Introduction to the de Jong's Alteration

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Abstract

This is the note for the talk on de Jong's alteration, at the Singularity Seminar in the University of Michigan, on December 6, 2018. We are going to sketch the existence of the de Jong's alteration to resolve the singularity. And as an application, we prove the non-negativity of the Serre's intersection multiplicity. Our main references are expository paper by Abramovich and Oort [AbOo], and Hochster's article on intersection multiplicities [Hoc97].

1 Introduction

Throughout the talk, we let k be a field.

Recall the famous result of Hironaka.

Theorem 1.1 (Hironaka, 1964). *Assume k is a field of characteristic 0, and X is a variety over k (geometrically integral scheme that is of finite type over k), and Z is a subvariety of X. Then there exists a finite sequences of blowups with nonsingular centers*

$$X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = X,$$

such that X_n is nonsingular, and the strict transform of Z is a normal crossing divisor.

We note that in the Hironaka's result, the obtained resolution is proper and birational over X; namely generically X_n and X are isomorphic. This is a very strong condition, and in fact we can even achieve an improvement in characteristic 0, such that the isomorphism can be extended to the whole nonsingular locus of X.

The result is powerful. And one hopes naturally if there exists such a result in characteristic p. Unfortunately, until now, the answer is still unclear: we have neither a confirmed proof for the existence of resolution, nor a counterexample.

However, if one replace the birationality by a weaker condition, allowing the function field to be different up to a finite extension (which is already enough in many applications), then we can actually resolve every singularity, even in characteristic p! This is just the renowned result of de Jong, stated as follows.

Theorem 1.2 (De Jong, 1995; [deJ95]). *Let X be a variety over an algebraically closed field k, Z a closed subvariety of X. Then there exists a separable proper surjective, generically finite morphism $X' \to X$, such that X' is quasi-projective nonsingular, and the essential pullback of Z is a normal crossing divisor in X'.*

Here we call a proper surjective, generically finite morphism an *alteration*.

Remark 1.3. Here we note that the essential pullback Z' of Z is defined as the union of components of $X' \times_X Z$ that are dominant to Z. This is called the strict transform in [AbOo], 3.1, and is slightly different from the usual strict transform defined in Hartshorne ([Har77]).

2 Sketch of the proof

We then sketch the proof of the Theorem 1.2.

The idea is to inductively reduce the case to the fiberation by nodal curves, using the moduli of stable nodal curves. To motivate it, we first look at the following example.

Example 2.1. Let P be the affine line $\mathbb{A}_1$ over k, and X is an affine cone over P, defined as

$$\begin{array}{c} X = \mathrm{Spec}(k[t][x,y]/(xy-t^2)) \\ \downarrow \\ P = \mathrm{Spec}(k[t]). \end{array}$$

Then it is straightforward to see that X is nonsingular everywhere except at $x = (0,0,0)$, which is inside of the fiber X_0 over $0 \in P$. The fiber $X_0 = \mathrm{Spec}(k[x,y]/(xy))$ is a nodal curve, defined over k.

So how do we resolve the singularity of X? In this case, we blowup at the singular point x in X, and get the projective variety $Bl_0(X)$ over X. Then the strict transform of X is nonsingular (after blowup $Bl_0(X)$ at the ideal defined by the exceptional divisor, which is an irreducible component in $Bl_0(X)$), and proper birational over X.

In this example, we see the singularity of the (relative) nodal curve is quite easy to resolve. And we remark that this does not depend on the characteristic of the base field.

De Jong's strategy is to reduce to this setting; in other words, after several alterations, we get a new variety $X' \to X$, together with a fiberation $X' \to P$ of relative one, such that the base P is regular, and $X' \to P$ is fiberwise nodal. Then the singularity of X' will be mild enough such that after finite many of blowups, we could achieve the resolution.

Now we begin to sketch the proof. Here we note before that each time we are doing an alteration $X' \to X$ with the essential pullback Z' of Z, we will replace (X,Z) by (X',Z') automatically to simplify the notation.

Proof of the Theorem 1.2. Assume X is of dimension d.

Step 1 (Reduction)

We first reduce the questions to the case when X is projective normal over k, and Z is a union of divisors.

We first blowup X at Z to get X', and take the strict transform Z' of Z, so that in the new pair (X',Z'), Z' is a divisor. Then recall the Chow's Lemma as follows:

Lemma 2.2 (Chow)**.** *Given an integral and finite type scheme over k, there exists a proper birational morphism (modification) $X' \to X$ such that X' is quasi-projective.*

By taking X' and the preimage of Z, we may assume X is quasi-projective. Besides, by taking the closure of X and Z in the open embedding $X \to X'$, where Y is projective, it suffices to assume X is projective. Furthermore, by taking the normalization of X, we can then assume X is normal. So we reduce to the situation.

Step 2 (Finding a projection)

In this step, we produce a projection of X onto a variety P so that $X \to P$ is generically a relative curve. Assume $X \to \mathbb{P}^N$ is a closed immersion, where $N \geq d = \dim(X)$. The way to find the projection is to use the Bertini's theorem and projection X step by step inside of $\mathbb{P}^N$.

Our tool is the following lemma:

Lemma 2.3. *(i) If X is of dimension $< N-1$ in $\mathbb{P}^N$, then there exists a non-empty open subset $U \in \mathbb{P}^N$, such that the projection from $x \in U$ sends Y birationally onto the image.*

(ii) *If X is of dimension $N-1$ in $\mathbb{P}^N$, then there exists a non-empty open subset $U \in \mathbb{P}^N$, such that the projection from $x \in U$ sends Y generically étale to $\mathbb{P}^{N-1}$.*

The lemma is proved by considering the secant/tangent space through a regular point y in Y, and consider their complement in $\mathbb{P}^N$.

By the Lemma, we can projects X onto $\mathbb{P}^d$, with $B \subset \mathbb{P}^d$ being the locus where $\pi : X \to \mathbb{P}^d$ is not étale. And there exists an open subset $U \in \mathbb{P}^d$ such that the projection from $x \in U$ maps $\pi(Z)$ generically étale to $\mathbb{P}^{d-1}$. Now we take $p \in U$ that is not in B, then the blowup of π at p gives

$$\begin{array}{ccc} X' := Bl_{\pi^{-1}(p)}(X) & \longrightarrow & X \\ \downarrow & & \downarrow \pi \\ Bl_p(\mathbb{P}^d) & \longrightarrow & \mathbb{P}^d, \end{array}$$

Note that $Bl_{\pi^{-1}(p)}(X)$ can be described as

$$\{(x, \ell) \in X \times \mathbb{P}^{d-1} \mid \pi(x) \in \ell\},$$

where $\mathbb{P}^{d-1}$ is the collection of lines through $p \in \mathbb{P}^d$. So the projection of $X \times \mathbb{P}^{d-1}$ induces the map $X' \to \mathbb{P}^{d-1}$. Then it is easy to check that the projection is of relative dimension 1, such that every fiber is generically smooth. And by the triviality of the $\pi_1(\mathbb{P}^{d-1})$ and the Stein factorization, $X' \to \mathbb{P}^{d-1}$ has connected fiber.

So we achieved the following: $X \to P$ is a generically smooth family of curves, with $Z \to P$ finite and generically étale.

Step 3 (Rigidify)

Since $\pi : X \to P$ is a family of generically smooth curves, in particular it is generically nodal. So we could find an open subset $U \in P$, such that the preimage $\pi^{-1}(U) \to U$ is a relative nodal curve. Then we are going to use the moduli of curves: if the moduli functor

$$T \longmapsto \{\textit{proper nodal curves } Y \textit{ over } T\}$$

is representable by the a universal pair of schemes $\mathcal{C} \to \mathcal{M}$, then we get a map $f : U \to \mathcal{M}$, such that $\pi^{-1}(U)$ is the pullback of the universal nodal curve $\mathcal{C}$ along f. And by taking the "closure", we could then extend $\pi^{-1}(U) \to U$ to a relative nodal curve that is birational over X.

This is roughly the thing we want to do; however, we need to be careful: the very first problem is that the moduli functor above is not representable. In order to achieve a representability, we will to rigidify the moduli problem, by adding level structure for the relative curves. In other words, instead of working on the curve itself, we want to consider the curve with n-marked points on it. And in the relative setting, the datum is given by the relative curve with n-sections on it.

Back to our setting of $X \to P$ and a divisor $Z \subset X$. In order to get the n-sections, we want to adjust our Z so that they become the disjoint union of sections of $X \to P$.

Lemma 2.4. *There exists a finite morphism $P_1 \to P$, satisfies: Let X_1 be the strict transform of X along $P_1 \to P$, and let Z_1 be the inverse image of Z in X_1. Then Z is a disjoint union n distinct sections $\sigma_i : P_1 \to X_1$.*

Remark 2.5. We actually need the section to satisfy one more condition: stability. We are not going to talk in detail here; the only point is that by Deligne & Mumford-Knudsen, the moduli of stable n-pointed curves of genus g admits a coarse moduli scheme and a tautological family.

Step 4 (Simplifying the fiber)

Now we are in the situation such that $\pi : X \to P$ is generically a stable n-pointed (nodal) curve. So there exists an open subset $U \subset P$, such that $\pi^{-1}(U) \to U$ is a family of stable n-pointed curve. We then want to complete the discussion in the last step: to extend the nodal curve from U to its closure. But in fact this is still not true in general; but we can achieve it after an alteration of the base:

Theorem 2.6 (Stable Extension Theorem)**.** *Let S be a locally noetherian integral scheme, $U \subset S$ an open dense subset, $C \to U$ with sections $s_i : U \to C$ being stable pointed curve. Then there exists an alteration $\varphi : S' \to S$, and a stable pointed curve $C' \to S'$ with sections $s_i' : S' \to C'$, such that the restriction of this stable pointed curve on $\varphi^{-1}(U) \subset S'$ is isomorphic to the pullback of $(C \to U, s_i)$ along ϕ.*

This theorem featuring the deepest ingredient of the proof: the moduli of stable n-pointed curves.

We then apply this theorem to our situation of $(X \to P, \sigma_i)$. Then we get an alteration $P_1 \to P$, and a family of stable n-pointed curve $X_1 \to P_1$, such that X_1 has an open subset isomorphic to one in $X \times_P P_1$:

$$\begin{array}{ccc} X_1 & \dashrightarrow & X \\ \downarrow & & \\ P_1 & \longrightarrow & P. \end{array}$$

Step 5 (Improve the rational map to a morphism) By the diagram above, it is clear that if the map $X_1 \to X \times_P P_1$ can be extended to a proper morphism, then we are in a very good status of the induction: after this alteration, $X \to P$ is a family of nodal curve. But to improve the condition from the rational map to the morphism, we need to be very careful.

To make the notation simple, we denote X_1 by C, replace $X \times_P P_1$ by X, and replace P_1 by P. Then we have the following diagram

$$\begin{array}{ccc} & & X \\ & \overset{\beta}{\nearrow} & \downarrow \\ C & \longrightarrow & P, \end{array}$$

where β is the rational map. Then we take T to be the closure of the graph of β in $C \times_P X$. It is then equipped with projection morphisms $pr_1 : T \to C$ and $pr_2 : T \to X$. Here the induced map $T \to P$ may not always have curve as fibers, so we will have to flattening it:

Lemma 2.7 (Raynaud-Gruson)**.** *Let $X \to Z$ be a proper, generically flat morphism of two finite type schemes. Then there exists a blowup $f : Z' \to Z$, such that the strict transform of X along f is flat over Z'.*

So we blowup P if necessary and by the strict transform, we assume $X \to P$ and $T \to P$ are flat (the base change of C is automatically flat)

And we come at the delicate discussion, which I am not going to do here:

Proposition 2.8. *The projection $\pi_1 : T \to C$ is an isomorphism, and $\pi_2 : T \to X$ is a birational projective morphism.*

So we can replace X by C and assume $X \to P$ is a stable n-pointed nodal curve, with Z being the union of sections σ_i and the preimage along the previous steps.

Step 6 (Induction and resolving)

At last, we use the induction. Let Σ be the locus in P where $\pi : X \to P$ is not relative smooth. Then there exists an alteration $f : P_1 \to P$ such that P_1 is nonsingular, and the strict transform of $\Sigma_1 = f^{-1}(\Sigma)$ is snc divisor. Now we take the union of the preimage of $\pi_1^{-1}(\Sigma_1)$ and $f^{-1}(Z)$ as Z_1. Then we are in the situation such that $(X_1 \to P_1, Z_1)$ is a nodal curve with regular base, such that the divisor Z_1 is snc. And the last step to resolve this singularity can be done by several exercises, which we leave it to the reader.

□

References

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