On the Multi-Dimensional Controller and Stopper Games

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Outline

Consider a zero-sum controller-and-stopper game:

- Two players: the "controller" and the "stopper".
- A state process X^α: can be manipulated by the controller through the selection of α.
- Given a time horizon $T > 0$. The stopper has
  - the right to choose the duration of the game, in the form of a stopping time τ in $[0, T]$ a.s.
  - the obligation to pay the controller the running reward $f(s, X_s^\alpha, \alpha_s)$ at every moment $0 \le s < \tau$, and the terminal reward $g(X_\tau^\alpha)$ at time τ.
- Instantaneous discount rate: $c(s, X_s^\alpha)$, $0 \le s \le T$.

VALUE FUNCTIONS

Define the lower value function of the game

$$V(t,x) := \sup_{\alpha \in \mathcal{A}_t} \inf_{\tau \in \mathcal{T}_{t,T}^t} \mathbb{E}\Bigg[\int_t^\tau e^{-\int_t^s c(u, X_u^{t,x,\alpha}) du} f(s, X_s^{t,x,\alpha}, \alpha_s) ds + e^{-\int_t^\tau c(u, X_u^{t,x,\alpha}) du} g(X_\tau^{t,x,\alpha}) \Bigg],$$

- $\mathcal{A}_t :=$ {admissible controls indep. of $\mathcal{F}_t$},
- $\mathcal{T}_{t,T}^t :=$ {stopping times in $[t, T]$ a.s. & indep. of $\mathcal{F}_t$}.

Note: the upper value function is defined similarly: $U(t,x) := \inf_\tau \sup_\alpha \mathbb{E}[\cdots]$. We say the game has a value if these two functions coincide.

Related Work

The game of control and stopping is closely related to some common problems in mathematical finance:

- pricing American contingent claims, see e.g. Karatzas & Kou [1998], Karatzas & Wang [2000] and Karatzas & Zamfirescu [2005].
- minimizing the probability of lifetime ruin, see Bayraktar & Young [2011].

But, it has not been studied to a great extent except certain particular cases.

Related Work (conti.)

One-dimensional case: Karatzas and Sudderth [2001] study the case where X^α moves along a given interval on $\mathbb{R}$. Under appropriate conditions, they

- show that the game has a value;
- construct explicitly a saddle-point of optimal strategies (α^*, τ^*).

But, difficult to extend their results to multi-dimensional cases (their techniques rely heavily on optimal stopping theorems for one-dimensional diffusions).

Related Work (conti.)

Multi-dimensional case: Karatzas and Zamfirescu [2008] develop a martingale approach to deal with this. Again, it is shown that the game has a value and a saddle point of optimal strategies is constructed, but under some STRONG assumptions:

- the volatility coefficient of X^α has to be nondegenerate.
- the volatility coefficient of X^α cannot be controlled!

Our Goal

We intend to investigate a much more general multi-dimensional controller-and-stopper game in which both the drift and the volatility coefficients of X^α can be controlled, and the volatility coefficient can be degenerate.

Main Result: The game has a value (i.e. $U = V$) and the value function is the unique viscosity solution to an obstacle problem of an HJB equation.

Methodology

- Show: V_* is a viscosity supersolution
  - prove continuity of an optimal stopping problem.
  - derive a weak DPP for V, from which the supersolution property follows.
- Show: U^* is a viscosity subsolution
  - prove continuity of an optimal control problem.
  - derive a weak DPP for U, from which the subsolution property follows.
- Prove a comparison result. Then $U^* \leq V_*$. Since $U^* \geq U \geq V \geq V_*$, we have $U = V$, i.e. the game has a value!!

Outline

Consider a fixed time horizon $T > 0$.

- $\Omega := C([0, T]; \mathbb{R}^d)$.
- $W = \{W_t\}_{t\in[0,T]}$: the canonical process, i.e. $W_t(\omega) = \omega_t$.
- $\mathbb{P}$: the Wiener measure defined on Ω.
- $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$: the $\mathbb{P}$-augmentation of $\sigma(W_s, s \in [0, T])$.

For each $t \in [0, T]$, consider

- $\mathbb{F}^t$: the $\mathbb{P}$-augmentation of $\sigma(W_{t\vee s} - W_t, s \in [0, T])$.
- $\mathcal{T}^t$:=$\{\mathbb{F}^t$-stopping times valued in $[0, T]$ $\mathbb{P}$-a.s.$\}$.
- $\mathcal{A}_t$:=$\{\mathbb{F}^t$-progressively measurable M-valued processes$\}$, where M is a separable metric space.
- Given $\mathbb{F}$-stopping times τ_1, τ_2 with $\tau_1 \leq \tau_2$ $\mathbb{P}$-a.s., define $\mathcal{T}^t_{\tau_1,\tau_2}$:=$\{\tau \in \mathcal{T}^t$ valued in $[\tau_1, \tau_2]$ $\mathbb{P}$-a.s.$\}$.

Concatenation

Given $\omega, \omega' \in \Omega$ and $\theta \in \mathcal{T}$, we define the concatenation of ω and ω' at time θ as

$$(\omega \otimes_\theta \omega')_s := \omega_r 1_{[0,\theta(\omega)]}(s) + (\omega'_s - \omega'_{\theta(\omega)} + \omega_{\theta(\omega)}) 1_{(\theta(\omega),T]}(s), \ s \in [0, T].$$

For each $\alpha \in \mathcal{A}$ and $\tau \in \mathcal{T}_{\theta,T}$, we define the shifted versions:

$$\begin{aligned} \alpha^{\theta,\omega}(\omega') &:= \alpha(\omega \otimes_\theta \omega') \\ \tau^{\theta,\omega}(\omega') &:= \tau(\omega \otimes_\theta \omega'). \end{aligned}$$

Properties of $\tau \in \mathcal{T}^t$ and $\alpha \in \mathcal{A}_t$

Property 0

Fix $t \in [0, T]$. If a r.v. ξ is $\mathcal{F}^t_T$-measurable, then ξ is indep. of $\mathcal{F}_t$.

Property 1

Fix $\theta \in \mathcal{T}$. For any $\tau \in \mathcal{T}_{\theta,T}$ and $\alpha \in \mathcal{A}$, $\tau^{\theta,\omega} \in \mathcal{T}^{\theta(\omega)}_{\theta(\omega),T}$ and $\alpha^{\theta,\omega} \in \mathcal{A}_{\theta(\omega)}$, $\mathbb{P}$-a.s.

Property 2

Given $\tau \in \mathcal{T}^t$ and $\alpha \in \mathcal{A}_t$,

- $\tau^{t,\omega} = \tau$, $\mathbb{P}$-a.s.
- For $(s, x) \in [0, T] \times \mathbb{R}^d$, $X^{s,x,\alpha^{t,\omega}}_r(\omega') = X^{s,x,\alpha}_r(\omega')$ for all $r \in [s, T]$, $\mathbb{P}$-a.s.

Assumptions on b and σ

Given $\tau \in \mathcal{T}$, $\xi \in \mathcal{L}_d^p$ which is $\mathcal{F}_\tau$-measurable, and $\alpha \in \mathcal{A}$, let $X^{\tau,\xi,\alpha}$ denote a $\mathbb{R}^d$-valued process satisfying the SDE:

$$dX_t^{\tau,\xi,\alpha} = b(t, X_t^{\tau,\xi,\alpha}, \alpha_t)dt + \sigma(t, X_t^{\tau,\xi,\alpha}, \alpha_t)dW_t, \tag{1}$$

with the initial condition $X_\tau^{\tau,\xi,\alpha} = \xi$ a.s.

Assume: $b(t, x, u)$ and $\sigma(t, x, u)$ are deterministic Borel functions, and continuous in (x, u); moreover, $\exists\ K > 0$ s.t. for $t \in [0, T]$, $x, y \in \mathbb{R}^d$, and $u \in M$

$$\begin{aligned} &|b(t,x,u) - b(t,y,u)| + |\sigma(t,x,u) - \sigma(t,y,u)| \le K|x-y|, \\ &|b(t,x,u)| + |\sigma(t,x,u)| \le K(1+|x|), \end{aligned} \tag{2}$$

This implies for any $(t, x) \in [0, T] \times \mathbb{R}^d$ and $\alpha \in \mathcal{A}$, (1) admits a unique strong solution $X_\cdot^{t,x,\alpha}$.

Assumptions on f, g, and c

f and g are rewards, c is the discount rate $\Rightarrow$ assume $f, g, c \geq 0$.

In addition, Assume:

- $f : [0, T] \times \mathbb{R}^d \times M \mapsto \mathbb{R}$ is Borel measurable, and $f(t, x, u)$ continuous in (x, u), and continuous in x uniformly in $u \in M$.
- $g : \mathbb{R}^d \mapsto \mathbb{R}$ is continuous,
- $c : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}$ is continuous and bounded above by some real number $\bar{c} > 0$.
- f and g satisfy a polynomial growth condition

$$|f(t, x, u)| + |g(x)| \leq K(1 + |x|^{\bar{p}}) \text{ for some } \bar{p} \geq 1. \qquad (3)$$

Reduction to the Mayer form

- Set $F(x,y,z) := z + yg(x)$. Observe that

$$\begin{aligned} V(t,x) &= \sup_{\alpha\in\mathcal{A}_t} \inf_{\tau\in\mathcal{T}^t_{t,T}} \mathbb{E}\left[Z^{t,x,1,0,\alpha}_\tau + Y^{t,x,1,\alpha}_\tau g(X^{t,x,\alpha}_\tau)\right] \\ &= \sup_{\alpha\in\mathcal{A}_t} \inf_{\tau\in\mathcal{T}^t_{t,T}} \mathbb{E}\left[F(\mathbf{X}^{t,x,1,0,\alpha}_\tau)\right], \end{aligned} \tag{4}$$

  where $\mathbf{X}^{t,x,y,z,\alpha}_\tau := (X^{t,x,\alpha}_\tau, Y^{t,x,y,\alpha}_\tau, Z^{t,x,y,z,\alpha}_\tau)$.
- More generally, for any $(x,y,z)\in\mathcal{S} := \mathbb{R}^d\times\mathbb{R}^2_+$, define

$$\bar{V}(t,x,y,z) := \sup_{\alpha\in\mathcal{A}_t} \inf_{\tau\in\mathcal{T}^t_{t,T}} \mathbb{E}\left[F(\mathbf{X}^{t,x,y,z,\alpha}_\tau)\right].$$

  Let $J(t,\mathbf{x};\alpha,\tau) := \mathbb{E}[F(\mathbf{X}^{t,\mathbf{x},\alpha}_\tau)]$. We can write V as

$$V(t,x) = \sup_{\alpha\in\mathcal{A}_t} \inf_{\tau\in\mathcal{T}^t_{t,T}} J(t,(x,1,0);\alpha,\tau).$$

Conditional expectation

Lemma 2.4

Fix $(t, \mathbf{x}) \in [0, T] \times \mathcal{S}$ and $\alpha \in \mathcal{A}$. For any $\theta \in \mathcal{T}_{t,T}$ and $\tau \in \mathcal{T}_{\theta,T}$,

$$\mathbb{E}[F(\mathbf{X}_{\tau}^{t,\mathbf{x},\alpha}) \mid \mathcal{F}_{\theta}](\omega) = J\left(\theta(\omega), \mathbf{X}_{\theta}^{t,\mathbf{x},\alpha}(\omega); \alpha^{\theta,\omega}, \tau^{\theta,\omega}\right) \ \mathbb{P}\text{-a.s.}$$

$$\left(= \mathbb{E}\left[F\left(\mathbf{X}_{\tau^{\theta,\omega}}^{\theta(\omega),\mathbf{X}_{\theta}^{t,\mathbf{x},\alpha}(\omega),\alpha^{\theta,\omega}}\right)\right]\right)$$

Outline

For $(t, x, p, A) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{M}^d$, define

$$H^a(t, x, p, A) := -b(t, x, a) - \frac{1}{2} Tr[\sigma\sigma'(t, x, a)A] - f(t, x, a),$$

and set

$$H(t, x, p, A) := \inf_{a \in M} H^a(t, x, p, A).$$

Supersolution Property of V_*

Proposition 3.2

The function V_* is a viscosity supersolution on $[0, T) \times \mathbb{R}^d$ to the obstacle problem of an HJB equation

$$\max\left\{c(t,x)w - \frac{\partial w}{\partial t} + H(t,x,D_x w, D_x^2 w),\ \ w - g(x)\right\} \geq 0. \tag{5}$$

Proof: Let $h \in C^{1,2}([0, T) \times \mathbb{R}^d)$ and $(t_0, x_0) \in [0, T) \times \mathbb{R}^d$ be s.t. $0 = (V_* - h)(t_0, x_0) < (V_* - h)(t, x)$, $\forall\, (t,x) \in [0, T) \times \mathbb{R}^d \setminus (t_0, x_0)$. If $V(t_0, x_0) = g(x_0)$, trivial. If $V(t_0, x_0) < g(x_0)$, want to show

$$0 \leq c(t_0,x_0)h(t_0,x_0) - \frac{\partial h}{\partial t}(t_0,x_0) + H(\cdot, D_x h, D_x^2 h)(t_0,x_0). \tag{6}$$

Proof (conti.)

- Assume the contrary. Then $\exists\ \zeta_0 \in M$ such that

$$0 > c(t_0,x_0)h(t_0,x_0) - \frac{\partial h}{\partial t}(t_0,x_0) + H^{\zeta_0}(\cdot, D_x h, D_x^2 h)(t_0,x_0).$$

- Define $\tilde{h}(t,x) := h(t,x) - |t-t_0|^2 - |x-x_0|^4$. Note that $(\tilde{h}, \partial_t \tilde{h}, D_x \tilde{h}, D_x^2 \tilde{h})(t_0,x_0) = (h, \partial_t h, D_x h, D_x^2 h)(t_0,x_0)$. Thus, can choose $r > 0$ with $t_0 + r < T$ s.t.

$$0 > c(t,x)\tilde{h}(t,x) - \frac{\partial \tilde{h}}{\partial t}(t,x) + H^{\zeta_0}(\cdot, D_x \tilde{h}, D_x^2 \tilde{h})(t_0,x_0), \quad (7)$$

for all $(t,x) \in \overline{B_r(t_0,x_0)}$.

Proof (conti.)

- Define $\zeta \in \mathcal{A}$ by $\zeta_t = \zeta_0$ for $t \geq 0$. Take (t_n, x_n) in $B_r(t_0, x_0)$ s.t. $(t_n, x_n, V(t_n, x_n)) \to (t_0, x_0, V_*(t_0, x_0))$. For $n \in \mathbb{N}$, set

$$\theta_n := \inf \left\{ s \geq t_n \,\middle|\, (s, X_s^{t_n, x_n, \zeta}) \notin B_r(t_0, x_0) \right\} \in \mathcal{T}_{t_n, T}^{t_n}.$$

- By applying the product rule to $Y_s^{t_n, x_n, 1, \zeta} \tilde{h}(s, X_s^{t_n, x_n, \zeta})$,

$$\tilde{h}(t_n, x_n) = \mathbb{E}\Bigg[Y_{\theta_n}^{t_n, x_n, 1, \zeta} \tilde{h}(\theta_n, X_{\theta_n}^{t_n, x_n, \zeta})$$

$$+ \int_{t_n}^{\theta_n} Y_s^{t_n, x_n, 1, \zeta} \left(c\tilde{h} - \frac{\partial \tilde{h}}{\partial t} + H^{\zeta_0}(\cdot, D_x \tilde{h}, D_x^2 \tilde{h}) + f \right)(s, X_s^{t_n, x_n}, \zeta_0) ds \Bigg]$$

$$< \mathbb{E}\left[Y_{\theta_n}^{t_n, x_n, 1, \zeta} h(\theta_n, X_{\theta_n}^{t_n, x_n, \zeta}) + \int_{t_n}^{\theta_n} Y_s^{t_n, x_n, 1, \zeta} f(s, X_s^{t_n, x_n, \zeta}, \zeta_0) ds \right]$$

$$- \eta.$$

Proof (conti.)

- More generally, it holds for all $\tau \in \mathcal{T}_{t_n,T}$ that

$$\tilde{h}(t_n, x_n) = \mathbb{E}\Bigg[Y^{t_n,x_n,1,\zeta}_{\theta_n\wedge\tau} \tilde{h}(\theta_n \wedge \tau, X^{t_n,x_n,\zeta}_{\theta_n\wedge\tau})$$

$$+ \int_{t_n}^{\theta_n\wedge\tau} Y^{t_n,x_n,1,\zeta}_s \left(c\tilde{h} - \frac{\partial \tilde{h}}{\partial t} + H^{\zeta_0}(\cdot, D_x\tilde{h}, D_x^2\tilde{h}) + f \right)(s, X^{t_n,x_n}_s, \zeta_0) ds \Bigg]$$

$$< \mathbb{E}\left[Y^{t_n,x_n,1,\zeta}_{\theta_n\wedge\tau} h(\theta_n \wedge \tau, X^{t_n,x_n,\zeta}_{\theta_n\wedge\tau}) + \int_{t_n}^{\theta_n\wedge\tau} Y^{t_n,x_n,1,\zeta}_s f(s, X^{t_n,x_n,\zeta}_s, \zeta_0) ds \right]$$

$$- \eta,$$

Proof (conti.)

- Since $(\tilde{h} - V)(t_n, x_n) \to 0$, can find $\hat{n} \in \mathbb{N}$ large enough s.t. for all $\tau \in \mathcal{T}_{t_{\hat{n}}, T}$

$$V(t_{\hat{n}}, x_{\hat{n}}) < \mathbb{E}\Bigg[Y^{t_{\hat{n}}, x_{\hat{n}}, 1, \zeta}_{\theta_{\hat{n}} \wedge \tau} h(\theta_{\hat{n}} \wedge \tau, X^{t_{\hat{n}}, x_{\hat{n}}, \zeta}_{\theta_{\hat{n}} \wedge \tau}) + \int_{t_{\hat{n}}}^{\theta_{\hat{n}} \wedge \tau} Y^{t_{\hat{n}}, x_{\hat{n}}, 1, \zeta}_s f(s, X^{t_{\hat{n}}, x_{\hat{n}}, \zeta}_s, \zeta_0) ds \Bigg] - \frac{\eta}{2} \qquad (8)$$

$$= \mathbb{E}\Bigg[Y^{t_{\hat{n}}, x_{\hat{n}}, 1, \zeta}_{\theta_{\hat{n}} \wedge \tau} h(\theta_{\hat{n}} \wedge \tau, X^{t_{\hat{n}}, x_{\hat{n}}, \zeta}_{\theta_{\hat{n}} \wedge \tau}) + Z^{t_{\hat{n}}, x_{\hat{n}}, 1, 0, \zeta}_{\theta_{\hat{n}} \wedge \tau} \Bigg] - \frac{\eta}{2}.$$

How to get a contradiction to this??

Proof (conti.)

By definition of V,

$$\begin{aligned} V(t_{\hat{n}}, x_{\hat{n}}) &\geq \inf_{\tau \in \mathcal{T}_{t_{\hat{n}},T}^{t_{\hat{n}}}} \mathbb{E}[F(\mathbf{X}_{\tau}^{\hat{n},(x_{\hat{n}},1,0),\alpha^*})] \\ &\geq \mathbb{E}[F(\mathbf{X}_{\hat{\tau}}^{\hat{n},(x_{\hat{n}},1,0),\alpha^*})] - \eta/4, \text{ for some } \hat{\tau} \in \mathcal{T}_{t_{\hat{n}},T}^{t_{\hat{n}}} \\ &\geq \mathbb{E}\left[Y_{\theta_{\hat{n}} \wedge \hat{\tau}}^{t_{\hat{n}},x_{\hat{n}},1,\zeta} h(\theta_{\hat{n}} \wedge \tau, X_{\theta_{\hat{n}} \wedge \hat{\tau}}^{t_{\hat{n}},x_{\hat{n}},\zeta}) + Z_{\theta_{\hat{n}} \wedge \hat{\tau}}^{t_{\hat{n}},x_{\hat{n}},1,0,\zeta} \right] - \frac{\eta}{4} - \frac{\eta}{4}. \end{aligned} \tag{9}$$

The blue part is the weak DPP we want to prove!

Weak DPP I

Proposition 3.1

Fix $(t, \mathbf{x}) \in [0, T] \times \mathcal{S}$ and $\varepsilon > 0$. Take arbitrary $\alpha \in \mathcal{A}_t$, $\theta \in \mathcal{T}^t_{t,T}$ and $\varphi \in USC([0, T] \times \mathbb{R}^d)$ with $\varphi \leq V$. We have the following:

(I) $\mathbb{E}[\bar{\varphi}^+(\theta, \mathbf{X}^{t,\mathbf{x},\alpha}_\theta)] < \infty$;

(II) If, moreover, $\mathbb{E}[\bar{\varphi}^-(\theta, \mathbf{X}^{t,\mathbf{x},\alpha}_\theta)] < \infty$, then there exists $\alpha^* \in \mathcal{A}_t$ with $\alpha^*_s = \alpha_s$ for $s \in [t, \theta]$ such that

$$\mathbb{E}[F(\mathbf{X}^{t,\mathbf{x},\alpha^*}_\tau)] \geq \mathbb{E}[Y^{t,x,y,\alpha}_{\tau\wedge\theta}\varphi(\tau \wedge \theta, X^{t,x,\alpha}_{\tau\wedge\theta}) + Z^{t,x,y,z,\alpha}_{\tau\wedge\theta}] - 4\varepsilon, \quad (10)$$

for any $\tau \in \mathcal{T}^t_{t,T}$.

In Bouchard & Touzi [2011],

- Problem: $\sup_{\alpha \in \mathcal{A}_t} J(t, x; \alpha)$.
- show weak DPP by using **LSC of** $J(t, x; \alpha)$ **in** (t, x).

In our case,

- Problem: $\sup_{\alpha \in \mathcal{A}_t} \inf_{\tau \in \mathcal{T}_{t,T}^t} J(t, x; \alpha, \tau)$.
- show weak DPP by using **LSC of** $\inf_{\tau \in \mathcal{T}_{t,T}^t} J(t, x; \alpha, \tau)$ **in** (t, x)**??**

Continuity of an Optimal Stopping Problem

Lemma 3.1

Fix $t \in [0, T]$. Then for any $\alpha \in \mathcal{A}_t$, the function

$$G^\alpha(s, \mathbf{x}) := \inf_{\tau \in \mathcal{T}^s_{s,T}} J(s, \mathbf{x}; \alpha, \tau)$$

is continuous on $[0, t] \times \mathcal{S}$.

Idea: Express optimal stopping problem as a solution to RBSDE; then use continuity results for RBSDE.

Proof: For $s \in [0, t]$ and $\mathbf{x} = (x, y, z) \in \mathcal{S}$, define the function $\tilde{f}^{(s,x)} : \Omega \times [s, T] \times \mathbb{R} \mapsto \mathbb{R}$ by

$$\tilde{f}^{(s,x)}(r, \mathfrak{y}) := f(r, X^{s,x,\alpha}_r, \alpha_r) - c(r, X^{s,x,\alpha}_r)\mathfrak{y}.$$

Proof (conti.)

Moreover, set $\xi := g(X_T^{s,x,\alpha})$ and $S_r := g(X_r^{s,x,\alpha})$ for $r \in [s, T]$. Let $(\mathfrak{Y}_r^{s,x}, \mathfrak{Z}_r^{s,x}, \mathfrak{K}_r^{s,x}; s \le r \le T)$ be the unique solution to the RBSDE associated with the data $(\xi, \tilde{f}, S)$.

By El Karoui, Kapoudjian, Pardoux, Peng & Quenez [1997],

$$\mathfrak{Y}_r^{s,x} = \operatorname*{ess\,inf}_{\tau \in \mathcal{T}_{r,T}} \mathbb{E}\Bigg[\int_r^\tau e^{-\int_r^l c(u, X_u^{s,x,\alpha})du} f(l, X_l^{s,x,\alpha}, \alpha_l) dl + e^{-\int_r^\tau c(u, X_u^{s,x,\alpha})du} g(X_\tau^{s,x,\alpha}) \mid \mathcal{F}_r \Bigg], \quad r \in [s, T].$$

and $\mathfrak{Y}_s^{s,x}$ is continuous on $[0, t] \times \mathbb{R}^d$.

Observe: $G^\alpha(s, (x, 1, 0)) = \mathfrak{Y}_s^{s,x}$.

Proof of Weak DPP I

Recall that we want to prove the following weak DPP:

Weak DPP

Fix $\varepsilon > 0$. For any $(t, \mathbf{x}) \in [0, T] \times \mathcal{S}$, $\alpha \in \mathcal{A}_t$, $\theta \in \mathcal{T}^t_{t,T}$ and $\varphi \in USC([0, T] \times \mathbb{R}^d)$ with $\varphi \leq V$, there exists $\alpha^* \in \mathcal{A}_t$ with $\alpha^*_s = \alpha_s$ for $s \in [t, \theta]$ s.t. for any $\tau \in \mathcal{T}^t_{t,T}$,

$$\mathbb{E}[F(\mathbf{X}^{t,\mathbf{x},\alpha^*}_\tau)] \geq \mathbb{E}[Y^{t,x,y,\alpha}_{\tau\wedge\theta}\varphi(\tau \wedge \theta, X^{t,x,\alpha}_{\tau\wedge\theta}) + Z^{t,x,y,z,\alpha}_{\tau\wedge\theta}] - 4\varepsilon.$$

Classical method for DPP: using measurable selection arguments; but very involved, not flexible...

We prove **“Weak” DPP**, which requires **NO** measurable selection arguments.

Proof of Weak DPP I (conti.)

Step 1: **Separate $[0,T]\times\mathcal{S}$ into small pieces.** By Lindelöf covering thm, take $\{(t_i,x_i)\}_{i\in\mathbb{N}}$ s.t. $\bigcup_{i\in\mathbb{N}} B(t_i,x_i;r^{(t_i,x_i)}) = (0,T]\times\mathcal{S}$,

$$\text{with } B(t_i,x_i;r^{(t_i,x_i)}) := (t_i - r^{(t_i,x_i)}, t_i] \times B_{r^{(t_i,x_i)}}(x_i).$$

Take a disjoint subcovering $\{A_i\}_{i\in\mathbb{N}}$ s.t. $(t_i,x_i)\in A_i$.

Step 2: **Construct desired control α^{t_i,x_i} in each A_i.** For each (t_i,x_i), by def. of $\bar{V}$, $\exists\ \alpha^{(t_i,x_i)}\in\mathcal{A}_{t_i}$ s.t.

$$\inf_{\tau\in\mathcal{T}^{t_i}_{t_i,T}} J(t_i,x_i;\alpha^{(t_i,x_i)},\tau) \geq \bar{V}(t_i,x_i)-\varepsilon. \tag{11}$$

Set $\bar{\varphi}(t,x,y,z) := y\varphi(t,x)+z$. For any $(t',x')\in A_i$,

$$G^{\alpha^{(t_i,x_i)}}(t',x') \underset{\text{lsc}}{\geq} G^{\alpha^{(t_i,x_i)}}(t_i,x_i)-\varepsilon \geq \bar{V}(t_i,x_i)-2\varepsilon$$

$$\geq \bar{\varphi}(t_i,x_i)-2\varepsilon \underset{\text{usc}}{\geq} \bar{\varphi}(t',x')-3\varepsilon.$$

Proof of Weak DPP I (conti.)

Step 3: Construct desired control α on the whole space $[0, T] \times \mathcal{S}$. For any $n \in \mathbb{N}$, set $B^n := \cup_{1 \le i \le n} A_i$ and define

$$\alpha_s^n := 1_{[t,\theta]}(s)\alpha_s + 1_{(\theta,T]}(s)\left(\alpha_s 1_{(B^n)^c}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha}) + \sum_{i=1}^n 1_{A_i}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})\alpha_s^{(t_i,x_i)}\right) \in \mathcal{A}_t.$$

Step 4: Estimations.

$$\mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha^n})] = \mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha})1_{\{\tau<\theta\}}] + \mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha^n})1_{\{\tau\ge\theta\}}]$$
$$\ge \mathbb{E}[1_{\{\tau<\theta\}}\bar{\varphi}(\tau, \mathbf{X}_\tau^{t,\mathbf{x},\alpha})] + \mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha^n})1_{\{\tau\ge\theta\}}],$$

because $g \ge V \ge \varphi$ implies

$$F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha}) = Y_\tau^{t,x,y,\alpha} g(X_\tau^{t,x,\alpha}) + Z_\tau^{t,x,y,z,\alpha}$$
$$\ge Y_\tau^{t,x,y,\alpha}\varphi(\tau, X_\tau^{t,x,\alpha}) + Z_\tau^{t,x,y,z,\alpha} = \bar{\varphi}(\tau, \mathbf{X}_\tau^{t,\mathbf{x},\alpha}).$$

PROOF OF WEAK DPP I (CONTI.)

By Lemma 2.4 and Properties 1 & 2,

$$\begin{aligned}
&\mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha^n})1_{\{\tau\geq\theta\}}|\mathcal{F}_\theta]1_{B^n}(\theta,\mathbf{X}_\theta^{t,\mathbf{x},\alpha}) \\
&= 1_{\{\tau\geq\theta\}}\sum_{i=0}^{n} J(\theta,\mathbf{X}_\theta^{t,\mathbf{x},\alpha};\alpha^{(t_i,x_i)},\tau^{\theta,\omega})1_{A_i}(\theta,\mathbf{X}_\theta^{t,\mathbf{x},\alpha}) \\
&\geq 1_{\{\tau\geq\theta\}}\sum_{i=0}^{n} G^{\alpha^{(t_i,x_i)}}(\theta,\mathbf{X}_\theta^{t,\mathbf{x},\alpha})1_{A_i}(\theta,\mathbf{X}_\theta^{t,\mathbf{x},\alpha}) \\
&\geq 1_{\{\tau\geq\theta\}}[\bar{\varphi}(\theta,\mathbf{X}_\theta^{t,\mathbf{x},\alpha})-3\varepsilon]1_{B^n}(\theta,\mathbf{X}_\theta^{t,\mathbf{x},\alpha}).
\end{aligned} \tag{12}$$

Thus,

$$\begin{aligned}
\mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha^n})1_{\{\tau\geq\theta\}}] &\geq \mathbb{E}\left[\mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha^n})1_{\{\tau\geq\theta\}}|\mathcal{F}_\theta]1_{B^n}(\theta,\mathbf{X}_\theta^{t,\mathbf{x},\alpha})\right] \\
&\geq \mathbb{E}[1_{\{\tau\geq\theta\}}\bar{\varphi}(\theta,\mathbf{X}_\theta^{t,\mathbf{x},\alpha})1_{B^n}(\theta,\mathbf{X}_\theta^{t,\mathbf{x},\alpha})]-3\varepsilon.
\end{aligned}$$

Proof of Weak DPP I (conti.)

Step 5: Conclusion.

$$\mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha^n})] = \mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha})1_{\{\tau<\theta\}}] + \mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha})1_{\{\tau\geq\theta\}}]$$

$$\geq \mathbb{E}[1_{\{\tau<\theta\}}\bar{\varphi}(\tau,\mathbf{X}_\tau^{t,\mathbf{x},\alpha})] + \mathbb{E}[1_{\{\tau\geq\theta\}}\bar{\varphi}(\theta,\mathbf{X}_\theta^{t,\mathbf{x},\alpha})1_{B^n}(\theta,\mathbf{X}_\theta^{t,\mathbf{x},\alpha})] - 3\varepsilon.$$

Take $n^* \in \mathbb{N}$ large enough s.t.

$$\begin{aligned}\mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha^{n^*}})] &\geq \mathbb{E}[1_{\{\tau<\theta\}}\bar{\varphi}(\tau,\mathbf{X}_\tau^{t,\mathbf{x},\alpha})] + \mathbb{E}[1_{\{\tau\geq\theta\}}\bar{\varphi}(\theta,\mathbf{X}_\theta^{t,\mathbf{x},\alpha})] - 4\varepsilon \\ &= \mathbb{E}[\bar{\varphi}(\tau\wedge\theta,\mathbf{X}_{\tau\wedge\theta}^{t,\mathbf{x},\alpha})] - 4\varepsilon \\ &= \mathbb{E}[Y_{\tau\wedge\theta}^{t,x,y,\alpha}\varphi(\tau\wedge\theta,X_{\tau\wedge\theta}^{t,x,\alpha}) + Z_{\tau\wedge\theta}^{t,x,y,z,\alpha}] - 4\varepsilon,\end{aligned}$$

for any $\tau \in \mathcal{T}_{t,T}^t$.

Done with the proof of weak DPP I!
Done with the proof of supersolution property of V_*!!

Outline

Subsolution Property of U^*

Proposition 4.2

The function U^* is a viscosity subsolution on $[0, T) \times \mathbb{R}^d$ to the obstacle problem of an HJB equation

$$\max\left\{c(t,x)w - \frac{\partial w}{\partial t} + H_*(t,x,D_x w, D_x^2 w), w - g(x)\right\} \leq 0.$$

Proof: Assume the contrary, i.e. $\exists\ h \in C^{1,2}([0, T) \times \mathbb{R}^d)$ and $(t_0, x_0) \in [0, T) \times \mathbb{R}^d$ s.t.

$$0 = (U^* - h)(t_0, x_0) > (U^* - h)(t, x),\ \forall\,(t,x) \in [0, T) \times \mathbb{R}^d \backslash (t_0, x_0),$$

and

$$\max\left\{c(t_0,x_0)h - \frac{\partial h}{\partial t} + H_*(t_0,x_0,D_x h, D_x^2 h), h - g(x_0)\right\}(t_0,x_0) > 0. \quad (13)$$

Proof (conti.)

Since by definition $U \leq g$, the USC of g implies $h(t_0, x_0) = U^*(t_0, x_0) \leq g(x_0)$. Then, we see from (13) that

$$c(t_0, x_0)h(t_0, x_0) - \frac{\partial h}{\partial t}(t_0, x_0) + H_*(\cdot, D_x h, D_x^2 h)(t_0, x_0) > 0.$$

Define the function $\tilde{h}(t, x) := h(t, x) + |t - t_0|^2 + |x - x_0|^4$. Note that $(\tilde{h}, \partial_t \tilde{h}, D_x \tilde{h}, D_x^2 \tilde{h})(t_0, x_0) = (h, \partial_t h, D_x h, D_x^2 h)(t_0, x_0)$. Then, by LSC of H_*, $\exists\ r > 0$ with $t_0 + r < T$ s.t.

$$c(t, x)\tilde{h}(t, x) - \frac{\partial \tilde{h}}{\partial t}(t, x) + H^a(\cdot, D_x \tilde{h}, D_x^2 \tilde{h})(t, x) > 0, \qquad (14)$$

for all $a \in M$ and $(t, x) \in \overline{B_r(t_0, x_0)}$.

Proof (conti.)

Define $\eta > 0$ by $\eta e^{\bar{c}T} := \min_{\partial B_r(t_0,x_0)}(\tilde{h} - h) > 0$.
Take $(\hat{t},\hat{x}) \in B_r(t_0,x_0)$ s.t. $|(U - \tilde{h})(\hat{t},\hat{x})| < \eta/2$. For $\alpha \in \mathcal{A}_{\hat{t}}$, set

$$\theta^\alpha := \inf\left\{ s \geq \hat{t} \;\middle|\; (s, X_s^{\hat{t},\hat{x},\alpha}) \notin B_r(t_0,x_0) \right\} \in \mathcal{T}_{\hat{t},T}^{\hat{t}}.$$

Applying the product rule to $Y_s^{\hat{t},\hat{x},1,\alpha}\tilde{h}(s, X_s^{\hat{t},\hat{x},\alpha})$, we get

$$\begin{aligned}
\tilde{h}(\hat{t},\hat{x}) &= \mathbb{E}\Bigg[Y_{\theta^\alpha}^{\hat{t},\hat{x},1,\alpha}\tilde{h}(\theta^\alpha, X_{\theta^\alpha}^{\hat{t},\hat{x},\alpha}) \\
&\quad + \int_{\hat{t}}^{\theta^\alpha} Y_s^{\hat{t},\hat{x},1,\alpha}\left(c\tilde{h} - \frac{\partial \tilde{h}}{\partial t} + H^\alpha(\cdot, D_x\tilde{h}, D_x^2\tilde{h}) + f \right)(s, X_s^{\hat{t},\hat{x},\alpha})ds \Bigg] \\
&> \mathbb{E}\left[Y_{\theta^\alpha}^{\hat{t},\hat{x},1,\alpha} h(\theta^\alpha, X_{\theta^\alpha}^{\hat{t},\hat{x},\alpha}) + \int_{\hat{t}}^{\theta^\alpha} Y_s^{\hat{t},\hat{x},1,\alpha} f(s, X_s^{\hat{t},\hat{x},\alpha}, \alpha_s)ds \right] + \eta
\end{aligned}$$

Proof (conti.)

By our choice of $(\hat{t},\hat{x})$, $U(\hat{t},\hat{x})+\eta/2>\tilde{h}(\hat{t},\hat{x})$. Thus,

$$U(\hat{t},\hat{x})>\mathbb{E}\left[Y_{\theta^\alpha}^{\hat{t},\hat{x},1,\alpha}h(\theta^\alpha,X_{\theta^\alpha}^{\hat{t},\hat{x},\alpha})+\int_{\hat{t}}^{\theta^\alpha}Y_s^{\hat{t},\hat{x},1,\alpha}f(s,X_s^{\hat{t},\hat{x},\alpha},\alpha_s)ds\right]+\frac{\eta}{2},$$

for any $\alpha\in\mathcal{A}_{\hat{t}}$.

How to get a contradiction to this??

Proof (conti.)

By the definition of U,

$$
\begin{aligned}
U(\hat{t},\hat{x}) &\leq \sup_{\alpha\in\mathcal{A}_{\hat{t}}} \mathbb{E}\left[F\left(\mathbf{X}_{\tau^*}^{\hat{t},\hat{x},1,0,\alpha}\right)\right]\\
&\leq \mathbb{E}\left[F\left(\mathbf{X}_{\tau^*}^{\hat{t},\hat{x},1,0,\hat{\alpha}}\right)\right]+\frac{\eta}{4},\ \text{for some } \hat{\alpha}\in\mathcal{A}_{\hat{t}}.\\
&\leq \mathbb{E}\left[Y_{\theta^{\hat{\alpha}}}^{\hat{t},\hat{x},1,\hat{\alpha}} h(\theta, X_{\theta^{\hat{\alpha}}}^{\hat{t},\hat{x},\hat{\alpha}}) + Z_{\theta^{\hat{\alpha}}}^{\hat{t},\hat{x},1,0,\hat{\alpha}}\right]+\frac{\eta}{4}+\frac{\eta}{4},
\end{aligned}
$$

The blue part is the weak DPP we want to prove!

Weak DPP II

Proposition 4.1

Fix $(t, \mathbf{x}) \in [0, T] \times \mathcal{S}$ and $\varepsilon > 0$. For any $\alpha \in \mathcal{A}_t$, $\theta \in \mathcal{T}^t_{t,T}$, and $\varphi \in LSC([0, T] \times \mathbb{R}^d)$ with $\varphi \geq U$, there exists $\tau^*(\alpha, \theta) \in \mathcal{T}^t_{t,T}$ such that

$$\mathbb{E}[F(\mathbf{X}^{t,\mathbf{x},\alpha}_{\tau^*})] \leq \mathbb{E}[Y^{t,x,y,\alpha}_\theta \varphi(\theta, X^{t,x,\alpha}_\theta) + Z^{t,x,y,z,\alpha}_\theta] + 4\varepsilon.$$

Weak DPP I:

- is for $V(t,x) = \sup_{\alpha \in \mathcal{A}_t} \inf_{\tau \in \mathcal{T}^t_{t,T}} J(t,x;\alpha,\tau)$.
- the proof relies on **LSC of** $\inf_{\tau \in \mathcal{T}^t_{t,T}} J(t,x;\alpha,\tau)$ **in** (t,x).

Weak DPP II,

- is for $U(t,x) = \inf_{\tau \in \mathcal{T}^t_{t,T}} \sup_{\alpha \in \mathcal{A}_t} J(t,x;\alpha,\tau)$.
- the proof might rely on **USC of** $\sup_{\alpha \in \mathcal{A}_t} J(t,x;\alpha,\tau)$ **in** (t,x)**??**

Continuity of an Optimal Control Problem

Lemma 4.3

Fix $t \in [0, T]$. For any $\tau \in \mathcal{T}^t_{t,T}$, the function $L^\tau : [0, t] \times \mathcal{S}$ defined by

$$L^\tau(s, \mathbf{x}) := \sup_{\alpha \in \mathcal{A}_s} J(s, \mathbf{x}; \alpha, \tau)$$

is continuous.

Idea of Proof: Generalize the arguments in Krylov[1980] for control problems with fixed horizon to our case with random horizon.

Proof of Weak DPP II

Step 1: Separate $[0,T]\times\mathcal{S}$ into small pieces. By Lindelöf covering thm, take $\{(t_i,x_i)\}_{i\in\mathbb{N}}$ s.t. $\bigcup_{i\in\mathbb{N}} B(t_i,x_i;r^{(t_i,x_i)}) = (0,T]\times\mathcal{S}$,

$$\text{with } B(t_i,x_i;r^{(t_i,x_i)}) := (t_i - r^{(t_i,x_i)}, t_i] \times B_{r^{(t_i,x_i)}}(x_i).$$

Take a disjoint subcovering $\{A_i\}_{i\in\mathbb{N}}$ s.t. $(t_i,x_i)\in A_i$.

Step 2: Construct desired stopping time $\tau^{(t_i,x_i)}$ in each A_i. For each (t_i,x_i), by def. of $\bar{U}$, $\exists\ \tau^{(t_i,x_i)} \in \mathcal{T}^{t_i}_{t_i,T}$ s.t.

$$\sup_{\alpha\in\mathcal{A}_{t_i}} J(t_i,x_i;\alpha,\tau^{(t_i,x_i)}) \le \bar{U}(t_i,x_i)+\varepsilon. \tag{15}$$

Set $\bar{\varphi}(t,x,y,z) := y\varphi(t,x)+z$. For any $(t',x')\in A_i$,

$$L^{\tau^{(t_i,x_i)}}(t',x') \underset{\text{usc}}{\le} L^{\tau^{(t_i,x_i)}}(t_i,x_i)+\varepsilon \le \bar{U}(t_i,x_i)+2\varepsilon$$
$$\le \bar{\varphi}(t_i,x_i)+2\varepsilon \underset{\text{lsc}}{\le} \bar{\varphi}(t',x')+3\varepsilon.$$

Proof of the Weak DPP II (conti.)

Step 3: Construct desired stopping time τ on the whole space $[0, T] \times \mathcal{S}$. For any $n \in \mathbb{N}$, set $B^n := \cup_{1 \le i \le n} A_i$ and define

$$\tau^n := T 1_{(B^n)^c}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha}) + \sum_{i=1}^{n} \tau^{(t_i,x_i)} 1_{A_i}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha}) \in \mathcal{T}_{t,T}^{t}.$$

Step 4: Estimations.

$$\mathbb{E}[F(\mathbf{X}_{\tau^n}^{t,\mathbf{x},\alpha})] = \mathbb{E}\left[F(\mathbf{X}_{\tau^n}^{t,\mathbf{x},\alpha}) 1_{B^n}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})\right] + \mathbb{E}\left[F(\mathbf{X}_{\tau^n}^{t,\mathbf{x},\alpha}) 1_{(B^n)^c}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})\right]$$

Proof of Weak DPP II (conti.)

By Lemma 2.4 and Properties 1 & 2,

$$\begin{aligned}
&\mathbb{E}[F(\mathbf{X}^{t,\mathbf{x},\alpha}_{\tau^n}) \mid F_\theta](\omega)\, 1_{B^n}(\theta(\omega), \mathbf{X}^{t,\mathbf{x},\alpha}_{\theta}(\omega)) \\
&= \sum_{i=0}^{n} J\left(\theta(\omega), \mathbf{X}^{t,\mathbf{x},\alpha}_{\theta}(\omega); \alpha^{\theta,\omega}, \tau^{(t_i,x_i)}\right) 1_{A_i}(\theta(\omega), \mathbf{X}^{t,\mathbf{x},\alpha}_{\theta}(\omega)) \\
&\leq \sum_{i=0}^{n} L^{\tau^{(t_i,x_i)}}\left(\theta(\omega), \mathbf{X}^{t,\mathbf{x},\alpha}_{\theta}(\omega)\right) 1_{A_i}(\theta(\omega), \mathbf{X}^{t,\mathbf{x},\alpha}_{\theta}(\omega)) \\
&\leq \left[\bar{\varphi}\left(\theta(\omega), \mathbf{X}^{t,\mathbf{x},\alpha}_{\theta}(\omega)\right) + 3\varepsilon\right] 1_{B^n}(\theta(\omega), \mathbf{X}^{t,\mathbf{x},\alpha}_{\theta}(\omega)).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathbb{E}\left[F(\mathbf{X}^{t,\mathbf{x},\alpha}_{\tau^n}) 1_{B^n}(\theta, \mathbf{X}^{t,\mathbf{x},\alpha}_{\theta})\right] = \mathbb{E}\left[\mathbb{E}[F(\mathbf{X}^{t,\mathbf{x},\alpha}_{\tau^{\varepsilon,n}}) \mid \mathcal{F}_\theta] 1_{B^n}(\theta, \mathbf{X}^{t,\mathbf{x},\alpha}_{\theta})\right] \\
&\leq \mathbb{E}[\bar{\varphi}(\theta, \mathbf{X}^{t,\mathbf{x},\alpha}_{\theta}) 1_{B^n}(\theta, \mathbf{X}^{t,\mathbf{x},\alpha}_{\theta})] + 3\varepsilon \leq \mathbb{E}[\bar{\varphi}(\theta, \mathbf{X}^{t,\mathbf{x},\alpha}_{\theta})] + 3\varepsilon.
\end{aligned}$$

Proof of Weak DPP II (conti.)

Step 5: Conclusion.

$$\mathbb{E}[F(\mathbf{X}^{t,\mathbf{x},\alpha}_{\tau^n})] \leq \mathbb{E}[\bar{\varphi}(\theta, \mathbf{X}^{t,\mathbf{x},\alpha}_{\theta})] + 3\varepsilon + \mathbb{E}[F(\mathbf{X}^{t,\mathbf{x},\alpha}_{T}) 1_{(B^n)^c}(\theta, \mathbf{X}^{t,\mathbf{x},\alpha}_{\theta})].$$

Now, take $n^* \in \mathbb{N}$ large enough s.t.

$$\begin{aligned} \mathbb{E}[F(\mathbf{X}^{t,\mathbf{x},\alpha}_{\tau^{n^*}})] &\leq \mathbb{E}[\bar{\varphi}(\theta, \mathbf{X}^{t,\mathbf{x},\alpha}_{\theta})] + 4\varepsilon \\ &= \mathbb{E}[Y^{t,x,y,\alpha}_{\theta} \varphi(\theta, X^{t,x,\alpha}_{\theta}) + Z^{t,x,y,z,\alpha}_{\theta}] + 4\varepsilon. \end{aligned}$$

Done with the proof of Weak DPP II!
Done with the proof of the subsolution property of U^*!!

Outline

To state an appropriate comparison result, we assume

A. for any $t, s \in [0, T]$, $x, y \in \mathbb{R}^d$, and $u \in M$,

$$|b(t,x,u)-b(s,y,u)|+|\sigma(t,x,u)-\sigma(s,y,u)| \leq K(|t-s|+|x-y|).$$

B. $f(t,x,u)$ is uniformly continuous in (t,x), uniformly in $u \in M$.

The conditions **A** and **B**, together with the linear growth condition on b and σ, imply that the function H is continuous, and thus $H = H_*$.

Comparison Result

Proposition 5.1

Assume **A** and **B**. Let u (resp. v) be an USC viscosity subsolution (resp. a LSC viscosity supersolution) with polynomial growth condition to (5), such that $u(T,x) \leq v(T,x)$ for all $x \in \mathbb{R}^d$. Then $u \leq v$ on $[0,T) \times \mathbb{R}^d$.

Idea of Proof: Generalize the arguments in Pham[2009].

$U^*(T,\cdot) = V_*(T,\cdot)$

Lemma 5.1

For all $x \in \mathbb{R}^d$, $V_*(T,x) \geq g(x)$.

Proof: Fix $\alpha \in \mathcal{A}$. Take $(t_m, x_m) \to (T, x)$ with $t_m < T$. Then

$$
\begin{aligned}
& V(t_m, x_m) \\
& \geq \inf_{\tau \in \mathcal{T}^{t_m}_{t_m,T}} \mathbb{E}\left[\int_{t_m}^{\tau} Y^{t_m,x_m,1,\alpha} f(s, X^{t_m,x_m,\alpha}, \alpha_s) ds + Y^{t_m,x_m,1,\alpha}_{\tau} g(X^{t_m,x_m,\alpha}_{\tau})\right] \\
& \geq \mathbb{E}\left[\int_{t_m}^{\tau_m} Y^{t_m,x_m,1,\alpha} f(s, X^{t_m,x_m,\alpha}, \alpha_s) ds + Y^{t_m,x_m,1,\alpha}_{\tau_m} g(X^{t_m,x_m,\alpha}_{\tau_m})\right] - \frac{1}{m}.
\end{aligned}
$$

Note that $\tau_m \to T$, since $\tau_m \in \mathcal{T}^{t_m}_{t_m,T}$ and $t_m \to T$. Then by Fatou's lemma, $\liminf_{m\to\infty} V(t_m, x_m) \geq g(x)$. Since (t_m, x_m) is arbitrarily chosen, conclude $V_*(T,x) \geq g(x)$.

Main Result

Theorem 5.1

Assume **A** and **B**. Then $U^* = V_*$ on $[0, T] \times \mathbb{R}^d$. In particular, $U = V$ on $[0, T] \times \mathbb{R}^d$, i.e. the game has a value, which is the unique viscosity solution to (5) with terminal condition $w(T, x) = g(x)$ for $x \in \mathbb{R}^d$.

Summary

- A much more general setting: We consider a zero-sum controller-and-stopper game in which
  - the state process X^α is multi-dimensional;
  - both the drift and the volatility coefficients of X^α can be controlled;
  - the volatility coefficient of X^α can be degenerate.
- Main Result: the game has a value and the value function is the unique viscosity solution to an obstacle problem of an HJB equation.
- Future Work:
  - How to construct a saddle-point of optimal strategies?

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Thank you very much for your attention!
Q & A