Journal of Thermal Stresses

ISSN: 0149-5739 (Print) 1521-074X (Online) Journal homepage: http://www.tandfonline.com/loi/uths20

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To cite this article: Abdullah M. Al-Shabibi & James R. Barber (2009) Transient Solution of the Unperturbed Thermoelastic Contact Problem, Journal of Thermal Stresses, 32:3, 226-243, DOI: 10.1080/01495730802507980

To link to this article: https://doi.org/10.1080/01495730802507980

Published online: 19 Feb 2009.

Journal of Thermal Stresses, 32: 226–243, 2009

ISSN: 0149-5739 print/1521-074X online
DOI: 10.1080/01495730802507980

TRANSIENT SOLUTION OF THE UNPERTURBED THERMOELASTIC CONTACT PROBLEM

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Automotive brake and clutch systems experience temperature and contact pressure variation due to frictional heat generation. Due to geometrical complexity and the coupled thermo-mechanical nature of this class of problems, direct finite element simulation is found to be computer-intensive. This paper explores a more time-efficient method that can be used to obtain the transient solution for the unperturbed clutch and brake system based on an eigenfunction expansion and a particular solution. An approximate solution can also be sought based on the same method in which only a subset of the eigenfunctions are used.

***Keywords*:** Brake; Clutch; Contact problem; Thermo-elastic instability; Transient problem

INTRODUCTION

Perturbations in the contact pressure or temperature field during thermoelastic contact in an automotive brake and clutch system are known to grow exponentially in time providing the sliding speed is sufficiently high [1]. Solving the thermo-mechanical perturbation problem leads to an eigenvalue problem for the exponential growth rate and the corresponding eigenfunctions. A more general transient solution of the homogenous (unloaded) problem can then be obtained as an eigenfunction expansion [2]. This solution, however, does not describe the underlying unperturbed process. For example, an axisymmetric clutch disk has an axisymmetric transient solution, which may be unstable to small non-axisymmetric perturbations associated with manufacturing errors or random variations in loading conditions. A complete solution of the problem then requires that the evolution of both the perturbed and unperturbed problems be determined. Also, when the initial perturbation and the unperturbed system have the same form (e.g., both are axisymmetric) the unperturbed problem has a substantial effect on the magnitude of the initial perturbation.

Due to geometrical complexity of this class of problems, analytical solutions are limited to few simple cases and in order to account for a practical brake or clutch geometry finite element solutions are usually sought [3–5]. Direct finite element simulation is found to be computer-intensive especially when small time

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steps and fine mesh are needed to capture the temporal and spatial variation of the temperature field. Zagrodzki [6] has suggested a method based on modal analysis. In this paper we explore an alternative method that can be used to solve the thermoelastic contact problem with frictional heating. The proposed method will be tested in the context of an axisymmetric problem of two sliding disks. The finite element method is used to discretize the system and to obtain the transient solution of the temperature field and the contact pressure. Both constant and varying sliding speed will be considered and the results will be compared with those of direct finite element simulation using the commercial package ABAQUS.

TRANSIENT SOLUTION OF THE UNPERTURBED HEAT CONDUCTION PROBLEM

The temperature field is determined by a system of inhomogeneous partial differential equations whose general solution can be written as the sum of a homogeneous and a particular solution, i.e.,

$$T(r, z, t) = T_h(r, z, t) + \theta_p(r, z, t) \tag{1}$$

where T_h and θ_p are, respectively, the homogenous and particular solutions for the temperature field in Cartesian coordinates x, y, z and t is time. The homogeneous solution is obtained by replacing the inhomogeneous boundary conditions (typically prescribed external loads) by their homogeneous equivalents, after which the resulting problem is solved by eigenfunction expansion. This solution has sufficient degrees of freedom to satisfy the initial conditions. The particular solution, on the other hand, consists of any function that satisfies the inhomogeneous equations and boundary conditions, without regard to initial conditions. If the boundary conditions are independent of time, one possible choice for the particular solution is to use the steady-state solution $\theta_p(r, z)$ [7]. Another alternative is the solution suggested by Zagrodzki [6]

$$\theta_p(r, z, t) = \theta_h(r, z, t)p(r, z, t)$$

where $p(r, z, t)$ is a new function to be determined. The two solutions are then superimposed to yield a general solution that satisfies both the initial and the boundary conditions.

A solution of exponential form is considered for the homogenous problem, leading to an eigenvalue problem for the exponential growth rate b_i and the associated eigenfunctions θ_h^i. A general solution for the transient evolution of the temperature field at constant sliding speed can then be written as

$$T(r, z, t) = \sum_{i=1}^{n} C_i e^{b_i t} \theta_h^i(r, z) + \theta_p(r, z), \tag{2}$$

where n is the number of eigenvalues (and also the number of degrees of freedom in the discretization) and C_i is a set of arbitrary constants to be determined from the initial condition $T(r, z, 0)$. This method is widely used to solve conventional heat conduction problems, in which case however all of the eigenvalues, b_i, are

negative. In the thermoelastic problem, positive eigenvalues will be obtained if the particular solution is unstable. Also notice that the difference between the initial temperature and the particular solution, θ_p, serves as the initial perturbation for the eigenfunction expansion.

SLIDING CONTACT OF TWO ELASTIC DISKS

Figure 1 shows two disks of dissimilar thermoelastic materials. The problem is axisymmetric and therefore the solution of the temperature field and contact pressure are independent of the circumferential direction. The domain of the two disks are denoted by Ω_1 and Ω_2, respectively, and their boundaries are denoted by Γ_1 and Γ_2. The two disks have a common contact interface $\Gamma_c = \Gamma_1 \cap \Gamma_2$.

Sliding friction occurs at the interface $z = 0$ with coefficient f, leading to the generation of frictional heat

$$q(r, t) = fVp(r, t), \tag{3}$$

where $p(r, t)$ is the contact pressure. The heat flowing in and out of the two bodies at the interface results in the thermo mechanical boundary condition

$$q_z(r, t) = -\left(K^1 \frac{\partial T^1}{\partial z}(r, 0, t) + K^2 \frac{\partial T^2}{\partial z}(r, 0, t)\right) = fVp(r, t) \tag{4}$$

where K^1 and K^2 are thermal conductivity of Ω_1 and Ω_2, respectively. The boundary condition

$$K^\beta \frac{\partial T^\beta}{\partial n} = hT^\beta(r, t), \quad (\beta = 1, 2) \tag{5}$$

can be used for the remaining boundary $\Gamma_1 \cup \Gamma_2 - \Gamma_c$, where n and h are the outward normal to the surface and the heat transfer coefficient, respectively. Heat transfer at this boundary has very little effect on the transient solution, but it is essential

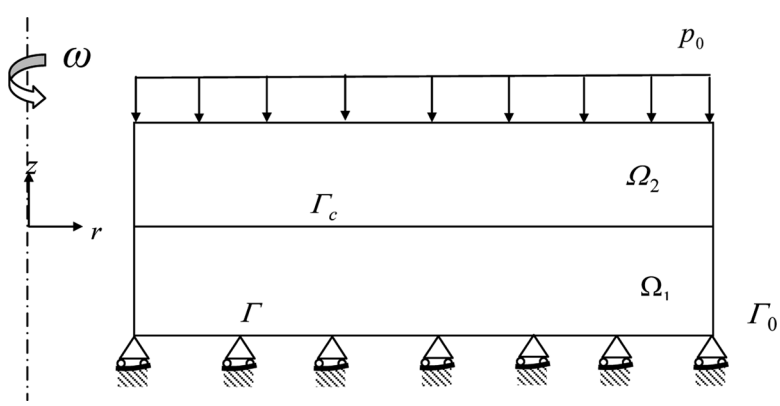

Figure 1 Sliding contact of two elastic disks.

for obtaining a bounded steady-state solution, since in order for the system to reach a steady state there must be a balance between the heat being generated at the interface, Γ_c, and the heat being dissipated at the boundary, $\Gamma_1 \cup \Gamma_2 - \Gamma_c$. Temperature continuity requires that the temperature of the two bodies at the interface to be equal

$$T^1(r, 0, t) = T^2(r, 0, t) \tag{6}$$

A distributed load P_0 is applied to the surface boundary Γ_p resulting in a mechanical boundary condition

$$\sigma_{nn} = -P_0 \tag{7}$$

where σ_{nn} is the traction normal to the surface. To support the distributed load, a displacement constraint

$$u_n = 0 \tag{8}$$

is specified at the boundary Γ_0, where u_n is the normal surface displacement. Furthermore, the contact continuity requires the displacement normal to the contact interface to be equal for the two disks, i.e.,

$$u_z^1(r, 0, t) = u_z^2(r, 0, t). \tag{9}$$

TRANSIENT HOMOGENOUS HEAT PROBLEM

All of the thermal and mechanical boundary conditions considered in the previous section are homogeneous except for equation (7). In order to obtain the homogenous solution, (7) is replaced by the equivalent homogeneous condition

$$\sigma_{nn} = 0 \tag{10}$$

The first step toward solving the homogeneous thermoelastic contact problem is to consider the solution of the heat conduction equation

$$K^\beta\left(\frac{\partial^2 T_h^\beta}{\partial r^2} + \frac{1}{r}\frac{\partial T_h^\beta}{\partial r} + \frac{\partial^2 T_h^\beta}{\partial z^2}\right) = \rho^\beta c_p^\beta \frac{\partial T_h^\beta}{\partial t}, \quad (\beta = 1, 2) \tag{11}$$

for the appropriate temperature field T_h^1 and T_h^2 that satisfies the homogeneous thermal boundary conditions (4–6), where ρ and c_p are the density and specific heat, respectively, for body β. Assuming a solution of the exponential form

$$T_h^\beta(r, z, t) = \theta_h^\beta(r, z)e^{bt} \tag{12}$$

and substituting it in equation (11) yields the equation

$$K^\beta\left(\frac{\partial^2 \theta_h^\beta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta_h^\beta}{\partial r} + \frac{\partial^2 \theta_h^\beta}{\partial z^2}\right) = \rho^\beta c_p^\beta b\theta_h^\beta. \tag{13}$$

for $\theta_h^\beta(r, z)$.

If the geometry is discretized by the finite element method, the instantaneous temperature field for each body can be characterized by a finite set of n^β nodal temperatures and it follows that there will be $n = n^1 + n^2$ terms in the eigenfunction series (2). To develop the eigenvalue problem, we first approximate the temperature $\theta_h^\beta(r, z)$ of equation (13) in the form

$$\theta_h^\beta(r, z) = \sum_{i=1}^{n^\beta} N_i(r, z)\Theta_i^\beta, \tag{14}$$

where Θ_i^β are the nodal temperatures in body β and $N_i(r, z)$ are a set of n^β shape functions. Applying the weighted residual method to equation (13), we obtain the set of equations

$$\int_{\Omega^\beta} W_j\left(K^\beta\left(\frac{\partial^2\theta_h^\beta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta_h^\beta}{\partial r} + \frac{\partial^2\theta_h^\beta}{\partial z^2}\right) - \rho^\beta c_p^\beta b\theta_h^\beta\right)d\Omega^\beta = 0, \tag{15}$$

where W_j is a set of linearly independent weight functions. The second derivative in equation (15) is replaced with the first derivative through integration by parts, giving

$$\begin{aligned} &-\int_{\Omega_\beta} K^\beta\left(\frac{\partial W_j}{\partial r}\frac{\partial\theta_h^\beta}{\partial r} + \frac{1}{r}W_j\frac{\partial\theta_h^\beta}{\partial r} + \frac{\partial W_j}{\partial z}\frac{\partial\theta_h^\beta}{\partial z} - b\rho^\beta c_p^\beta W_j\theta_h^\beta\right)d\Omega_\beta \\ &+\int_{\Omega_\beta}\left(K^\beta\frac{\partial}{\partial r}\left(W_j\frac{\partial\theta_h^\beta}{\partial r}\right) + K^\beta\frac{\partial}{\partial z}\left(W_j\frac{\partial\theta_h^\beta}{\partial z}\right)\right)d\Omega_\beta = 0 \end{aligned} \tag{16}$$

The second integral in (16) can then be replaced by a surface integral using Gauss' theorem to yield, after considering the boundary conditions (4–5),

$$\begin{aligned} &-\int_{\Omega^\beta}\left(K^\beta\left(\frac{\partial W_j}{\partial r}\frac{\partial\theta_h^\beta}{\partial r} + \frac{1}{r}W_j\frac{\partial\theta_h^\beta}{\partial r} + \frac{\partial W_j}{\partial z}\frac{\partial\theta_h^\beta}{\partial z}\right) - b\rho^\beta c_p^\beta W_j\theta_h^\beta\right)d\Omega^\beta \\ &+\int_{(\Gamma^\beta-\Gamma_c)}\left(hW_j\theta_h^\beta\right)d\left(\Gamma^\beta - \Gamma_c\right) + \int_{\Gamma_c}\left(W_j q^\beta\right)d\Gamma_c \end{aligned} \tag{17}$$

where

$$q^\beta = -K^\beta\frac{\partial\theta^\beta(r, z)}{\partial n}, \quad (r, z \in \Gamma_c) \tag{18}$$

Substituting (14) into (17) and using the same functions N_i as both shape and weight functions, we obtain the matrix equation

$$(\boldsymbol{C}^\beta + \boldsymbol{H}^\beta)\boldsymbol{\Theta}^\beta + \boldsymbol{q}^\beta = b\boldsymbol{M}^\beta\boldsymbol{\Theta}^\beta \tag{19}$$

where

$$C_{ji}^\beta = \int_{\Omega^\beta} K^\beta\left(\frac{\partial N_j}{\partial r}\frac{\partial N_i}{\partial r} + \frac{1}{r}N_j\frac{\partial N_i}{\partial r} + \frac{\partial N_j}{\partial z}\frac{\partial N_i}{\partial z}\right)d\Omega^\beta \tag{20}$$

$$M_{ji}^{\beta} = \int_{\Omega^{\beta}} \left(\rho^{\beta} c_p^{\beta} N_j N_i\right) d\Omega^{\beta} \tag{21}$$

$$H_{ji}^{\beta} = \begin{cases} \int_{(\Gamma^{\beta}-\Gamma_c)} (hW_j W_i) d(\Gamma^{\beta} - \Gamma_c) & i, j \in \Gamma^{\beta} - \Gamma_c \\ 0 & i, j \notin \Gamma^{\beta} - \Gamma_c \end{cases} \tag{22}$$

and

$$q_j^{\beta} = \begin{cases} \int_{\Gamma_c} (W_j q^{\beta}) d\Gamma_c, & j \in \Gamma_c \\ 0 & j \notin \Gamma_c \end{cases} \tag{23}$$

Adding the matrix equations for the two bodies yields the assembled matrix equation for the whole system

$$(\boldsymbol{C} + \boldsymbol{H})\boldsymbol{\Theta} + \boldsymbol{q} = b\boldsymbol{M}\boldsymbol{\Theta} \tag{24}$$

where the extra degrees of freedom resulting from the boundary condition (6) are eliminated by adding their associate matrix elements. The nodal heat fluxes of the two bodies are added to yield a new nodal heat flux vector $\boldsymbol{q}$. This vector is zero everywhere except at the contact interface. It might be more convenient to express (24) in the form

$$(\boldsymbol{C} + \boldsymbol{H})\boldsymbol{\Theta} + \boldsymbol{A}\boldsymbol{q}_c = b\boldsymbol{M}\boldsymbol{\Theta} \tag{25}$$

where

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{I}_c \\ \boldsymbol{0} \end{bmatrix} \tag{26}$$

$\boldsymbol{I}_c$ is the identity matrix of order $n_c \times n_c$ and n_c is the number of the contact nodes.

THE THERMOELASTIC PROBLEM

The nodal heat flux $\boldsymbol{q}_c$ is proportional to the corresponding nodal contact pressure through the discrete equivalent of equation (3). A second equation linking the contact pressure to the temperature distribution can be obtained from the finite element solution of the thermoelastic contact problem. We define a quasi-static displacement field where the time variable has been eliminated from the displacement field since the thermoelastic governing equations are time-independent. The displacement functions u, v are written in the discrete form

$$\begin{aligned} u^{\beta}(r, z) &= \sum_{i=1}^{n^{\beta}} N_i(r, z) U_i^{\beta} \\ v^{\beta}(r, z) &= \sum_{i=1}^{n^{\beta}} N_i(r, z) V_i^{\beta} \\ w^{\beta}(r, z) &= \sum_{i=1}^{n^{\beta}} N_i(r, z) W_i^{\beta} \end{aligned} \tag{27}$$

where U_i^β, V_i^β and W_i^β are the components of the nodal displacement vector $\boldsymbol{U}^\beta$. The potential energy for the body β can then be written

$$\Pi^\beta = \frac{1}{2}\int_{\Omega^\beta} \left(\boldsymbol{\varepsilon}^{\beta^T}\boldsymbol{\sigma}^\beta - \boldsymbol{\varepsilon}_0^{\beta^T}\boldsymbol{\sigma}^\beta\right) d\Omega^\beta - \int_{\Gamma_c} u_y^\beta p^\beta d\Gamma_c \tag{28}$$

where

$$\begin{aligned} \boldsymbol{\sigma} &= \{\sigma_r, \sigma_\theta, \sigma_z, \tau_{r\theta}, \tau_{rz}, \tau_{\theta z}\} \\ \boldsymbol{\varepsilon} &= \{\varepsilon_r, \varepsilon_\theta, \varepsilon_z, \gamma_{r\theta}, \gamma_{rz}, \gamma_{\theta z}\} \\ \boldsymbol{\varepsilon}_0 &= \alpha T(r, z)\{1, 1, 1, 0, 0, 0\} \end{aligned} \tag{29}$$

are, respectively, the stress, strain and thermal strain vectors. The and strain are related by

$$\boldsymbol{\sigma}^\beta = \boldsymbol{D}^\beta\left(\boldsymbol{\varepsilon}^\beta - \boldsymbol{\varepsilon}_0^\beta\right) \tag{30}$$

where

$$\boldsymbol{D} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-2\nu)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-2\nu)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-2\nu)/2 \end{bmatrix} \tag{31}$$

Substituting (14, 27) into the strain-displacement relations we obtain the discrete form of the strains as

$$\boldsymbol{\varepsilon}^\beta = \sum_{i=1}^{n} \boldsymbol{B}_i \boldsymbol{U}_i^\beta \tag{32}$$

$$\boldsymbol{\varepsilon}_0^\beta = \sum_{i=1}^{n} N_i \Theta_i^\beta \{1, 1, 1, 0, 0, 0\} \tag{33}$$

where

$$B_i = \begin{bmatrix} \dfrac{\partial N_i}{\partial r} & \dfrac{N_i}{r} & 0 & 0 & \dfrac{\partial N_i}{\partial z} & 0 \\ 0 & 0 & 0 & \dfrac{\partial N_i}{\partial r} - \dfrac{N_i}{r} & 0 & \dfrac{\partial N_i}{\partial z} \\ 0 & 0 & \dfrac{\partial N_i}{\partial z} & 0 & \dfrac{\partial N_i}{\partial r} & 0 \end{bmatrix}^T . \tag{34}$$

We then substitute (32–34) into (28) and perform the integrations. Minimizing the resulting expressions with respect to the nodal displacements $\boldsymbol{U}$ then yields the

system of equations

$$\boldsymbol{K}^{\beta}\boldsymbol{U}^{\beta} = \boldsymbol{G}^{\beta}\boldsymbol{\Theta}^{\beta} + \boldsymbol{P}^{\beta} \tag{35}$$

where

$$\boldsymbol{K}^{\beta} = \int_{\Omega^{\beta}} \boldsymbol{B}^{T}\boldsymbol{D}^{\beta}\boldsymbol{B}d\Omega^{\beta}; \quad \boldsymbol{G}^{\beta} = \int_{\Omega^{\beta}} \boldsymbol{B}^{T}\boldsymbol{D}\boldsymbol{C}d\Omega^{\beta}; \quad \boldsymbol{P}^{\beta} = \int_{\Gamma_c} \boldsymbol{N}pd\Gamma_c \tag{36}$$

and

$$\boldsymbol{B} = [B_1\ B_2\ \cdots\ B_{n^{\beta}}]; \quad \boldsymbol{C} = [C_1\ C_2\ \cdots\ C_n]; \quad \boldsymbol{N} = [N_1\ N_2\ \cdots\ N_{n^{\beta}}] \tag{37}$$

with

$$C_i = N_i\{1\ 1\ 1\ 0\ 0\ 0\}^T \tag{38}$$

Adding the matrix equations for the two bodies yield assembled matrix equations for the whole system

$$\boldsymbol{KU} = \boldsymbol{GT} + \boldsymbol{P} \tag{39}$$

where the extra degrees of freedom resulted from the boundary condition are eliminated from the assembled system of matrices. The nodal contact forces at the contact interface are equal and opposite and therefore they disappeared from the assembled nodal forces vector $\boldsymbol{P}$. The vector $\boldsymbol{P}$ is zero everywhere except at the surface of zero displacement where it represents the nodal reaction forces. The zero displacement boundary condition can be eliminated from the (39) to solve for the unknown nodal displacement which yields a modified system of equations.

$$\widetilde{\boldsymbol{K}}\widetilde{\boldsymbol{U}} = \widetilde{\boldsymbol{G}}\widetilde{\boldsymbol{\Theta}} \tag{40}$$

The nodal displacements of body 1 can be extracted from (40) to yield a system of equations for $\boldsymbol{P}_c$ in terms of $\widetilde{\boldsymbol{\Theta}}$ which can be written in the symbolic form

$$\boldsymbol{P_c} = \boldsymbol{L}\widetilde{\boldsymbol{\Theta}} \tag{41}$$

The frictional heating relation is $\boldsymbol{q_c} = f\boldsymbol{V}\boldsymbol{p_c}$ where $V_{ji} = \omega r_i \delta_{ji}$ and r_i is the radial coordinate for the i-th node. Equations (25, 41) can then be used to eliminate $\boldsymbol{p_c}$, $\boldsymbol{q_c}$, leading to the generalized linear eigenvalue equation

$$(fV\boldsymbol{AL} - \boldsymbol{C} - \boldsymbol{H})\boldsymbol{\Theta}_h = b\boldsymbol{M}\Theta_h \tag{42}$$

for b.

If the eigenvalues and eigenfunctions of this equation are denoted by b_k, $\widehat{\Theta}_i^k$ respectively, a general solution for the evolution of the nodal homogeneous temperatures $\Theta_i^h(t)$ at constant speed can be written

$$\Theta_i^h(t) = \sum_{k=1}^{n} C_k \widehat{\Theta}_i^k e^{b_k t} \tag{43}$$

PARTICULAR SOLUTION

The particular solution is obtained by setting the right hand side of the transient heat equation (13) to zero. This is also achievable by making the growth rate b in the homogeneous problem equal to zero, to yield the finite element solution of the steady-state heat conduction problem

$$(\boldsymbol{C} + \boldsymbol{H})\boldsymbol{\Theta}_p - \boldsymbol{A}\boldsymbol{q}_c = \boldsymbol{0} \tag{44}$$

where $\boldsymbol{\Theta}_p$ is the vector of nodal temperatures in the the particular solution.

A relationship linking the contact pressure to the particular solution nodal temperature and the applied load, P_0, can be obtained from the finite element solution of the thermoelastic contact problem. This solution proceeds as in thermoelastic problem section with a modified potential energy relation for body 2,

$$\Pi = \frac{1}{2}\int_{\Omega^\beta} \left(\boldsymbol{\varepsilon}^T\boldsymbol{\sigma} - \boldsymbol{\varepsilon}_0^T\boldsymbol{\sigma}\right) d\Omega - \int_{\Gamma_c} u_z p d\Gamma_c - \int_{\Gamma_p} u_z p_0 d\Gamma_p \tag{45}$$

The last term in (45) corresponds to the non-homogeneous boundary condition (7). This modification yields an extra term in the nodal contact force, which can be written in the symbolic form

$$\boldsymbol{P}_c = \boldsymbol{L}\boldsymbol{\Theta} + \boldsymbol{P}_0 \tag{46}$$

The frictional heating relation $\boldsymbol{q}_c = fV\boldsymbol{p}_c$ and equations (44, 46) can then be used to eliminate $\boldsymbol{p}_c$, $\boldsymbol{q}_c$, leading to the particular solution for the nodal temperatures

$$\boldsymbol{\Theta}_p = [\boldsymbol{C} + \boldsymbol{H} - fV\boldsymbol{L}]^{-1}(fV\boldsymbol{P}_0) \tag{47}$$

A general solution for the evolution of the nodal temperatures at constant speed can then be written by superposing the homogeneous temperature (43) and the particular solution temperature (47)

$$T^i(t) = \sum_{k=1}^{n} C_k \widehat{\Theta}_i^k e^{b_k t} + \Theta_p^i \tag{48}$$

where the constants C_k are to be determined from the initial conditions $T^i(0)$.

NATURE OF THE APPROXIMATION FOR A VARYING SLIDING SPEED

If the sliding speed varies in time, a piecewise-constant approximation can be used. During the time period, $t_{j-1} < t < t_j$, the sliding speed is constant and equal to V_j and the temperature field evolution can be written as

$$T^i(t) = \sum_{k=1}^{n} C_k \widehat{\Theta}_i^k e^{b_k(t-t_{j-1})} + \Theta_p^i \tag{49}$$

where b_i, $\widehat{\Theta}_i$, Θ_p^i are the eigenvalues, eigenfunctions and particular solution respectively appropriate to speed V_j. The constants C_k are determined considering the temperature field at the end of the previous time steps. This is to be repeated for as many times as the number of time steps used to approximate the sliding speed time history.

During the j-th time period, the sliding speed is constant and equal to V_j and we can write the temperature field in the eigenfunction series

$$T^{(j)} = \sum_{i=1}^{n} C_i^{(j)} e^{b_i(t-t_{j-1})} \widehat{\Theta}_i^{(j)} \tag{50}$$

where b_i, $\widehat{\Theta}_i(x, y, z)$ are the eigenvalues and eigenfunctions respectively appropriate to speed V_j and we have chosen to reset the zero for time in each time step.

Continuity of temperature at time t_j then requires that

$$T^{(j+1)}\left(t_j^+\right) = T^{(j)}\left(t_j^-\right) \tag{51}$$

and hence

$$\sum_{i=1}^{n} C_i^{(j+1)} \widehat{\Theta}_i^{(j+1)} = \sum_{i=1}^{n} C_i^{(j)} e^{b_i(t-t_{j-1})} \widehat{\Theta}_i^{(j)} \tag{52}$$

In other words, at the end of each time step we need to re-expand the instantaneous temperature field as a series of the eigenfunctions appropriate to the speed during the next time step.

RESULTS AND DISCUSSION

The Two-Disk Model

The method established above was tested in the context of a system of two clutch disks that are in contact and slide relatively (Figure 1). The two disks are regarded to be made of two different materials: steel and friction material, whose thermal and mechanical properties are given in Table 1. The boundary condition of zero axial displacement is considered for the steel disk surface opposite to the sliding surface (as shown in Figure 1). A uniform pressure is applied to the opposite surface of the friction disk. The dimensions of the two disks are that of a typical clutch system (Table 1).

Table 1 Material and geometrical properties of the steel and friction disk

Property	Steel disk	Friction disk
Outer radius (mm)	57.0	57.0
Inner radius (mm)	44.5	44.5
Disk thickness (mm)	2.5	2.5
Young's modulus (GPa)	125	0.30
Poisson's ratio	0.25	0.25
Thermal conductivity	54.0	1.0
Specific heat	532	120
Density	7800	2000
Thermal expansion coefficient	12×10^{-6}	12×10^{-6}

Critical Speed

The linear eigenvalue equation (42) was solved for several different sliding speeds. At each speed the maximum exponential growth rate was checked for a positive value. A positive growth rate indicates operation above critical speed. A critical speed (ω_c) of about 500 rad/s was found for this disk system.

Particular Solution

Considering a constant sliding speed and using equation (47), the particular solution (i.e., the steady-state solution) for the contact pressure distribution along the contact interface was computed. Figure 2 shows the particular solutions for

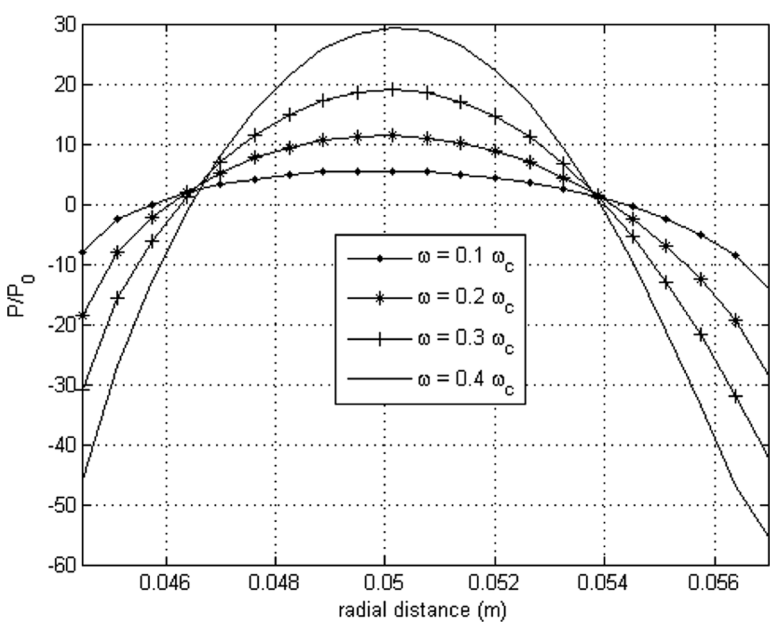

Figure 2 Particular solution for the contact pressure distribution ($\omega < \omega_c$).

the contact pressure distribution at different operating speeds ranging between 0.04 and $0.4\,\omega_c$. Notice that the pressure distribution is non-uniform, even for speeds below the critical value, with a peak value occurring near the mid-radius. This is mainly because of the thermo-elastic distortion that modifies the contact pressure distribution. The maximum value of the contact pressure depends solely on the operating speed, where a 3000% increase in the contact pressure can be reached for $\varpi = 0.4\,\varpi_c$. Furthermore, a negative pressure is noted at the two edge points of the contact surface. In the physical system, this would be an indication that separation would occur in the steady state. However, the bilateral solution assuming full contact remains acceptable as a component of the complete transient solution as long as the total instantaneous pressure is greater than zero.

Transient Solution for a Constant and Varying Sliding Speed

The transient solution proposed here will cease to apply once the predicted contact pressure falls below zero, since this is an indication of separation, which changes the effective boundary conditions of the problem and hence the eigenvalues and eigenfunctions in equation (42). Figure 3 shows the contact pressure distribution as a function of radius for a constant sliding speed $\varpi = 0.1\,\omega_c$ at various times. This process can be run for as long as 30 s without developing negative pressures. Figure 4 shows the corresponding results for $\varpi = 0.5\,\omega_c$ and in this case separation is indicated after 0.35 s. Typical automatic transmission clutch engagements last around 0.5 s, but during this time the relative sliding speed falls to zero. Figure 5

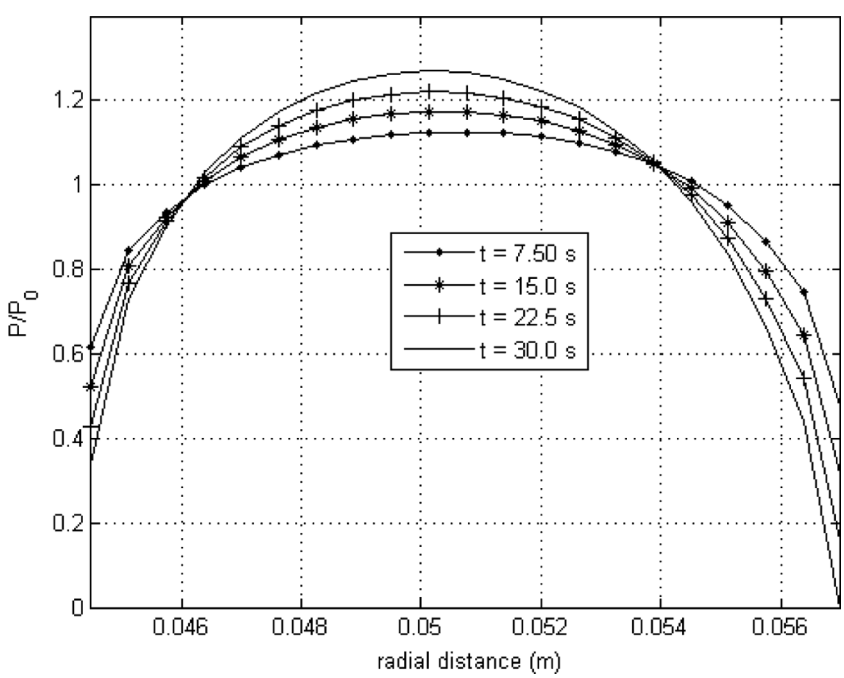

Figure 3 Contact pressure distribution at different instants of time for $\omega = 0.1\omega_c$.

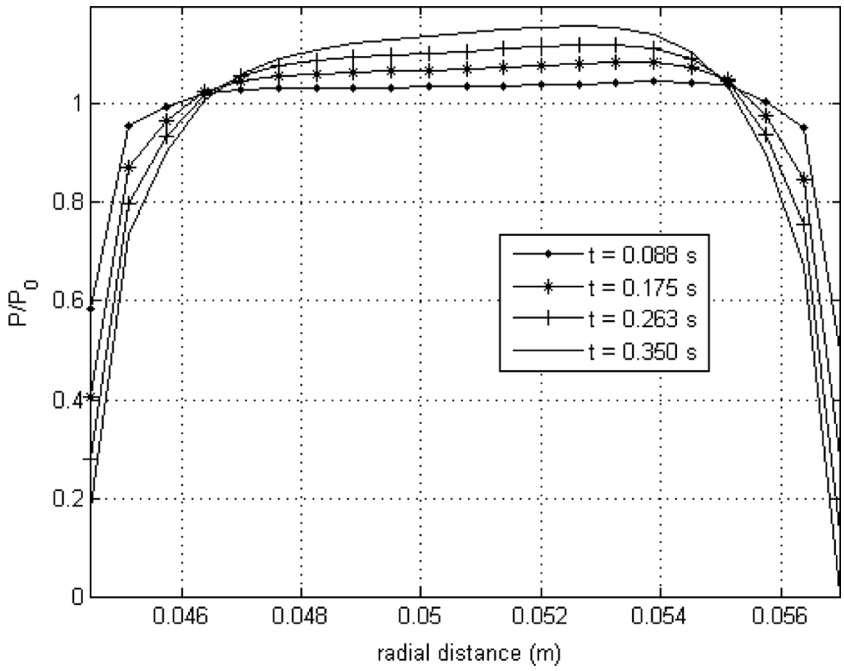

Figure 4 Contact pressure distribution at different instants of time for $\omega = 0.5\,\omega_c$.

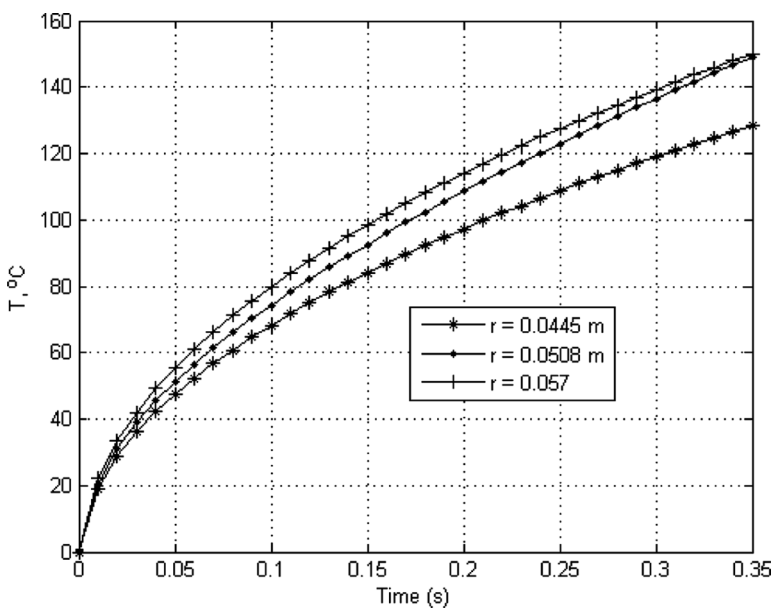

Figure 5 Temperature time history for $\omega = 0.5\,\omega_c$ at different radial distance.

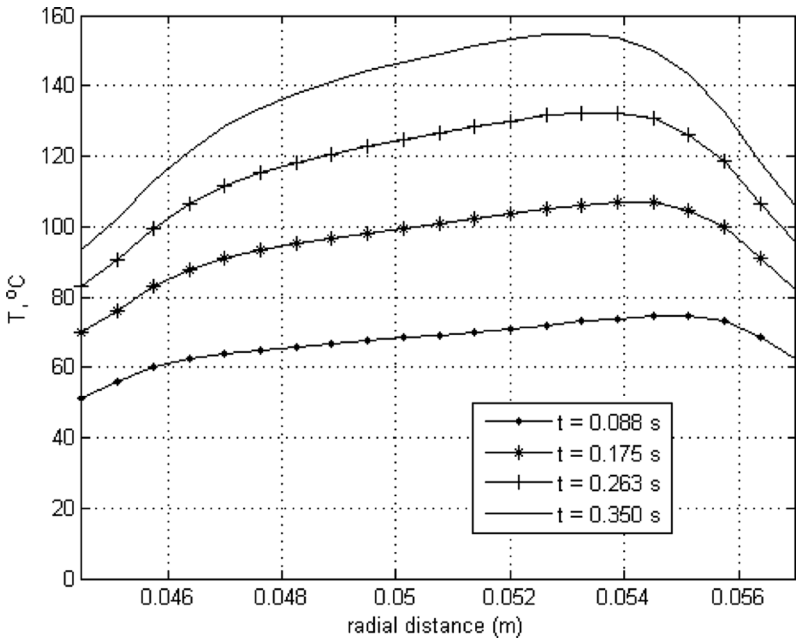

Figure 6 Contact surface temperature distribution at different instants of time ($\omega = 0.5\,\omega_c$).

shows the nodal temperature time history at three different locations (inner radius, outer radius and mean radius) along the contact interface. The temperature at the mean radius consistently exceeds that at the other two radii mainly because of the higher contact pressure (as seen in Figure 4, which implies a higher rate of heat generation. The temperature at the outer radius is greater than that at the inner radius because of the higher sliding speed that depends on the radial distance. Figure 6, on the other hand, shows the temperature distribution across the contact surface at different instants of time for $\varpi = 0.5\,\omega_c$. It can be seen from this figure that the maximum temperature is not exactly at the mean radius and this is due to the fact that the frictional heat is the product of the sliding speed and contact pressure. Although the contact pressure is higher at the mean radius, the sliding speed increased linearly with the radial distance.

The general solution of eigenfunction expansion and steady state problem provides an exact solution to the transient problem as long as the sliding speed is constant. However, the piecewise constant representation of equation (48) can be adopted for cases involving variable speed $\omega(t)$. This approximation was validated using direct numerical simulation of the same problem. Figure 7 shows the predicted contact pressure time history at three different locations for a uniform deceleration with initial speed $\omega(0) = 0.8\,\omega_c$ and stopping time $t_0 = 0.5$ sec. The contact pressure distribution initially increases and toward the end of the engagement time it starts to decrease.

The final contact pressure distribution retains the non-uniformity because of the thermal distortions induced by the presence of the temperature field. Figure 8 shows the temperature evolution at three positions along the contact surface. Here

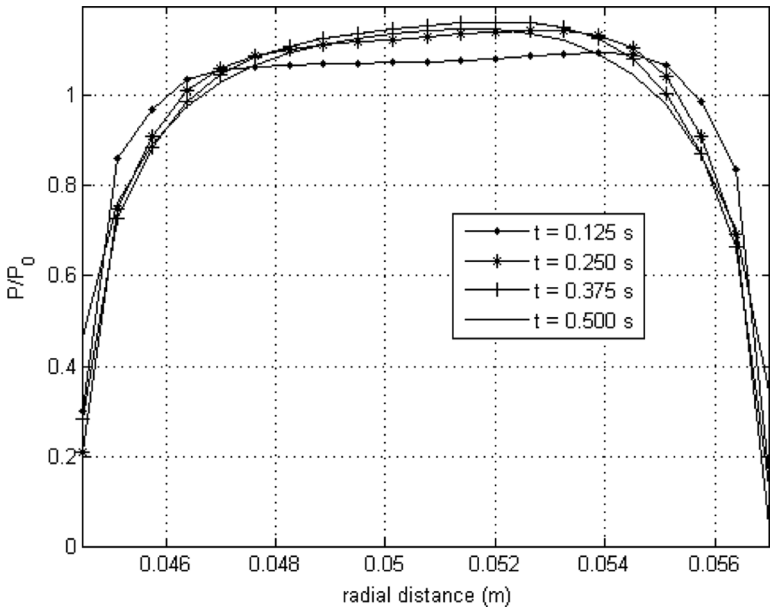

Figure 7 Contact pressure distribution at different instants of time for $\omega_0 = 0.8\,\omega_c$.

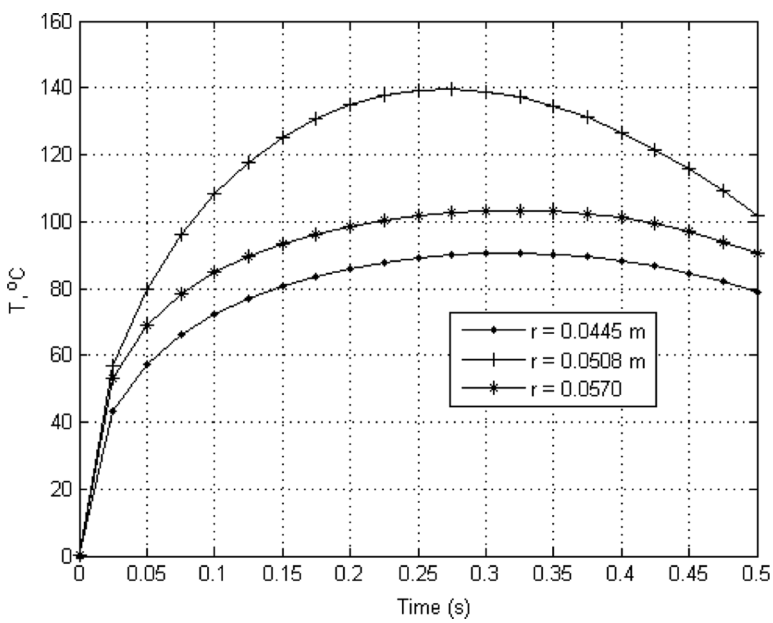

Figure 8 Temperature time history for $\omega_0 = 0.8\,\omega_c$ at different radial distance.

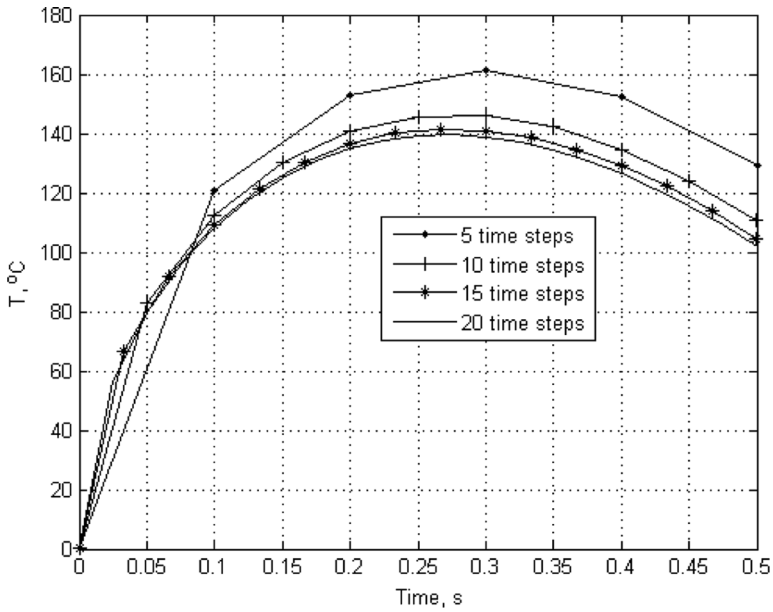

Figure 9 Effect of number of time steps on nodal temperature time history, with decreasing sliding speed ($\omega_0 = 0.8\,\omega_c$).

the maximum temperature is reached at $t = 0.275$ sec. It should be stated that the proposed solution can not be used for a higher sliding speed because that would results into a negative contact pressure or contact separation. However, it can still be used to predict the maximum temperature as long as that takes place before contact separation. The sliding speed considered in this example might be considered a medium operating speed for typical clutch systems.

The appropriate number of time steps was investigated (Figure 9) and compared against the exact solution, defined as the situation where the use of more time steps produces no significant change in the solution. If the stopping time is divided into 5 equal time steps, an acceptable solution is obtained (error of 15%). More accurate results were obtained by using 15 time steps.

At speeds above the critical value, some of the eigenfunctions of the homogeneous solution grow with time and hence dominate the output of the transient problem. Retaining the dominant eigenfunctions in the expansion series can create a reduced order model of the system. This was investigated to determine the appropriate number of terms in the truncated eigenfunction expansion. Figure 10 shows the surface temperature predicted by the reduced order model for various numbers of terms in the truncated series. The use of a 50-term series gives quite good prediction for the surface temperature time history (underestimating the maximum temperature by less that 5%). The use of a 100-term series is shown to give a solution undistinguishable from the exact one (corresponding to a 420-term series). The operating speed plays a very important role in determining the number of terms required in the truncated series. When the speed is above the critical the truncated series overestimates the nodal temperature. This is expected since

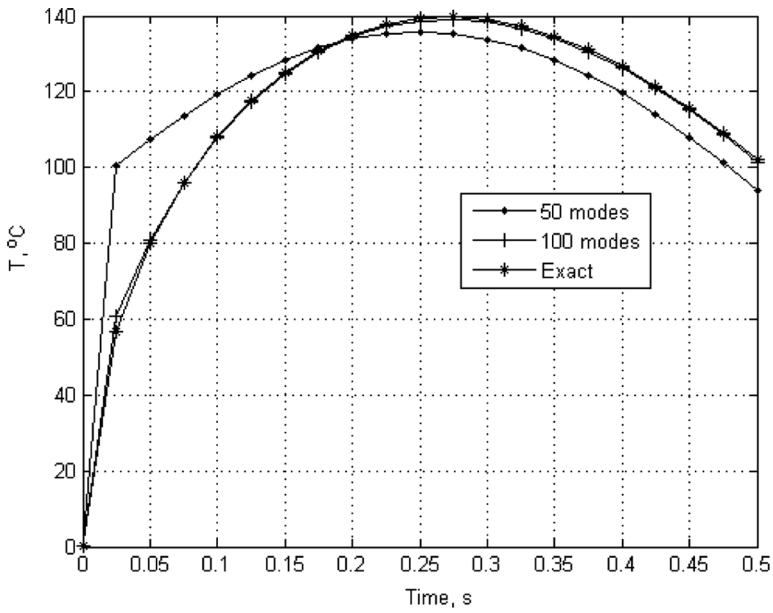

Figure 10 Effect of number of modes on nodal temperature time history, $\omega_0 = 0.8\,\omega_c$.

the truncated series involve the growing modes and relatively few decaying modes. When the speed, however, falls below the critical value a relatively large number of modes is required as in Figure 10. Furthermore, a relatively large number of modes is required to capture the initial condition.

CONCLUSIONS

In this paper, the solution to the transient non-homogeneous thermoelastic contact problem with frictional heating was explored. The problem is divided into homogeneous and particular systems and the solutions of the two systems are superposed to provide the degrees of freedom needed to satisfy the initial and boundary conditions of the problem. The proposed solution methodology was tested for a constant speed where it showed that the solution can be applied for low to medium range clutch operating speeds. The proposed solution however ceases to apply for higher operating speeds because it would then predict negative contact pressure which is an indication of separation. An approximate solution was also discussed for a varying sliding speed where a piecewise constant representation was used for the sliding speed time history. This approximate solution was validated against that of direct finite element simulation using commercial software. The appropriate number of time steps was investigated and it was shown that a sufficiently accurate solution can be obtained with the use of relatively small number of time steps.

A reduced-order model was tested to determine the appropriate number of terms in the truncated eigenfunction series. Using some instead of all terms in the expansion series, an approximate solution can be acquired without jeopardizing the

accuracy. The initial temperature field is shown to be crucial since it represents the particular solution, which can have quite irregular form. This is especially true when the system operates above the critical speed. Misrepresenting the initial temperature may result in an inaccurate representation of the transient process.

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