NON-$\mathbb{Q}$-FACTORIAL POSITIVE CHARACTERISTIC FLIPS IN DIMENSION THREE

1. Some results and definitions from [HW20]

Definition 1.1. Let X be an F-finite scheme defined over a field of characteristic $p > 0$. Given an effective $\mathbb{Q}$-divisor B, we say that (X, B) is *globally F-regular* (resp. *purely globally F-regular*) if for every effective divisor D on X (resp. every $D \geq 0$ intersecting $\lfloor B \rfloor$ properly) and every integer $e \gg 0$, the natural homomorphism of $\mathcal{O}_X$-modules

$$\mathcal{O}_X \to F^e_* \mathcal{O}_X(\lfloor (p^e - 1)B \rfloor + D)$$

splits.

If the above splittings hold locally on X, then we refer to these notions as *strong F-regularity*, and *pure F-regularity*, respectively (they should be thought of as analogues of klt and plt). Given a morphism $f \colon X \to Y$, we say that (X, B) is relatively (over Y) F-regular, or purely F-regular, if the above splittings hold locally over Y.

Lemma 1.2 ([HW20, Lemma 2.4])**.** *Let $(X, S+B)$ be a plt pair where $S = \lfloor S+B \rfloor$ is a prime divisor and let $f \colon X \to Z$ be a proper birational morphism of normal varieties over an F-finite field of characteristic $p > 0$. Assume that $-(K_X + S + B)$ is f-ample and (S^ν, B_{S^ν}) is relatively F-regular over $f(S)$, where $S^\nu \to S$ is the normalisation and $K_{S^\nu} + B_{S^\nu} = (K_X + S + B)|_{S^\nu}$. Then $(X, S + B)$ is purely globally F-regular over a neighborhood of $f(S) \subset Z$.*

Proposition 1.3 ([HW20, Proposition 2.5])**.** *Suppose that $(X, S + B)$ is a purely F-regular pair over an F-finite field of characteristic $p > 0$ where $\lfloor S + B \rfloor = S$ is a prime divisor. Then S is normal.*

Proof. Apply [Das, Theorem A] to the base change over the algebraic closure of the base field k. □

Proposition 1.4 ([HW20, Proposition 3.1])**.** *Let $(X, S+A+B)$ be a dlt pair over an F-finite field of characteristic $p > 0$ where $\lfloor S + A + B \rfloor = S + A$, the $\mathbb{Q}$-Cartier Weil divisor A is ample, and the Weil divisor S is irreducible. Further, let $f \colon X \to Z$ be a contraction with Z affine such that $(X, S + (1 - \epsilon)A + B)$ is relatively purely F-regular over Z for any $\epsilon > 0$. Write $K_S + B_S = (K_X + S + B)|_S$ and $A_S = A|_S$.*

Then for every $k \geq 1$ such that $k(K_X + S + B + A)$ is Cartier, we have

$$|k(K_X + S + A + B)|_S = |k(K_S + A_S + B_S)|.$$

2. Existence of flips

Lemma 2.1 (cf. [HW19a, Lemma 3.3], [HW20, Lemma 4.1])**.** *Let $(S, C+B)$ be a two-dimensional plt pair defined over an F-finite field of characteristic $p > 0$, where $f\colon S \to T$ is a birational morphism, C is an irreducible divisor with $f|_C\colon C \to f(C)$ birational, and $-(K_S + C + B)$ is an f-ample $\mathbb{Q}$-divisor. Then $(S, C+B)$ is relatively purely F-regular over a neighbourhood of $f(C) \subset T$. In particular, $(S, (1-\epsilon)C + B)$ is relatively F-regular for every $0 < \epsilon \leq 1$.*

Proof. Since S is two-dimensional, C is normal. Write $K_C + B_C = (K_S + C + B)|_C$. The klt pair (C, B_C) is strongly F-regular and hence relatively F-regular over $f(C)$ as $f|_C$ is affine. Thus, by Lemma 1.2, $(S, C+B)$ is relatively purely F-regular over a neighbourhood of $f(C) \subseteq T$. ☐

Proposition 2.2. *Let $g\colon X \to Y$ be a projective birational morphism of varieties defined over an F-finite field k of characteristic $p > 0$ and let (X, Δ) be a dlt pair such that $\operatorname{Supp}\operatorname{Exc}(g) = \lfloor \Delta \rfloor$. Let E_i be all irreducible exceptional divisors, and suppose that they are $\mathbb{Q}$-Cartier. Further assume that $K_X + \Delta \sim_{\mathbb{Q},Y} \sum e_i E_i$ for $e_i \in \mathbb{Q}$ and that there exists a relatively anti-ample effective exceptional divisor F.*

Let $H \sim_{\mathbb{Q},Y} \sum_i h_i E_i$ be a $\mathbb{Q}$-divisor such that $K_X + \Delta + H$ is nef and induces a small birational morphism $h\colon X \to Z$ for which $-(K_X+\Delta)$ is h-ample. Suppose that all the exceptional divisors E_i are $\mathbb{Q}$-equivalent to each other over Z up to a multiple. Then the canonical ring $R(K_X + \Delta)$ is finitely generated over Z.

Proof. Since F is relatively anti-ample over Y (and so over Z), there exists an effective irreducible exceptional divisor $S := E_k$ which is h-anti-ample for some $k > 0$. Furthermore, by the negativity lemma, the nef exceptional $\mathbb{Q}$-divisor $\sum_i (e_i + h_i)E_i$ is anti-effective and $e_i + h_i < 0$ for every $i > 0$. Since $\sum_i (e_i + h_i)E_i$ is numerically relatively trivial over Z and E_k is relatively anti-ample, there exists a g-exceptional irreducible divisor $A := E_l$ which is relatively ample over Z.

By a small perturbation by exceptional divisors, we may assume that $\Delta = S + A + B$, where $\lfloor B \rfloor = 0$. Then $(X, S + (1-\epsilon)A + B)$ is relatively purely F-regular over Z by Lemma 2.1 and inversion of F-adjunction (Lemma 1.2). Thus S is normal and the assumptions of Proposition

1.4 are satisfied. In particular,

$$|k(K_X + S + A + B)|_S = |k(K_S + A_S + B_S)|.$$

for $(K_X+S+B)|_S = K_S+B_S$ and $A_S = A|_S$. Since $R(K_S+A_S+B_S)$ is finitely generated over $h(S)$ by the two-dimensional MMP (see [Tanaka, Theorem 1.1 and Theorem 4.2]), so is $R(K_X + S + A + B)$ over Z by the usual argument (cf. [Corti, Lemma 2.3.6]). □

References

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[Das] O. Das *On strongly F-regular inversion of adjunction*

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