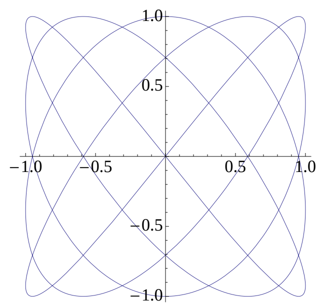
Math 116-031 – Quiz 3 **Name**:

Polar & Parametric Curves + a review problem

1. Jake is pacing around his office, trying desperately to understand Michel Brion's article on positivity in the K-theory of vector bundles on complex flag manifolds. Jake's position is given by the parametric equations $\big(x(t), y(t)\big) = (\sin(4\pi t), \cos(5\pi t))$ for $0 \le t \le 2$.



Jake's path :-(

(a) [2 points] Find the tangent line to Jake's path at $t = 1.4$. (You may give parametric equations or an equation of the form $y = f(x)$.)

Solution. First, we need the slope, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$. So,

$$\left.\frac{dy/dt}{dx/dt}\right|_{t=1.4} = \left.\frac{-5\pi\sin(5\pi t)}{4\pi\cos(4\pi t)}\right|_{t=1.4} \approx 0,$$

(the exact value is around 3×10^{-15}.)

Next, we need the coordinates: we see that $(x, y) \approx (-0.95, -1)$. So the tangent line is given by

$$y = (3 \times 10^{-15})(x + 0.95) - 1, \text{ or just } y = -1,$$

if we take the slope to be zero.

(b) [4 points] Write an integral that computes the total distance Jake walks while reading the article. You do not have to simplify or compute the integral.

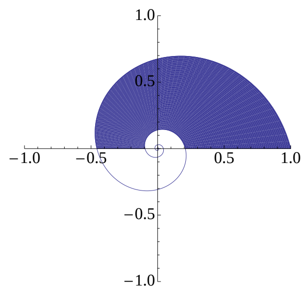
From the arc length formula for parametric curves, we see that the distance is

$$\int_{t=0}^{t=2} \sqrt{\big(4\pi\cos(4\pi t)\big)^2 + \big(-5\pi\sin(5\pi t)\big)^2}\,dt.$$

2. Jake gives up on Michel Brion's article after encountering the equation

$$R^i\psi_*(\omega_{\tilde{X}_w} \otimes \psi^*\mathcal{L}_{X_w}(\rho)) = R^i\psi_*(\omega_{\tilde{X}_w}) \otimes \mathcal{L}_{X_w}(\rho)$$

and decides to draw the spiral given by $r = e^{-\theta/4}$ for $\theta \geq 0$. Here's what he draws:



(a) [6 points] Write an expression involving definite integrals that computes the total shaded area above. You do not need to simplify or compute the integral(s).

Solution. $\dfrac{1}{2}\displaystyle\int_0^{\pi} (e^{-\theta/4})^2 d\theta - \dfrac{1}{2}\int_{2\pi}^{3\pi} (e^{-\theta/4})^2 d\theta.$

(The first integral gives all the area above the x-axis, and the second integral subtracts the inner portion we need to remove.)

(b) [4 points] Find the minimum y-value on the spiral.

Solution. We know that $y = r\sin(\theta) = e^{-\theta/4}\sin(\theta)$. We set its derivative to zero:

$$\begin{aligned} 0 = \frac{dy}{d\theta} &= -\frac{1}{4}e^{-\theta/4}\sin(\theta) + e^{-\theta/4}\cos(\theta) \\ &= e^{-\theta/4}\Big(-\frac{1}{4}\sin(\theta) + \cos(\theta)\Big). \end{aligned}$$

The $e^{-\theta/4}$ factor is always positive, so we look at the other factor. It works out to $\tan\theta = -4$, which gives $\theta = -1.32, 1.82, 4.95, \ldots$. From the picture, we want the value between π and 2π, so $\theta = 4.95$, which gives $y = -0.28$.

(c) [6 points] Write an expression that computes the length of the boundary of the shaded region. You do not need to simplify or compute any integral(s) in your answer.

There are four edges to work out: the top edge, the two horizontal edges, and the inner edge. These are:

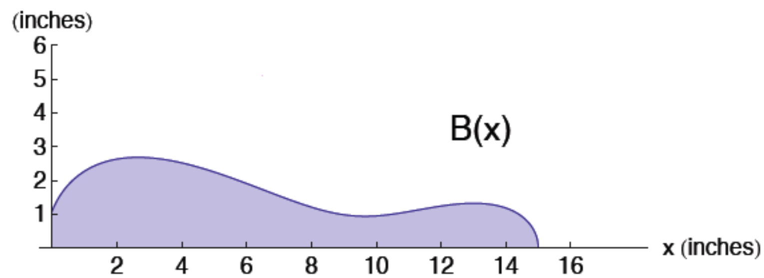
$$\begin{aligned} \text{top edge} &= \int_0^{\pi} \sqrt{(e^{-\theta/4})^2 + \big(-\tfrac{1}{4}e^{-\theta/4}\big)^2}\, d\theta, \\ \text{inner edge} &= \int_{2\pi}^{3\pi} \sqrt{(e^{-\theta/4})^2 + \big(-\tfrac{1}{4}e^{-\theta/4}\big)^2}\, d\theta, \\ \text{right horizontal edge} &= r(0) - r(2\pi) = 1 - e^{-\pi/2}, \\ \text{left horizontal edge} &= r(\pi) - r(3\pi) = e^{-\pi/4} - e^{-3\pi/4}. \end{aligned}$$

The total length is the sum of these four sides.

3. During a friendly game of bowling, your friends Walter and Smokey begin to argue over whether Walter's toe slipped over the foul line. You decide to pass the time by finding a mathematical model for the shape of a bowling pin. After some careful thought, you find that a knocked-over pin is a solid of revolution given by rotating the region under the curve

$$B(x) = \sqrt{1.2 + 5.32x - 1.485x^2 + 0.135x^3 - 0.004x^4}$$

over the interval $[0, 15]$, rotated about the x-axis. You draw this picture:



Your friend Jeff "The Dude" Lebowski informs you that the top end of the bowling pin is heavier, to make it easier to knock over. Its density in the diagram above is $\delta(x) = 2x$ grams per cubic inch.

(a) [6 points] Write a definite integral that gives the mass of the bowling pin. You do not have to evaluate this integral. Where necessary, you can just refer to "$B(x)$" rather than copying the formula above.

Solution. $\displaystyle\int_0^{15} \pi B(x)^2 \; 2x \; dx.$

(b) [2 points] Walter uses bowling pins with density $\delta(x) = 4x$ grams per cubic inch, twice the density of the pin in part (a). How does this affect the position $\bar{x}$ of the pin's center of mass? (Limit your answer to a sentence or two.)

The center of mass is unchanged, because the new bowling pin is *uniformly* heavier. In other words, doubling the mass everywhere doesn't change the mass *distribution*.

Algebraically, you can see this by the fact that the constant factors cancel out in the equation for $\overline{x}$:

$$\overline{x} = \frac{\displaystyle\int_0^{15} \pi B(x)^2 \; 4x \; dx}{\displaystyle\int_0^{15} \pi B(x)^2 \; 4x \; dx} = \frac{4\displaystyle\int_0^{15} \pi B(x)^2 \; x \; dx}{4\displaystyle\int_0^{15} \pi B(x)^2 \; x \; dx} = \frac{\displaystyle\int_0^{15} \pi B(x)^2 \; x \; dx}{\displaystyle\int_0^{15} \pi B(x)^2 \; x \; dx},$$

the same for both pins. (For the first pin, the 2's cancel out the same way.)

WALTER SOBCHAK
OVER THE LINE!
SMOKEY
Huh?
WALTER SOBCHAK
I'm sorry, Smokey. You were over the line, that's a foul.

SMOKEY
Bullshit. Mark it 8, Dude.
WALTER SOBCHAK
Uh, excuse me. Mark it zero. Next frame.
SMOKEY
Bullshit, Walter. Mark it 8, Dude.
WALTER SOBCHAK
Smokey, this is not 'Nam. This is bowling. There are rules.