{3}{2}$, and $z = -x^2$. Equation (A.11) becomes

$$\left(1 - \frac{\mu}{2}\right)\Phi\left(\frac{3}{2} - \frac{\mu}{4}, \frac{3}{2}; -x^2\right) = \Phi\left(\frac{1}{2} - \frac{\mu}{4}, \frac{1}{2}; -x^2\right) - \frac{\mu}{2}\Phi\left(\frac{1}{2} - \frac{\mu}{4}, \frac{3}{2}; -x^2\right) \quad \text{(A.12)}$$

Substituting Eq. (A.12) into Eq. (A.10), we find

$$\beta_2'(\mu, x) = \Phi\left(\frac{1}{2} - \frac{\mu}{4}, \frac{1}{2}; -x^2\right) \quad \text{(A.13)}$$

We let $\mu \to \mu - 2$ in Eq. (A.2). This gives

$$\beta_1(\mu - 2, x) = \Phi\left(\frac{1}{2} - \frac{\mu}{4}, \frac{1}{2}; -x^2\right) \quad \text{(A.14)}$$

Equations (A.13) and (A.14) give

$$\beta_2'(\mu, x) = \beta_1(\mu - 2, x) \quad \text{(A.15)}$$

Therefore

$$\beta_2''(\mu, x) = x(\mu - 2)\Phi\left(\frac{3}{2} - \frac{\mu}{2}, \frac{3}{2}; -x^2\right) = (\mu - 2)\beta_2(\mu - 4, x) \quad \text{(A.16)}$$

and, from Eq. (A.6)

$$\beta_1''(\mu, x) = \Phi\left(1 - \frac{\mu}{4}, \frac{1}{2}; -x^2\right) = \mu\beta_1(\mu - 4, x) \tag{A.17}$$

Following Li and Barber [2], we let $\mu \to 4n$. Equations (A.2) and (A.3) become

$$\beta_1(4n, x) = e^{-x^2}\Phi\left(n + \frac{1}{2}, \frac{1}{2}; x^2\right) \tag{A.18}$$

$$\beta_2(4n, x) = xe^{-x^2}\Phi\left(n + 1, \frac{3}{2}; x^2\right) \tag{A.19}$$

where we have used the Kummer transformation. Let us define the functions

$$R_{n_1}(x) = \beta_1(4n, x) \tag{A.20}$$

$$R_{n_2}(x) = \beta_2(4n, x) \tag{A.21}$$

Therefore, following (A.16) and (A.17) we have

$$R''_{n_2}(x) = 2(2n - 1)R_{n-1_2}(x) \tag{A.22}$$

$$R''_{n_2}(x) = 2(2n - 1)R_{n-1_2}(x) \tag{A.23}$$

Let us define the functions

$$\psi_{n_1} = \tau^n e^{-\tau} R_{n_1}(Z) \cos(mx) \tag{A.24}$$

$$\psi_{n_2} = \tau^n e^{-\tau} R_{n_2}(Z) \cos(mx) \tag{A.25}$$

We have

$$\nabla^2\psi_{n_1} = m^2(n\psi_{n-1_1} - \psi_{n_1}) \tag{A.26}$$

and

$$\nabla^2\psi_{n_2} = m^2\left[\left(n - \frac{1}{2}\right)\psi_{n-1_2} - \psi_{n_2}\right] \tag{A.27}$$

We write the general function ψ where

$$\psi = \sum_{n=1}^{\infty} \{C_n\psi_{n_1} + D_n\psi_{n_2}\} \tag{A.28}$$

where C_n and D_n are unknown constants. From Eqs. (2) and (19*a*) we also know that ψ must satisfy

$$\nabla^2\psi = \frac{E\alpha}{1-\nu}\sum_{n=0}^{\infty}\{A_n\psi_{n_1} + B_n\psi_{n_2}\} \tag{A.29}$$

where A_n and B_n are defined by Eqs. (19*c*) and (19*d*). From Eqs. (A.26), (A.27), and (A.28) we have

$$\nabla^2\psi = m^2\left\{C_1\psi_{0_1} + \frac{D_1}{2}\psi_{0_2} + \sum_{n=1}^{\infty}\left\{[(n+1)C_{n+1} - C_n]\psi_{n_1} + \left[\left(n+\frac{1}{2}\right)D_{n+1} - D_n\right]\psi_{n_2}\right\}\right\} \tag{A.30}$$

When we equate Eqs. (A.29) and (A.30), we arrive at Eqs. (28).

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