The note on de Jong's conjecture

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Abstract

This is the note for the talk at June 26, 2018, in the University of Michigan, as a digression for the serial lectures on Drinfeld's geometric Langlands correspondence (from Galois to automorphic)[Dri83]. We will mostly follow de Jong's paper [deJ01], discuss the relation of Drinfeld's theorem and the de Jong's conjecture, and prove the structure result for the deformation ring of mod-ℓ representations of fundamental groups.

1 De Jong's conjecture

In this section, we will formulate the conjecture and sketch the proof for GL_2-case using Drinfeld's thoerem.

Statement of the conjecture

Let $\mathbb{F}$ be a finite field of characteristic ℓ, $F = \mathbb{F}((t))$. Assume X is a normal scheme of finite type over a finite field k of characteristic p, where $p \neq \ell$. Let $\overline{X}$ be the base change of X to $\overline{k}$, and we fix a geometric point $\overline{x}$ of X. Then there exists a natural exact sequence of étale fundamental groups

$$0 \longrightarrow \pi_1(\overline{X}, \overline{x}) \longrightarrow \pi_1(X, \overline{x}) \longrightarrow \mathrm{Gal}(\overline{k}/k) \longrightarrow 0.$$

The de Jong's conjecture is about the finiteness of the image of $\pi_1(\overline{X})$ in a given continuous $\mathbb{F}((t))$-representation of $\pi_1(X)$.

Conjecture 1.1 (De Jong, [deJ01] 2.3)**.** *Assume $\rho : \pi_1(X) \to \mathrm{GL}(V)$ is a continuous representation, where V is a finite dimensional F-vector space and $\mathrm{GL}(V)$ has the topology induced from $\mathrm{End}(V)$. Then the image $\rho(\pi_1(\overline{X}))$ of the geometric fundamental group is finite.*

The conjecture is equivalent to the statement that the geometric monodromy group of any given lisse $\overline{\mathbb{F}((t))}$-sheaf over X is finite.

Remark 1.2. The higher dimensional conjecture was proved in most of the cases by Bockle-Khare, and Gaitsgory separately.

We first note that the proof for GL_1 does not need a lot of efforts granting an fact about the abelianization of fundamental groups:

Proposition 1.3. *When $\dim(V)$ is 1, the above conjecture holds.*

Proof. By the geometric class field theory, the abelianization $\pi_1^{ab}(X)$ is of the form

$$(pro-p) \times (finite) \times \widehat{\mathbb{Z}},$$

such that $\pi_1^{ab}(\overline{X})$ maps surjectively onto the first two factors. (The case when X is a curve comes from the geometric class field; the general case was proved by Deligne [Del80], 1.3.1.) But note that $\mathrm{GL}_1(F) = \mathbb{F}((t))^*$ is of the form

$$(pro-\ell) \times (finite) \times \mathbb{Z}.$$

From this, by the continuity, we see the image of $\pi_1(\overline{X})$ can only be a finite group. □

Proof of the Conjecture for GL_2

Here we sketch how to prove the conjecture for $\dim(V) = 2$, granting the geometric Langlands correspondence of GL_2 by Drinfeld [Dri83].

The very first reduction is to reduce the conjecture to the case when X is a smooth projective curve. More precisely, we have

Theorem 1.4 ([deJ01], 2.17). *The conjecture for* $\dim(V) \leq n$ *holds if and only if it holds for every case* (X, V, ρ)*, where* X *is a smooth projective curve,* $\dim(V) \leq n$*, and* $\rho|_{\pi_1(\overline{X})}$ *is absolutely Lie irreducible, with* $\det(\rho) = 1$.

Here we note that the absolutely Lie irreducible for a representation of G is defined as for any open subgroup $U \subset G$, the restriction of the representation on U is (algebraically) irreducible.

*Proof*Step 1 We first consider the reduction from a normal scheme of dimension n to a curve.

Assume the conjecture for X of dimension strictly smaller than n holds. Since $\pi_1(X)$ is the quotient of the Galois group $\mathrm{Gal}(k(\overline{X})^s/k(\overline{X}))$ of the function field, for any dominant map $X' \to X$ of the same dimension, the corresponding $\pi_1(\overline{X}') \to \pi_1(\overline{X})$ is open. Note that the conjecture preserves by taking an open subgroup of $\pi_1(X)$ (which corresponds to a finite étale covering X), so we may assume X is smooth affine and geometrically connected. Besides, since the pro-ℓ subgrou $1 + t\,\mathrm{End}_F(V)$ is open in $\mathrm{GL}(V)$, we could take an open subgroup such that ρ factors through the pro-ℓ-quotient of $\pi_1(X)$.

Then by shrinking X if necessary, we may assume there exists an elementary fibration ([SGA4], XI section 3):

$$X \xrightarrow{f} X' \xrightarrow{g} Y,$$

where f is an open immersion, g is smooth proper of relative dimension 1, such that $X' \to X$ is étale over Y. Moreover, by taking an étale map of Y and base change if necessary, we may assume $X \to Y$ has a section. Then there exists a short exact sequence of pro-ℓ-fundamental groups ([SGA1], XIII, 4.3 and 4.4)

$$0 \longrightarrow \pi_1^{\ell}(X_{\overline{y}}, \overline{y}) \longrightarrow \pi_1^{\ell}(X, \overline{x}) \longrightarrow \pi_1^{\ell}(Y, \overline{y}) \longrightarrow 0,$$

where $\overline{y}$ is a geometric point of Y over $\overline{x}$.

From this, by taking the bigger diagram consisting of arithmetic and geometric fundamental groups, we see it suffices to prove the finiteness for the image of $\pi_1^{\ell}(\overline{Y})$ and $\pi_1^{\ell}(X_{\overline{y}})$, which is true by induction.

Step 2 The second reduction is to reduce the case of normal curves to projective curves. For this, we consider the completion $X \subset X'$ for X' smooth projective. Then

□

Then we recall a variant of Drinfeld's main theorem. We fix Λ to be the ring $\mathbb{F}[[t]]$.

Theorem 1.5 (Drinfeld). *Assume* $\rho : \pi_1(X) \to \mathrm{GL}_2(\mathbb{F}((t)))$ *is a two dimensional continuous representation with trivial determinant, such that* $\rho|_{\pi_1(\overline{X})}$ *is absolutely irreducible. Then there exists an eigenform* $f : \mathrm{GL}_2(\mathbb{A}_X) \to \Lambda$ *such that*

$$T_v f = \mathrm{Tr}(\rho(F_v)) f;$$
$$U_v f = q_v^{-1} f.$$

Remark 1.6. The statement is slightly different from the Main theorem in [Dri83]: the coefficient of the representation here is the field $\mathbb{F}((t))$ of characteristic ℓ, while that in Drinfeld's paper is a finite extension of $\mathbb{Q}_\ell$. Besides, the irreduciblility here is stronger than that in [Dri83]. We refer to [deJ01], section 4 for the adjustment from Drinfeld's paper to the version we need.

Next we note that given the condition in the Drinfeld's theorem 1.5 (especially the requirement that the determinant of ρ is trivial), eigenvalues of T_v are in fact algebraic:

Fact 1.7 ([deJ01], 4.8)**.** Assume f is an eigenform over Λ with central character $u_v f = q_v^{-1} f$ for each $v \in X$. Then all the eigenvalues t_v are algebraic, namely there exists a finite extension $E/\mathbb{F}$ such that all of the t_v are contained in E.

Namely the Fact says that when f comes from the representation ρ in the Theorem 1.5, the trace of Frobenius Fr_v on the representation is inside $\mathbb{F}$. Granting this, the conjecture for $\dim(V) = 2$ is done assuming the following criterion of finiteness:

Proposition 1.8 ([deJ01], 2.8)**.** *Under the condition of the conjecture 1.1, the following two are equivalent:*

(i) $\rho(\pi_1(X))$ *is finite.*

(ii) For each Frobenius element F_v *(well-defined up to conjugacy) of* $v \in X$*, its characteristic polynomial under* ρ *has coefficients in* $\mathbb{F}$*.*

Proof. By the Cheboterav's density theorem for the fundamental group (reference?), the second condition implies the characteristic polynomial

$$P_g(T) = a_n(g)T^n + a_{n-1}(g)T^{n-1} + \cdots + a_0(g)$$

of any $\rho(g)$ with $g \in \pi_1(X)$ has coefficients in $\mathbb{F}$. Then we observe that by the continuity of the ρ, the function

$$\pi_1(X) \to \mathbb{F};\ g \mapsto a_i(g)$$

is also continuous, since a_i can be given as the trace function of the i-th wedge product of ρ. For this reason, by the finiteness of $\mathbb{F}$, each a_i is locally constant and there exists an open subgroup $H = \pi_1(Y) \subset \pi_1(X)$ such that $\rho(g)$ has one as their only eigenvalues for $g \in H$.

Now we claim that the image $\rho(H)$ is finite. We first note that the subgroup $\rho(H)$ in $\mathrm{GL}(V)$ is unipotent. So by the general fact about the unipotent subgroup of the GL_n, there exists a basis of V such that $\rho(H)$ are all of the upper triangular (Consider the irreducible component W of $\rho|_H$, then the $F[H]$ maps surjectively onto $\mathrm{End}(W)$. But for any $h \in H$, and any $\sum a_i h_i$, the trace $\mathrm{Tr}((h-1)(\sum a_i h_i))$ is 0. Thus $h - 1 = 0$.) Moreover, since $F = \mathbb{F}((t))$ is of characteristic ℓ, any element in H is nilpotent by a ℓ-power. We note that there exists a decreasing filtration of $\rho(H)$ by the derived series $\rho(H)^{(i)} = \textit{closure of } [H^{(i-1)}, H]$, such that each factor is nilpotent by ℓ^n. Thus $\rho(H)$ is a successive extension of pro-ℓ nilpotent groups. So to prove the finiteness of the pro-ℓ unipotent subgroup $\rho(H)$, it suffices to show that $\rho(H)$ is topologically finite generated (thus so are $\rho(H)^{(i)}$)

The last statement is true: since the image of $\pi_1(Y) = H$ is pro-ℓ, it factors through the pro-ℓ-quotient $\pi_1(Y) \to \pi_1^{\ell}(Y)$. And by the equality

$$\mathrm{Hom}(\pi_1^{\ell}(Y), \mathbb{Z}/\ell\mathbb{Z}) = \mathrm{H}^1(\pi_1^{\ell}(Y), \mathbb{Z}/\ell\mathbb{Z}) = \mathrm{H}^1(Y_{\acute{e}t}, \mathbb{Z}/\ell\mathbb{Z}),$$

the result then follows from the finiteness of the étale cohomology. □

2 Deformation rings and their structure

In this section, we study the deformation ring for mod-ℓ representations of fundamental groups $\pi_1(\overline{X})$ and $\pi_1(X)$. We will use the Conjecture 1.1 to show the structure theorem for the deformation ring of $\pi_1(X)$.

We fix $\mathcal{O}$ to be the ring of integers in a finite totally ramified extension of the fraction field of the Witt vector $\mathrm{Frac}(W(\mathbb{F}))$. Let $\mathcal{C}$ be the category of complete Noetherian local rings R over $\mathcal{O}$ such that $R/m_R = \mathbb{F}$.

Let Γ be a profinite group. Assume $\rho_0 : \Gamma \to \mathrm{GL}_n(\mathbb{F})$ to be a continuous residual representation, and $\epsilon : \Gamma \to \mathcal{O}^*$ be a continuous character such that $\epsilon \equiv \det \rho_0 \mod m_{\mathcal{O}}$. Then the *deformation functor* is given by the equivalent classes of liftings

$$Def(\Gamma, \rho_0, \epsilon) : \mathcal{C} \longrightarrow Set;$$
$$R \longmapsto \{\rho_R : \Gamma \to \mathrm{GL}_n(R) \mid \det \rho_R = \epsilon;\ \rho_R \equiv \rho_0 \ mod\ m_R\}/\sim,$$

where two liftings $\rho_R \sim \rho'_R$ if one can be conjugated to another by an element in $\mathrm{GL}_n(R)$.

Now we state our main theorem.

Theorem 2.1 ([deJ01], 3.5)**.** *Let X be a smooth geometrically connected curve over* $\mathrm{Spec}(k)$ *for k finite of characteristic $p \neq \ell$. Let $\rho_0 : \pi_1(X) \to \mathrm{GL}_n(\mathbb{F})$ be a continuous representation, and $\epsilon : \pi_1(X) \to \mathcal{O}^*$ compatible with ρ_0. Assuming the following conditions*

(i) $\ell \nmid n$;

(ii) $\rho_0|_{\pi_1(\overline{X})}$ is absolutely irreducible.

(iii) The conjecture holds for X and $\dim(V) \leq n$.

Then the functor $Def(\pi_1(X), \rho_0, \epsilon)$ is representatble by a finite flat $\mathcal{O}$-algebra R which is a complete intersection.

The strategy of the proof is to compare the deformation problem for $Def(\pi_1(X), \rho_0, \epsilon)$ and $Def(\pi_1(\overline{X}), \rho_0|_{\pi_1(\overline{X})}, \epsilon|_{\pi_1(\overline{X})})$. The very first question is the representability of the above two functor. Since both $\pi_1(X)$ and $\pi_1(\overline{X})$ are profinite whose pro-ℓ quotient is topologically finitely generated, and by assumption $\rho_0|_{\pi_1(\overline{X})}$ and ρ_0 are absolutely irreducible, due to the Schlessinger-Mazur theorem ([Maz89], Section 1.2) both of the functor are representable. We denote by (R, ρ) and $(R, \overline{rho})$ to be the representable pairs for the above two deformation problem, together with universal representations. Here we note that since $\rho|_{\pi_1(\overline{X})}$ is a deformation of $\rho_0|_{\pi_1(\overline{X})}$ on $R \in \mathcal{C}$, by the universal property of $(\overline{R}, \overline{\rho})$ there exists a morphism

$$\psi : \overline{R} \longrightarrow R.$$

The first step is to describe the structure of the ring $\overline{R}$, which is relatively easier.

Proposition 2.2. *The $\mathcal{O}$-algebra $\overline{R}$ is isomorphic to $\mathcal{O}[[x_1, \ldots, x_s]]$ for some $s \in \mathbb{N}$.*

Proof. The idea is to study the obstruction space of deformation ring, as in [Maz89], section 1.6. Let $Ad(\rho_0|_{\pi_1(\overline{X})})$ be the adjoint representation of $\rho_0|_{\pi_1(\overline{X})}$, and d_i be the dimension of $\mathrm{H}^i(\pi_1(\overline{X}), Ad(\rho_0|_{\pi_1(\overline{X})}))$. Then by [Maz89], Proposition 2 in the Section 1.6, $\overline{R}$ is a power series ring of dimension d_1 if d_2 equals 0.

We then compute d_2 by using the étale cohomology. Since $\overline{X}$ is a $K(\pi_1, 1)$ for étale cohomology, we have

$$\mathrm{H}^i(\pi_1(\overline{X}), Ad(\rho_0|_{\pi_1(\overline{X})})) = \mathrm{H}^2(\overline{X}_{et}, \mathcal{F}),$$

where $\mathcal{F}$ is the locally constant $\mathbb{F}$-étale sheaf associated to the adjoint representation $sl_n(\mathbb{F})$ with adjoint action. Since X is a curve, when $\overline{X}$ is affine, $\mathrm{H}^2 = 0$. And if X is projective, by the Poincaré duality and the observation that $\mathcal{F}$ is self-dual, d_2 equals to the dimension of

$$\mathrm{H}^0(\overline{X}, \mathcal{F}),$$

which corresponds to the dimension of subvector space of $\mathbb{F}^n$ that is invariant under the adjoint representation $\rho_0|_{\pi_1(\overline{X})}$. But by assumption $\rho_0|_{\pi_1(\overline{X})}$ is absolutely irreducible. Hence $d_2 = 0$, and we get the result. □

Here we remark that the Poincaré duality only works for $\overline{X}$ over $\overline{k}$, so we cannot apply the same method to X.

Frobenius twist Our next step is to study R with the help of $\overline{R}$ and the Frobenius twist.

Pick $F \in \pi_1(X)$ be a Frobenius element, i.e. a lift of the topological generator of $\mathrm{Gal}(\overline{k}/k)$ to $\pi_1(X)$. Let h_1 be an element in $\mathrm{GL}_n(R)$ such that $h_1 \equiv \rho_0(F) \bmod m_R$. Define the Frobenius twist $\overline{\rho}^F$ to be the representation of $\pi_1(\overline{X})$ over $\overline{R}$ given by

$$\pi_1(\overline{X}) \ni \gamma \longmapsto h_1^{-1}\overline{\rho}(F\gamma F^{-1})h_1.$$

By the choice of h_1 and F, $\overline{\rho}^F$ is a deformation of $\rho_0|_{\pi_1(\overline{X})}$, which leads to an endomorphism $\Phi : \overline{R} \to \overline{R}$ together with an element $h_2 \in \mathrm{GL}_n(\overline{R})$ such that

$$\Phi \circ \overline{\rho} = c_{h_2} \circ \overline{\rho}^F.$$

Moreover we could construct its inverse $\overline{\rho}^{F^{-1}}$ in the same way. So by the universal property of $\overline{R}$ again, the endomorphism Φ is thus an $\mathcal{O}$-linear isomorphism.

We could also define the F-twist of ρ. We then notice that the two representations $\psi \circ \rho|_{\pi_1(\overline{X})}$ and $\psi \circ \rho^F|_{\pi_1(\overline{X})}$ are equivalent by a conjugation, since $\rho(F\gamma F^{-1})$ can be written as $\rho(F)\rho(\gamma)\rho(F)^{-1}$. So by the universal property of $\overline{R}$ again we have the equality

$$\psi \circ \Phi = \psi : \overline{R} \to R.$$

This leads to the observation that $\overline{R} \to R$ factors through $\overline{R} \to \overline{R}/I_\Phi$, where I_Φ is the ideal

$$I_\Phi = (r - \Phi(r), r \in \overline{R}).$$

And since Φ is a $\mathcal{O}$-linear ring homomorphism, I_Φ is generated by s elements $x_i - \Phi(x_i)$.

In fact, the quotient ring is exactly R:

Proposition 2.3. *The map $\phi : \overline{R} \to R$ identifies R as*

$$\mathcal{O}[[x_1, \dots, x_s]]/(x_1 - \Phi(x_1), \dots, x_s - \Phi(x_s)).$$

Proof. The proof has two steps: we first show that ϕ is surjective, then construct a deformation of ρ_0 on the quotient $\overline{R}/I_\Phi$, compatible with $\overline{\rho}|_{\pi_1(\overline{X})}$ on $\overline{R}$. Granting this, we get a composition

$$\overline{R} \xrightarrow{\psi} R \longrightarrow \overline{R}/I_\Phi,$$

where the second map is given by the universal property of R. Note that by the compatibility, the composition is the natural quotient map. Hence $R \to \overline{R}/I_\Phi$ is an isomorphism.

For the first assertion about the surjection, by the completeness and the identity of residues, it suffices to show the surjection:

$$m_{\overline{R}}/(m_{\overline{R}}^2 + m_{\mathcal{O}}\overline{R}) \longrightarrow m_R/(m_R^2 + m_{\mathcal{O}}R).$$

Then those two are $\mathbb{F}$-vector spaces which are dual to the tangent spaces

$$\mathrm{H}^1(\pi_1(X), Ad(\rho_0)) \longrightarrow \mathrm{H}^1(\pi_1(\overline{X}), Ad(\rho_0|_{\pi_1(\overline{X})}))$$

that are associated to the two deformation problems ([Maz89], Section 1.2). So we only need to show the injectivity of the dual map. Then by the inflation-restriction sequence, it is equivalent to the vanishing of

$$\mathrm{H}^1(\mathrm{Gal}(\overline{k}/k), \mathrm{H}^0(\pi_1(\overline{X}), Ad(\rho_0|_{\pi_1(\overline{X})}))).$$

But by assumption, $\rho_0|_{\pi_1(\overline{X})}$ is absolutely irreducible. Thus $\pi_1(\overline{X})$-invariance of $Ad(\rho_0|_{\pi_1(\overline{X})})$ is zero, and we get the assertion.

We next construct a representation of $\pi_1(X)$ over $\overline{R}/I_\Phi$. This will be the place where we need the condition $\ell \nmid n$ in the Theorem 2.1. Note first that since the image of F in $\pi_1(X)/\pi_1(\overline{X})$ is the topological generator, it suffices to extend $\overline{\rho}$ to the Frobenius element F. Recall that $\square$

Proof for the main theorem

At last, we are reaching the proof of the main theorem 2.1.

We first use some results about commutative algebra. By what we have proved, the deformation ring R is isomorphic to $\mathcal{O}[[x_1,\dots,x_s]]/(x_1-\Phi(x_1),\dots,x_s-\Phi(x_s))$. This is a complete intersection if $R/\pi_{\mathcal{O}}R$ is of dimension 0, for $(x_1-\Phi(x_1),\dots,x_s-\Phi(x_s),\pi_{\mathcal{O}})$ is a sequence of the power series ring (which is CM local) of dimension $s+1$. In particular, $\mathcal{O}[[x_1,\dots,x_s]]/(f_1,\dots,f_s)$ has no $\mathcal{O}$-torsion, which then implies the flatness. And by the Nakayama's Lemma with the $\pi_{\mathcal{O}}$-completeness of R, we see R is finite over $\mathcal{O}$.

So it suffices to show that $R/\pi_{\mathcal{O}}$ is of dimension 0. Suppose not, then there exists an integral quotient $R/\pi_{\mathcal{O}} \longrightarrow A$ of dimension 1, where A is a complete local domain of characteristic ℓ. We take the integral closure A' of A, then we get a morphism $R/\pi_{\mathcal{O}} \to A' \cong \mathbb{F}'[[t]]$, whose composition becomes

$$R \to \mathbb{F}'[[t]].$$

This leads to a representation of $\pi_1(X)$ over $\mathbb{F}'[[t]]$.

Now by the Conjecture 1.1, the image of $\pi_1(\overline{X})$ is finite. Thus by the Lemma 2.4 about finite dimensional algebras below, the restriction to $\pi_1(\overline{X})$ factors through $\mathrm{GL}_n(\mathbb{F}')$, namely the representation is given by $\rho_0 \otimes_{\mathbb{F}} \mathbb{F}'[[t]]$. If $\mathbb{F}'$ is exactly $\mathbb{F}$, then by the universal property of $\overline{R}$ the induced morphism

$$\overline{R} \to \mathbb{F}'[[t]]$$

factors through $\overline{R} \to \mathbb{F} \to \mathbb{F}'[[t]]$, which contradicts to the construction of A and A'. For the general case where $\mathbb{F}'$ is finite over $\mathbb{F}$, we only need to note that the universal deformation ring over $\mathcal{O}' = \mathcal{O} \otimes_{W(\mathbb{F})} W(\mathbb{F}')$ is exactly $\overline{R} \otimes_{\mathcal{O}} \mathcal{O}'$. And by the same argument we get the composition $\overline{R} \to \overline{R} \otimes \mathcal{O}' \to \mathbb{F}' \to \mathbb{F}'[[t]]$. So we are done.

Lemma 2.4. *Let G be a finite group, and k be any field Assume $\rho : G \to \mathrm{GL}_n(k[[t]])$ is a deformation of an absolutely irreducible representation $\rho_0 : G \to \mathrm{GL}_n(k)$. Then ρ is the base change $\rho_0 \otimes_k k[[t]]$.*

Proof. The idea is to look at the finite-dimensional k algebra $k[[t]][G]$ together with the representation $\widetilde{\rho} : k[G] \to M_n(k)$. We denote by K to be the field $k((t))$. Then ρ induces a representation $K[G] \to M_n(K)$. Let m_0 be the kernel of the composition $k[G] \to K[G] \to M_n(K)$. Then by the basic fact of finite-dimensional algebras, m_0 is a maximal bisided ideal such that $k[G]/m_0 = M_r(D_0)$, for D_0 a skew field over k. Moreover, we have a surjection

$$D_0 \otimes_k K \longrightarrow K.$$

But note that the surjection above is splits, while the map of the Brauer group $Br(k) \to Br(K)$ is injective. In this way, the map $D_0 \otimes_k K \to K$ is isomorphism, and $D_0 = k$ such that the kernel of $K[G] \to M_n(K)$ is $m_0 \otimes_k K$. And by comparing the dimension, $r = n$. For that reason, the representation $\rho \otimes K$ comes from a representation $\rho_0' \otimes_k K$; in particular by the injection of $k[[t]] \to k((t)) = K$, ρ is also the base change $\rho_0' \otimes_k k[[t]]$. In this way, since its reduction mod t is ρ_0, we have

$$\rho_0' = \rho_0,$$

and ρ is exactly $\rho_0 \otimes_k k[[t]]$. □

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