An introduction to the tautological ring and double ramification cycles

Felix Janda

February 13, 2019

1 Tautological rings

Tautological classes are certain cohomology classes on the moduli space of curves. There are many surveys on this subject [32, 29, 28, 20].

1.1 Motivation

Tautological rings are part of the study of intersection theory of moduli spaces. We give two motivations: First, Schubert calculus is a succesful application of the study of the cohomology of moduli spaces to enumerative problems. Second, relations between tautological classes are of enormous importance for computations in Gromov–Witten theory.

1.1.1 Schubert calculus

Enumerative geometry is concerned with counting the number of solutions to geometric problems. A problem raised and solved by Schubert[26] is the following: How many lines pass through 4 generic lines $\ell_1, \ell_2, \ell_3, \ell_4$ in $\mathbb{P}^3$?

Schubert calculus gives a systematic way of solving similar problems. For this specific problem, consider the Grassmannian $\mathrm{Gr}(2, 4)$ of lines ℓ in $\mathbb{P}^3$, and the subsets

$$H_i = \{\ell | \ell \cap \ell_i \neq \emptyset\} \subset \mathrm{Gr}(2, 4)$$

for $i \in \{1, 2, 3, 4\}$. Then to solve the enumerative problem, we need to compute the number of elements in

$$H_1 \cap H_2 \cap H_3 \cap H_4 = \{\ell | \ell \cap \ell_i \neq \emptyset \forall i\}.$$

Assuming that the H_i intersect transversally, this is the intersection number

$$\deg([H_1] \cup [H_2] \cup [H_3] \cup [H_4]),$$

where $[H_i]$ is the cohomology class dual to H_i.

The following facts make the computation doable:

- $A^*(\mathrm{Gr}(2,4)) \cong H^*(\mathrm{Gr}(2,4)) \cong \mathbb{Q}[\sigma_1, \sigma_2]/(\sigma_1^3 - 2\sigma_1\sigma_2, \sigma_2^2 - \sigma_1^2\sigma_2)$[1]
- $\deg(\sigma_2^2) = 1$

Here, $\sigma_1 = [H_i]$, and σ_2 is the class dual to the locus of lines passing through a specific point.

Exercise 1. Show from the above description, that $\dim H^*(\mathrm{Gr}(2,4)) = 6$, and that $\deg(\sigma_1^4) = 2$.

Therefore, the number of lines passing through 4 given lines is 2.

The key points enabling this computation are:

- Have an explicit description of $H^*(\mathrm{Gr}(2,4))$.
- Can compute $[H_i] = \sigma_1$.
- Know how to compute $\deg(-)$.

1.1.2 Gromov–Witten Theory

Gromov–Witten theory may be regarded as the study of the "virtual" intersection theory of the moduli space of stable maps $\{(C, f\colon C \to X)\}$ to a smooth, projective variety X. Therefore, facts about the intersection theory of the moduli space of curves $\{C\}$ give rise to universal (for any target X) relations (such as the WDVV equation) between Gromov–Witten invariants. In case of some targets (for example $X = \mathbb{P}^2$), they suffice for the computation of all Gromov–Witten invariants.

1.2 Definition of the tautological ring

1.2.1 Moduli space of smooth curves

The Grassmannian $\mathrm{Gr}(2,4)$ is an example of a *moduli space*, in the sense that it parameterizes geometric configurations (here projective lines in $\mathbb{P}^3$). Furthermore, algebraic families of lines in $\mathbb{P}^3$ correspond to maps from the base of the family to $\mathrm{Gr}(2,4)$.

One famous example of a moduli space is the moduli space M_g of curves. For $g \geq 2$, the moduli space M_g parameterizes smooth, connected algebraic curves C of genus g up to isomorphism. Because of the existence of automorphisms, M_g is not a scheme, but an orbifold (or smooth, separated Deligne–Mumford stack of finite type), though the reader unfamiliar with orbifolds may pretend that it is a smooth variety. The dimension of M_g is $3g-3$, but in general it is difficult to describe M_g explicitly. On the other hand, we can say a lot about M_g from its description as a moduli space, in particular, there is a *universal (family of) curves* $\pi\colon C_g \to M_g$, which is defined such that the fiber of C_g over a point $[C] \in M_g$ is C itself.

We may also think of C_g as the moduli space $M_{g,1}$ of pairs (C, p), where $[C] \in M_g$ together with a point $p \in C$. More generally, for any non-negative

[1] By convention, we work with rational (Chow) cohomology.

integers g and n such that $2g-2+n>0$, we may define a moduli space $M_{g,n}$ of smooth algebraic curves C of genus g together with n pairwise distinct marked points $p_1,\dots,p_n \in C$ up to automorphisms compatible with the markings.

Exercise 2. Prove that $M_{0,3}$ is a point, and that $M_{0,4} \cong \mathbb{P}^1 \setminus \{0,1,\infty\}$.

The moduli space $M_{g,n}$ has dimension $3g-3+n$, and its own universal curve $\pi\colon C_{g,n} \to M_{g,n}$, which now comes with n sections $s_i\colon M_{g,n} \to C_{g,n}$ defined by $[C,p_1,\dots,p_n] \mapsto ([C,p_1,\dots,p_n],p_i)$. In addition, the $M_{g,n}$ are connected via the *forgetful maps* $\pi\colon M_{g,n+1} \to M_{g,n}$ defined by $[C,p_1,\dots,p_n,p_{n+1}] \to [C,p_1,\dots,p_n]$.

1.2.2 Tautological classes on $M_{g,n}$

We can use the universal curve to come up with interesting cycles on M_g. The cotangent spaces T_p^*C can be glued to a line bundle ω_π called *cotangent line bundle* on C_g. We may then define

$$\begin{aligned}\psi_i =& s_i^*(c_1(\omega_\pi)) \in A^1(M_{g,n}),\\ \widetilde{\kappa}_i =& \pi_*\left((c_1(\omega_\pi))^{i+1}\right), \qquad \kappa_i = \widetilde{\kappa}_i + \psi_1^i + \cdots + \psi_n^i \in A^i(M_{g,n}).\end{aligned}$$

Exercise 3. Prove that $\kappa_0 = 2g-2+n$. [Hint: observe that a differential on a genus g curve has $2g-2$ zeros (counted with multiplicities).]

Definition 1. A *tautological class* on $M_{g,n}$ is a polynomial in the κ_i and ψ_j.

While not all algebraic cycles on $M_{g,n}$ are tautological [30], most cycles that come up in practice are. For example, another interesting set of classes are the *Hodge classes* $\lambda_i \in A^i(M_{g,n})$. They are the Chern classes of the *Hodge bundle* $\mathbb{E}_g = \pi_*\omega_\pi$. More, concretely, the fiber of $\mathbb{E}_g$ over $[C] \in M_g$ is the space of holomorphic one forms on C. Therefore $\mathbb{E}_g$ is a vector bundle of rank g, so that $\lambda_0 = g$ and $\lambda_i = 0$ for $i > g$.

Theorem 1 (Mumford [19])**.** *The class λ_i is tautological for any i, g and n.*

Example 1. In $A^*(M_{g,n})$, we have

$$\lambda_1 = \frac{\widetilde{\kappa}_1}{12}, \qquad \lambda_2 = \frac{\widetilde{\kappa}_1^2}{288}, \qquad \lambda_3 = \frac{\widetilde{\kappa}_1^3}{10368} - \frac{\widetilde{\kappa}_3}{360},$$

Other interesting cycles are defined using loci of special curves. For example, given integers $a_1,\dots,a_n$ summing up to 0 but not all zero, we may define the *double ramification (DR) locus*

$$\mathrm{DRL}^o_{g,\mathbf{a}} = \{[C,p_1,\dots,p_n] : a_1[p_1] + \cdots + a_n[p_n] \sim 0\} \subset M_{g,n}.$$

Here $\sim$ stands for rational equivalence of divisors. Taking the class of a DR locus givs a *double ramification (DR) cycle*. As we will see later, DR cycles are tautological classes.

Exercise 4. Prove that on $M_{g,n}$, we have $\pi^*\psi_i = \psi_i$, and that $\pi^*\widetilde{\kappa}_i = \widetilde{\kappa}_i$.[2]

Exercise 5. Prove that there are no positive degree tautological classes on $M_{0,n}$. [Hint: Use the previous exercise]

[2] These relations are *only* true on $M_{g,n}$, but not the Deligne–Mumford compactification.

1.2.3 Deligne–Mumford compactification

The moduli of smooth curves is not compact, which is inconvenient for many reasons, for example it has no intersection numbers. As Angelo Vistoli once said "working with a non-complete moduli space is like keeping change in a pocket with holes". There are several compactifications of $M_{g,n}$, the most common is the *Deligne–Mumford compactification*[4] $\overline{M}_{g,n}$, which is both a compactification of $M_{g,n}$, and also a moduli space of its own.

Definition 2. An *n-marked stable curve* is a connected, but not necessarily irreducible algebraic curve C with at most nodal singularities[3] together with n disjoint markings $p_1, \ldots, p_n \in C$ away from the nodes such that C has only finitely many automorphisms compatible with the markings.

More concretely, the automorphism condition is equivalent to saying that any component of (arithmetic) genus 0 of C has at least 3 *special points* (that is nodes or markings)[4], and that any component of arithmetic genus 1 has at least 1 special point. (Note that this is necessary because there are infinitely many automorphisms of $\mathbb{P}^1$, or in other words, Mobius transformations fixing less than three points in $\mathbb{P}^1$, and because any elliptic curve has an infinity translation symmetry.) The Deligne–Mumford moduli space $\overline{M}_{g,n}$ is the moduli space of all stable curves of arithmetic genus g with n marked points.

Exercise 6. Prove that every stable curve of genus g with n markings has at most $3g - 3 + n$ nodes.

Exercise 7. Prove that $\overline{M}_{0,3} = M_{0,3}$, and that $\overline{M}_{0,4} = \mathbb{P}^1$.

The moduli spaces $\overline{M}_{g,n}$ are connected via many *tautological maps*. One type of tautological maps, are forgetful maps $\pi\colon \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ analogous to the ones we have seen before, with one difference: when naively forgetting about a marking, it might happen that there is a genus zero component $Z \subset C$ with only two special points. In this case, we need to *stabilize* C by contracting Z to a point. The forgetful map $\pi\colon \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ may also be regarded as the universal curve over $\overline{M}_{g,n}$. (Why?)

Another type of tautological maps, are the *gluing maps*

$$\begin{aligned} \overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} &\to \overline{M}_{g_1+g_2,n_1+n_2}, \\ \overline{M}_{g-1,n+2} &\to \overline{M}_{g,n}, \end{aligned}$$

which glue curves along a pair of markings.

The moduli space $\overline{M}_{g,n}$ splits into several strata according to the *topological type* or *dual graph* of a stable curve. Given $[C, p_1, \ldots, p_n] \in \overline{M}_{g,n}$, its dual graph consists of the following data $\Gamma = (V, H, v, \iota, g)$:

[3] A nodal singularity looks (étale or analytically) locally like the coordinate cross $\{xy = 0\} \subset \mathbb{C}^2$.

[4] There is one subtlety here: We should count self-nodes with multiplicity two, because we should actually look at the normalization $\widetilde{C}$ of C instead of C itself. Here the normalization is explicitly given by separating the branches at each node.

- A set V of a vertex v_Z for each component $Z \subset C$.
- A set H of a half-edge $h_{Z,p}$ for each component $Z \subset C$ with a special point p
- A map $v\colon H \to V$ defined by $h_{Z,p} \mapsto v_Z$.
- An involution $\iota\colon H \to H$ whose fixed points correspond to the markings of C. The edge set E is defined as the set of two-element orbits of ι. The one-element orbits are called *legs* and in bijection with $\{1, \dots, n\}$.
- A map $g\colon V \to \mathbb{Z}_{\geq 0}$ assigning to v_Z the geometric genus of Z.

The data of Γ describes how C is glued together from its components (together with an ordering of the branches at each node of C). Because of stability, each vertex $v \in V$ satisfies $2g(v) - 2 + n(v) > 0$. Accordingly, each dual graph Γ defines a more general gluing map

$$\xi_\Gamma\colon \prod_{w \in V} \overline{M}_{g(w),n(w)} \to \overline{M}_{g,n}.$$

Here, $n(w) = |v^{-1}(w)|$ is the *valency* of w, and $g = \sum_{w \in V} g(w) + h^1(\Gamma)$ with $h^1(\Gamma)$ the number of loops in Γ. The simple gluing maps we have seen before correspond to the dual graphs with a single edge. The image of ξ_Γ is the closure of the locus of curves with dual graph Γ, and the degree of ξ_Γ is $|\operatorname{Aut}(\Gamma)|$, the number of automorphisms of Γ.

Exercise 8. Show that there are only finitely many dual graphs Γ for fixed g and n (satisfying the stability condition $2g(v) - 2 + n(v) > 0$ for all $v \in V$).

1.2.4 The tautological ring

The classes ψ_i, κ_j and λ_k can be defined on $\overline{M}_{g,n}$ in the same way as for $M_{g,n}$ as long as we define ω_π as the relative dualizing sheaf. In addition, we want to include classes of (closure of) strata, and compatibility of the gluing and forgetful maps. This leads to the following (surprisingly) compact definition (first stated in [5]).

Definition 3. The *tautological rings* $R^*(\overline{M}_{g,n})$ is the smallest system of $\mathbb{Q}$-subalgebra of $A^*(\overline{M}_{g,n})$ that is closed under push-forward via tautological (gluing or forgetful) maps. A *tautological class* is an element of the tautological ring.

By definition, $R^*(\overline{M}_{g,n})$ contains the unit, therefore by closure under push-forward via gluing maps, it also contains classes of strata. As we will see below, we can get psi classes when intersecting strata classes. Furthermore, there is the (non-trivial) identity[5]

$$\kappa_i = \pi_*(\psi_{n+1}^{i+1}),$$

[5] This is often used as a definition of κ_i. The different definitions are discussed in [1] and the references therein. If you want to think about how to prove this yourself, start with the case $n = 0$.

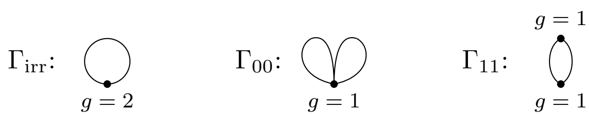

Figure 1: Some dual graphs of $\overline{M}_3$

and so tautological classes on $\overline{M}_{g,n}$ do also include kappa classes. The tautological rings also turn out to be closed under pull-back via tautological maps (see [7]).

What makes the tautological ring of $\overline{M}_{g,n}$ much more managable than the full Chow ring is that there is a concrete set of additive generators $R^*(\overline{M}_{g,n})$. Each generator is of the form

$$\xi_{\Gamma,*}\left(\prod_{v\in V} P_v\right),$$

where Γ is a dual graph, and each P_v is a monomial in all kappa and psi-classes (including those corresponding to edges of Γ) of $\overline{M}_{g(v),n(v)}$.

Exercise 9. The tautological ring $R^*(M_{g,n})$ is defined as the image under the restriction map $A^*(\overline{M}_{g,n}) \to A^*(M_{g,n})$. Show that $R^*(M_{g,n})$ consists of polynomials of psi and kappa-classes.

Multiplication of the generators is a bit subtle. Say, we want to multiply ξ_{Γ_A} and ξ_{Γ_B}. The result is a sum over (A,B)-graphs[6], or in other words, dual graphs Γ together with a coloring of each edge in at least one of the colors A and B such that contracting all edges not colored in A results in a graph isomorphic to Γ_A, and contracting all edges not colored in B results in Γ_B. Then, if E_{AB} denotes the set of edges of Γ colored in both A and B, we have

$$\frac{\xi_{\Gamma_A,*}(1)\cdot\xi_{\Gamma_B,*}(1)}{|\operatorname{Aut}(\Gamma_A)|\cdot|\operatorname{Aut}(\Gamma_B)|} = \sum_{\Gamma}\frac{1}{|\operatorname{Aut}(\Gamma)|}\xi_{\Gamma,*}\left(\prod_{\{h,h'\}\in E_{AB}}(-\psi_h-\psi_{h'})\right),$$

where ψ_h denotes the psi-class corresponding to an half-edge h. We note that sometimes a dual graph might be an (A,B)-graph in different ways, as in the following example.

Example 2. Let Γ_{irr} be the dual graph with a single vertex and a single edge, which is a loop, and let $\delta_{\mathrm{irr}} = \frac{1}{2}\xi_{\Gamma_{\mathrm{irr}},*}(1)$. Assume $g = 3$ and $n = 0$. Then

$$\delta_{\mathrm{irr}}^2 = \frac{1}{2}\xi_{\Gamma_{\mathrm{irr}},*}(-\psi_1-\psi_2) + \frac{2}{8}\xi_{\Gamma_{00},*}(1) + \frac{2}{4}\xi_{\Gamma_{11},*}(1).$$

The dual graphs Γ_{irr}, Γ_{00} and Γ_{11} are illustrated in Figure 1. To make Γ_{irr} an (A,B)-graph the single edge must be of color A and B. For Γ_{00} and Γ_{11} there are two possibilities to color the edges in A and B.

[6] We present the multiplication here in a slightly different form compared to the literature[7, 20].

Exercise 10. This exercise gives a flavor of the formula for the DR cycle (see below) and for its origin. Let z be a formal variable, and let δ_{irr} be as in the example. Show that

$$\exp(z\delta_{\mathrm{irr}}) = \sum_{\Gamma} \frac{1}{|\operatorname{Aut}(\Gamma)|}\xi_{\Gamma,*}\left(\prod_{(h,h')\in E} \frac{1-\exp(-z(\psi_h+\psi_{h'}))}{\psi_h+\psi_{h'}}\right),$$

where the sum is over all dual graphs Γ that do not become disconnected when cutting any single edge. [Hint: Prove that the z-derivatives of both sides are equal.]

Exercise 11. Let $\delta_{i,n+1}$ be the divisor class on $\overline{M}_{g,n+1}$ corresponding to curves with a genus zero component with a single node, and containing only the markings i and $n+1$. Show that

$$-\psi_i = \pi_*(\delta_{i,n+1}^2).$$

The identities from Exercise 4 do no longer hold on $\overline{M}_{g,n}$ because the cotangent lines of $\overline{M}_{g,n}$ and $\overline{M}_{g,n+1}$ do not agree on the locus where the component of C containing i is contracted under stabilization. Instead, we have

$$\psi_i = \pi^*\psi_i + \delta_{i,n+1}, \tag{1}$$

where $\delta_{i,n+1}$ is as in the previous exercise.

Exercise 12. Show $\psi_i^k = \pi^*\psi_i^k + \delta_{i,n+1}\pi^*\psi_i^{k-1}$ for $k \geq 1$.

1.3 Intersection numbers

Unlike $M_{g,n}$, the moduli space $\overline{M}_{g,n}$ is compact, and we thus can define intersection numbers such as

$$\int_{\overline{M}_{g,n}} \psi_1^{a_1}\cdots\psi_n^{a_n} \in \mathbb{Q}$$

for $a_1,\ldots,a_n$ summing up to $3g-3+n$.

Witten's conjecture[31] is a way of recursively computing all these intersection numbers. It says that their generating series is a solution to the KdV *integrable hierarchy.* More explicitly, this gives a set of recursive relations. We only discuss the two simplest examples here:

Exercise 13. Use (1) to prove the *string equation*

$$\int_{\overline{M}_{g,n+1}} \psi_1^{a_1}\cdots\psi_n^{a_n} = \int_{\overline{M}_{g,n}} \psi_1^{a_1-1}\psi_2^{a_2}\cdots\psi_n^{a_n} + \cdots + \int_{\overline{M}_{g,n}} \psi_1^{a_1}\cdots\psi_n^{a_n-1},$$

and the *dilaton equation*

$$\int_{\overline{M}_{g,n+1}} \psi_1^{a_1}\cdots\psi_n^{a_n}\psi_{n+1} = (2g-2+n)\int_{\overline{M}_{g,n}} \psi_1^{a_1}\cdots\psi_n^{a_n}.$$

In the string equation factors ψ_i^{-1} on the right hand side are defined to be zero.

The other equations explain how to remove higher powers of ψ_{n+1}, and boil down the computation of all intersection numbers to just

$$\int_{\overline{M}_{0,3}} 1 = 1.$$

Exercise 14. Use just the string and dilaton equations and

$$\int_{\overline{M}_{1,1}} \psi_1 = \frac{1}{24}.$$

to compute all the ψ-integrals for $g = 0$ and $g = 1$.

The first proof of Witten's conjecture was given by Kontsevich[17], but by now there exists a multitude of other proofs.

Another important property of tautological classes is that all integrals involving tautological classes may be explicitly computed using Witten's conjecture. First, integrals involving κ-classes may can be computed in terms of ψ-integrals with additional markings, for example

$$\int_{\overline{M}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} \kappa_j = \int_{\overline{M}_{g,n+1}} \psi_1^{a_1} \cdots \psi_n^{a_n} \psi_{n+1}^{j+1}$$

Exercise 15. Derive this equation.

Second, for integrals of general tautological classes, we have

$$\int_{\overline{M}_{g,n}} \xi_{\Gamma *} \left(\prod_{v \in V} P_v \right) = \prod_{v \in V} \int_{\overline{M}_{g(v),n(v)}} P_v,$$

reducing to a computation of integrals only involving ψ and κ-classes.

Comparing to the Grassmannian, we see that we can compute any intersection number involving tautological classes. Furthermore, the DR cycle will be an example of a geometrically defined class that can be computed explicitly in the tautological ring. What we do not have for the tautological ring is an explicit description in terms of generators and relations. We have seen explicit generators but describing the set of all relations (often called *tautological relations*) remains an active subject of research. On the other hand, in many (and conjecturally all) cases, the relations are Pixton's generalized Faber–Zagier relations[22, 21, 14], and the tautological ring can be explicitly computed. The program admcycles.sage[24] (based on code by Aaron Pixton) is a nice way to explore computations in the tautological ring.

2 Double ramification cycles

Double ramification (DR) cycles are interesting examples of tautological classes on $\overline{M}_{g,n}$, and in order to work with them, it is important to find explicit formulas for them. In Section 1.2.2, we already encountered the definition of their restriction to $M_{g,n}$.

We recall that the DR locus $\mathrm{DRL}^o_{g,\mathbf{a}} \subset M_{g,n}$ is the locus of marked curves $[C, p_1, \dots, p_n]$ such that the (degree zero) divisor $a_1[p_1] + \cdots + a_n[p_n]$ is linearly equivalent to zero. In other words, there exists a map $f\colon C \to \mathbb{P}^1$ (which we may also think of as a meromorphic function) such that

$$f^{-1}([0]) = \sum_{i|a_i>0} a_i[p_i], \qquad f^{-1}([0]) = \sum_{i|a_i<0} |a_i|[p_i].$$

This is the origin of the terminology "double ramification" — we look at maps to $\mathbb{P}^1$ with specified ramification over two points.

Returning to the original description of the double ramification locus, we notice that it is about imposing a condition in the Jacobian of C. Let $J_{g,n} \to M_{g,n}$ be the *universal Jacobian*[7] of $C_{g,n} \to M_{g,n}$. The family $J_{g,n}$ is of relative dimension g, and has a zero section $0\colon M_{g,n} \to J_{g,n}$ and an *Abel-Jacobi section* $s_{\mathbf{a}}\colon M_{g,n} \to J_{g,n}$ defined by

$$[C, p_1, \dots, p_n] \mapsto [C, p_1, \dots, p_n, a_1[p_1] + \cdots + a_n[p_n]].$$

Then, we may write $\mathrm{DRL}^o_{g,\mathbf{a}} = s_{\mathbf{a}}^{-1}([0])$, and we see that $\mathrm{DRL}^o_{g,\mathbf{a}}$ is closed of codimension g. We can thus define the double ramification cycle

$$\mathrm{DRC}^o_{g,\mathbf{a}} = [\mathrm{DRL}^o_{g,\mathbf{a}}] = s_{\mathbf{a}}^*([0]) \in A^g(M_{g,n}).$$

Richard Hain[11] used this definition of the DR-cycle to find the formula[8]

$$\mathrm{DRC}^o_{g,\mathbf{a}} = \frac{1}{2^g g!}(a_1^2\psi_1 + \cdots + a_n^2\psi_n)^g.$$

The story becomes much more interesting when we try to extend from $M_{g,n}$ to $\overline{M}_{g,n}$. Eliashberg first raised the question of finding a good extension of $\mathrm{DRC}^o_{g,\mathbf{a}}$ to $\overline{M}_{g,n}$. The simplest way to extend would be to define the DR cycle as the class of the closure of the DR locus. There has been some work [27, 2, 25] on understanding this extension, but it is hard to compute. For some partial compactifications of $M_{g,n}$, such as the moduli of curves of compact type, the above discussion on the Abel–Jacobi map essentially extends [9, 8], but things become very difficult at some point [13]. Because of this, the usual definition of the double ramification cycle on $\overline{M}_{g,n}$ instead starts by viewing DRL as a moduli of marked curves $(C, p_1, \dots, p_n)$ together with a map $f\colon C \to \mathbb{P}^1$ defined up to scaling, and with prescribed ramification at 0 and ∞. There is a compactification of this moduli space $\overline{M}_{g,\mathbf{a}}(\mathbb{P}^1, 0, \infty)^\sim$ of *rubber stable maps to* $\mathbb{P}^1$ considered in Gromov–Witten theory, which comes with a (proper) forgetful map

$$p\colon \overline{M}_{g,\mathbf{a}}(\mathbb{P}^1, 0, \infty)^\sim \to \overline{M}_{g,n},$$

and there is a *virtual class*

$$[\overline{M}_{g,\mathbf{a}}(\mathbb{P}^1, 0, \infty)^\sim]^{\mathrm{vir}} \in A_{2g-3+n}(\overline{M}_{g,\mathbf{a}}(\mathbb{P}^1, 0, \infty)^\sim).$$

[7] The fiber of $[C, p_1, \dots, p_n]$ is the Jacobian of degree zero divisors on C.

[8] Actually all tautological classes of degree g on $M_{g,n}$ (including the DR cycle) are zero. Hain's paper actually deals with compact-type curves.

Then, we can define the DR cycle

$$\mathrm{DRC}_{g,\mathbf{a}} = p_* \left([\overline{M}_{g,\mathbf{a}}(\mathbb{P}^1, 0, \infty)^\sim]^{\mathrm{vir}} \right) \in A^g(\overline{M}_{g,n}).$$

The definition of the DR cycle in terms of a virtual cycle is not very explicit or directly useful for computations. Using techniques from Gromov–Witten theory (virtual localization and orbifold Gromov–Witten theory) in [15], a formula for the DR cycle (first conjectured by Pixton) was deduced. We state the formula here.

The formula involves an auxiliary cycle $P^r_{g,\mathbf{a}}$ depending polynomially on an extra parameter r. The cycle $P^r_{g,\mathbf{a}}$ is defined for all sufficiently large r by the formula

$$P^r_{g,\mathbf{a}} = \sum_{\Gamma} \sum_{w} \frac{1}{|\operatorname{Aut}(\Gamma)|} \frac{1}{r^{h^1(\Gamma)}} \xi_{\Gamma,*} \Bigg(\prod_{i=1}^{n} \exp\left(\frac{1}{2} a_i^2 \psi_i\right) \cdot \prod_{(h,h')\in E} \frac{1 - \exp(-\frac{1}{2} w(h) w(h')(\psi_h + \psi_{h'}))}{\psi_h + \psi_{h'}} \Bigg),$$

in which the second sum is over functions, called *weightings modulo* r, $w\colon H \to \{0, \ldots, r-1\}$ satisfying

- $w(h) + w(h') = 0$ for every $(h, h') \in E$;
- $\sum_{h \text{ at } v} w(h) = 0$ for every $v \in V$;
- $w(\ell_i) = a_i$ if ℓ_i is the ith leg.

Then, the formula for the DR cycle states that $\mathrm{DRC}_{g,\mathbf{a}}$ is the part of $P^0_{g,\mathbf{a}}$ of degree g. (Note that $P^0_{g,\mathbf{a}}$ is not the same as setting $r = 0$ in the above formula for $P^r_{g,\mathbf{a}}$ for $r \gg 0$.)

Exercise 16. Show that the restriction of Pixton's formula to $M_{g,n}$ agrees with $\mathrm{DRC}^o_{g,\mathbf{a}}$.

Exercise 17. Show that for any dual graph Γ, there are $r^{h^1(\Gamma)}$ weightings modulo r.

Exercise 18. Show that $\mathrm{DRC}_{1,(0)} = -\frac{\delta_{\mathrm{irr}}}{12} \in R^1(\overline{M}_{1,1})$.

Furthermore, it has been proven in [3] that the part of $P^0_{g,\mathbf{a}}$ in degree greater than g vanishes. This vanishing is not obvious, and instead implies relations in the tautological ring.

To finish, we mention that there are variants of the DR cycle related to considering meromorphic differentials instead of meromorphic functions [6, 10, 12, 18, 23], and a version relative to a target variety [16].

References

[1] E. Arbarello and M. Cornalba. Combinatorial and algebro-geometric cohomology classes on the moduli spaces of curves. *J. Algebraic Geom.*, 5(4):705–749, 1996. arXiv:alg-geom/9406008.

[2] R. Cavalieri and N. Tarasca. Classes of Weierstrass points on genus 2 curves, 2017. arXiv:1711.00486.

[3] E. Clader and F. Janda. Pixton's double ramification cycle relations. *Geom. Topol.*, 22(2):1069–1108, 2018. arXiv:1601.02871.

[4] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Inst. Hautes Études Sci. Publ. Math.*, (36):75–109, 1969.

[5] C. Faber and R. Pandharipande. Relative maps and tautological classes. *J. Eur. Math. Soc. (JEMS)*, 7(1):13–49, 2005. arXiv:math/0304485.

[6] G. Farkas and R. Pandharipande. The moduli space of twisted canonical divisors. *J. Inst. Math. Jussieu*, 17(3):615–672, 2018. arXiv:1508.07940.

[7] T. Graber and R. Pandharipande. Constructions of nontautological classes on moduli spaces of curves. *Michigan Math. J.*, 51(1):93–109, 2003. arXiv:math/0104057.

[8] S. Grushevsky and D. Zakharov. The double ramification cycle and the theta divisor. *Proc. Amer. Math. Soc.*, 142(12):4053–4064, 2014. arXiv:1206.7001.

[9] S. Grushevsky and D. Zakharov. The zero section of the universal semiabelian variety and the double ramification cycle. *Duke Math. J.*, 163(5):953–982, 2014. arXiv:1206.3534.

[10] J. Guéré. A generalization of the double ramification cycle via log-geometry, 2016. arXiv:1603.09213.

[11] R. Hain. Normal functions and the geometry of moduli spaces of curves. In *Handbook of moduli. Vol. I*, volume 24 of *Adv. Lect. Math. (ALM)*, pages 527–578. Int. Press, Somerville, MA, 2013. arXiv:1102.4031.

[12] D. Holmes. Extending the double ramification cycle by resolving the Abel-Jacobi map, 2017. arXiv:1707.02261.

[13] D. Holmes, J. L. Kass, and N. Pagani. Extending the double ramification cycle using Jacobians. *Eur. J. Math.*, 4(3):1087–1099, 2018. arXiv:1712.07098.

[14] F. Janda. Relations on $\overline{M}_{g,n}$ via equivariant Gromov-Witten theory of $\mathbb{P}^1$. *Algebr. Geom.*, 4(3):311–336, 2017. arXiv:1509.08421.

[15] F. Janda, R. Pandharipande, A. Pixton, and D. Zvonkine. Double ramification cycles on the moduli spaces of curves. *Publ. Math. Inst. Hautes Études Sci.*, 125:221–266, 2017. arXiv:1602.04705.

[16] F. Janda, R. Pandharipande, A. Pixton, and D. Zvonkine. Double ramification cycles with target varieties, 2018. arXiv:1812.10136.

[17] M. Kontsevich. Intersection theory on the moduli space of curves and the matrix Airy function. *Comm. Math. Phys.*, 147(1):1–23, 1992.

[18] S. Marcus and J. Wise. Logarithmic compactification of the Abel-Jacobi section, 2017. arXiv:1708.04471.

[19] D. Mumford. Towards an enumerative geometry of the moduli space of curves. In *Arithmetic and geometry, Vol. II*, volume 36 of *Progr. Math.*, pages 271–328. Birkhäuser Boston, Boston, MA, 1983.

[20] R. Pandharipande. A calculus for the moduli space of curves. In *Algebraic geometry: Salt Lake City 2015*, volume 97 of *Proc. Sympos. Pure Math.*, pages 459–487. Amer. Math. Soc., Providence, RI, 2018. arXiv:1603.05151.

[21] R. Pandharipande, A. Pixton, and D. Zvonkine. Relations on $\overline{M}_{g,n}$ via 3-spin structures. *J. Amer. Math. Soc.*, 28(1):279–309, 2015. arXiv:1303.1043.

[22] A. Pixton. Conjectural relations in the tautological ring of $\overline{M}_{g,n}$, 2012. arXiv:1207.1918.

[23] J. Schmitt. Dimension theory of the moduli space of twisted k-differentials. *Doc. Math.*, 23:871–894, 2018. arXiv:1607.08429.

[24] J. Schmitt and J. van Zelm. admcycles - a Sage package for computations with tautological classes and admissible cover cycles. https://people.math.ethz.ch/ schmittj/admcycles.html.

[25] J. Schmitt and J. van Zelm. Intersections of loci of admissible covers with tautological classes, 2018. arXiv:1808.05817.

[26] H. Schubert. *Kalkül der abzählenden Geometrie*. Springer-Verlag, Berlin-New York, 1979. Reprint of the 1879 original, With an introduction by Steven L. Kleiman.

[27] N. Tarasca. Double total ramifications for curves of genus 2. *Int. Math. Res. Not. IMRN*, (19):9569–9593, 2015. arXiv:1401.3057.

[28] M. Tavakol. The moduli space of curves and its invariants, 2016. arXiv:1610.09589.

[29] R. Vakil. The moduli space of curves and Gromov-Witten theory. In *Enumerative invariants in algebraic geometry and string theory*, volume 1947 of *Lecture Notes in Math.*, pages 143–198. Springer, Berlin, 2008.

[30] J. van Zelm. Nontautological bielliptic cycles. *Pacific J. Math.*, 294(2):495–504, 2018. arXiv:1612.01206.

[31] E. Witten. Two-dimensional gravity and intersection theory on moduli space. In *Surveys in differential geometry (Cambridge, MA, 1990)*, pages 243–310. Lehigh Univ., Bethlehem, PA, 1991.

[32] D. Zvonkine. An introduction to moduli spaces of curves and their intersection theory. In *Handbook of Teichmüller theory. Volume III*, volume 17 of *IRMA Lect. Math. Theor. Phys.*, pages 667–716. Eur. Math. Soc., Zürich, 2012.