HH, HKR, THH, and BMS2

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Abstract

This expository note has two goals. We first introduce the Hochschild homology of a ring, and study the degeneracy and the non-degeneracy of its HKR spectral sequence following [ABM]. After that, we introduce the Topological Hochschild homology, and give a detailed computation of the topological Hochschild homology for perfectoid rings and general quasi-regular semi-perfectoid rings, following [BMS2, Section 6-8].

Contents

Conventions

Depending on the focus of each section, both of the notations π_* and H^* appear in this note. We fix the convention that $\pi_i = \mathrm{H}^{-i}$ for $i \in \mathbb{Z}$. Under the identification, the homological canonical truncation $\tau_{\leq i}$ is the same as the cohomological canonical truncation $\tau^{\geq -i}$.

1 HH and its basic properties

The Hochschild homology was invented as a generalization of the differential forms to non-commutative geometry. In this section, we introduce the Hochschild homology together with its basic properties.

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1.1 Conctruction

Let k be a commutative ring. For an associative k-algebra A, we define the *Hochschild complex of A over k* to be

$$\mathrm{HH}(A/k) := A \otimes_{A \otimes_k^L A^{\mathrm{op}}} A^{\mathrm{op}},$$

as an algebra object in the derived category $\mathcal{D}(k)$ of k-modules.

Remark 1.1.1. When A is commutative, the ring A^{op} is equal to A itself, and the Hochschild complex is the self intersection of the diagonal Δ in the product $X \times X$, for the affine scheme $X = \mathrm{Spec}(A)$.

The Hochschild homology can be written more explicitly as below. Recall the *bar complex* of the left $A \otimes_k^L A^{\mathrm{op}}$ algebra A is defined as

$$B_\bullet(A/k) := \quad A \otimes_k^L A \xleftarrow{b'} A \otimes_k^L A \otimes_k^L A \xleftarrow{b'} \cdots,$$

where the map b' is given by (in the case when A is flat over k)

$$a_0 \otimes \cdots a_n \longmapsto \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_{i-1}a_i \otimes \cdots \otimes a_n.$$

The bar complex has a natural $A \otimes_k^L A^{\mathrm{op}}$-linear structure by acting A (resp. A^{op}) on the far left (resp. far right) component from the left (resp. right). It is acyclic except at the degree 0 where the 0-th homotopy group is A. So bar complex $B_\bullet(A/k)$ gives a flat resolution of A as a left $A \otimes_k^L A^{\mathrm{op}}$-algebra, and we get a natural quasi-isomorphism

$$A \otimes_{A \otimes_k^L A^{\mathrm{op}}} B_\bullet(A/k) \longrightarrow \mathrm{HH}(A/k).$$

By writing down the tensor product explicitly, we obtain a model for the Hochschild homology as below:

$$\mathrm{HH}(A/k) \cong \left(A \xleftarrow{b} A \otimes_k^L A \xleftarrow{b} A \otimes_k^L A \otimes_k^L A \longleftarrow \cdots \right)$$

where the map b is given by (in the case when A is flat over k)

$$b(a_0 \otimes \cdots a_n) = a_0a_1 \otimes \cdots \otimes a_n - a_0 \otimes a_1a_2 \otimes \cdots \otimes a_n + \cdots + (-1)^n a_0 \otimes \cdots \otimes a_{n-1}a_n + (-1)^{n+1} a_na_0 \otimes \cdots \otimes a_{n-1}.$$

Moreover, when A is commutative, the derived tensor product $A \otimes_k^L A$ above can be represented by the non-derived tensor product $P_\bullet \otimes_k P_\bullet$ for a simplicial flat k-algebra resolution $P_\bullet \to A$.

From the construction above, the homotopy groups of $\mathrm{HH}(A/k)$ of lower degrees are quite computatble.

Example 1.1.2. Let A be a flat k-algebra.

(i) The 0-th Hochschild homology $\pi_0\mathrm{HH}(A/k)$ is $A/[A, A]$.

(ii) Assume A is commutative. Then we have $\pi_1\mathrm{HH}(A/k)$ is naturally isomorphic to the Kähler differential $\Omega^1_{A/k}$, via the map

$$\pi_1\mathrm{HH}(A/k) \longrightarrow \Omega^1_{A/k};\ a \otimes b \longmapsto a \cdot db.$$

1.2 Properties

We then introduce some basic properties of the Hochschild homology.

Proposition 1.2.1 (Étale base change)**.** *Let A be a commutative k-algebra, and let A' be an étale A-algebra. Then the following canonical map is a quasi-isomorphism*

$$\mathrm{HH}(A/k) \otimes_A A' \longrightarrow \mathrm{HH}(A'/k).$$

Proposition 1.2.2 (Transitivity). *Let $A \to B$ be a map of k-algebra such that A is commutative. Then the functoriality of the Hochschild homology induces the following natural quasi-isomorphism*

$$\mathrm{HH}(B/k) \otimes^L_{\mathrm{HH}(A/k)} A \longrightarrow \mathrm{HH}(B/A).$$

In the case when A is commutative, the Hochschild homology encodes the information of the Kähler differentials. More precisely, we have the following result.

Theorem 1.2.3 (Hockschild-Kostant-Rosenberg). *Let A be a commutative k-algebra. Then the Hochschild homology $\mathrm{HH}(A/k)$ admits a natural complete descending $\mathbb{N}$-indexed filtration whose i-th graded pieces is $\mathbb{L}^i_{A/k}[i]$, for $i \geq 0$. In particular, when A/k is smooth, we get a natural isomorphism*

$$\Omega^i_{A/k} \cong \mathrm{HH}_i(A/k), \ i \geq 0.$$

Here $\mathbb{L}^i_{A/k} = L \wedge^i \mathbb{L}_{A/k}$ is the i-th cotangent complex for A/k, which is a derived generalization of the i-th Kähler differential to all algebras over k, not necessary smooth.

The filtration above, which is the called the *HKR filtration*, is defined as follows. Pick any simplicial k-polynomials resolution $P_\bullet$ of A. Then the Postnikov filtration on each $\mathrm{HH}(P_n/k)$ (defined by the canonical truncation $\tau_{\leq i}$ on the Hochschild homology $\mathrm{HH}(P_n/k)$) induces a filtration on $\mathrm{HH}(A/k)$, whose i-th graded pieces is quasi-isomorphic to the complex $\mathrm{HH}_i(P_\bullet/k)[i]$. It can be showed that the filtration is independent of the choice of $P_\bullet$.[1]

1.3 The action by sphere

As seen above, the Hochschild homology serves as a non-commutative analogue of Kähler differential forms of all degrees in the non-commutative world. It is then natural to ask if we cam also define a version of the differential operator, generalizing the natural differential operator $d : \Omega^i_{A/k} \to \Omega^{i+1}_{A/k}$ for a commutative k-algebra A. In fact, there exists such a differential operator, coming from a natural operator by the sphere.

More precisely, the Hochschild complex is equipped with a natural S^1-action. This can be seen as follows: As an object in the symmetric monoidal ∞-category $\mathrm{CAlg}(\mathcal{D}(k))$, the derived tensor product $A \otimes^L_{A \otimes^L_k A^{\mathrm{op}}} A^{\mathrm{op}}$ can be regarded as the tensor product

$$A \otimes^L_k \operatorname{colim} \left(\begin{array}{ccc} * \sqcup * & \longrightarrow & * \\ \downarrow & & \\ * & & \end{array} \right)$$

Note that as a point $*$ is quasi-isomorphic to a line, the colimit of the diagram $\begin{array}{ccc} * \sqcup * & \longrightarrow & * \\ \downarrow & & \\ * & & \end{array}$ is nothing but identifying the two ends of a line together, which is exactly S^1. So this gives the identification

$$\mathrm{HH}(A/k) := A \otimes^L_{A \otimes^L_k A^{\mathrm{op}}} A^{\mathrm{op}} \cong A \otimes^L_k S^1.$$

Here the tensor product $A \otimes^L_k S^1$ is the object in $\mathrm{CAlg}(\mathcal{D}(k))$ satisfying for any $B \in \mathrm{CAlg}(\mathcal{D}(k))$, we have a weak equivalence of mapping spaces (Kan complexes)

$$\mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}(A \otimes_k S^1, B) \cong \mathrm{Map}_{Space}(S^1, \mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}(A, B)).$$

[1] In a slightly fancier language, we can first define the Postnikov filtration on $\mathrm{HH}(P/k)$ for smooth algebras P over k. Then the HKR filtration is the *left Kan extension* of this filtration from smooth k-algebras to all (simplicial) k-algebras.

As S^1 is a topological group, the action of S^1 on itself induces an action of S^1 on $X = \mathrm{HH}(A/k) \cong A \otimes_k^L S^1$. Namely we have a map of ring object in the infinity category *Space* of Kan complexes

$$S^1 \longrightarrow \mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}(X, X).$$

Note that since the right side above is a commutative ring object in the *Space*, the adjoint map to the forgetful functor from k-algebras to sets induces a map of commutative ring objects

$$C_*(S^1, k) \longrightarrow \mathrm{Map}_{\mathrm{CAlg}(\mathcal{D}(k))}(X, X),$$

where the left side is the k-linear singular complexes of S^1. Note that since S^1 is connective with $\pi_1(S^1) = \mathbb{Z}$, and $\pi_i(S^1) = 0$ for $i \geq 2$, the chain complex $C_*(S^1, k)$ is quasi-isomorphic to the ring $k[\epsilon]/\epsilon^2$ in $\mathrm{CAlg}(\mathcal{D}(k))$, where ϵ is of homological degree 1. So explicitly the S^1-action on $X = \mathrm{HH}(A/k)$ is equivalent to a map $\epsilon : X \to X[1]$ of complexes, such that $\epsilon^2 = 0$.

The operator ϵ on the Hochschild homology is called the *Connes operator*. It turned out that this is exactly the relative differential operator:

Proposition 1.3.1 ([Mor19], Remark 2.12)**.** *Let A/k be a smooth commutative algebra. Then the S^1-action on $\mathrm{HH}(A/k)$ induces a commutative differential algebra structure on the homology ring $\pi_*\mathrm{HH}(A/k)$. Moreover, the universal property of the de Rham complex induces the following natural commutative diagram with vertical maps being isomorphisms:*

$$\begin{array}{ccc} \Omega^n_{A/k} & \xrightarrow{d} & \Omega^{n+1}_{A/k} \\ \downarrow & & \downarrow \\ \pi_n\mathrm{HH}(A/k) & \xrightarrow{\epsilon} & \pi_{n+1}\mathrm{HH}(A/k). \end{array}$$

Remark 1.3.2. Here we note that the tensor product $A \otimes_k^L S^1$ of an object $A \in \mathrm{CAlg}(\mathcal{D}(k))$ with a topological space S^1 above can be defined in a very general framework, where we can replace $\mathcal{D}(k)$ by any symmetric monoidal presentable ∞-category $\mathcal{C}$ and S^1 by any topological space.

2 (Non)degeneracy of HKR filtration

Recall in the last section that there exists a natural filtration on $\mathrm{HH}(A/k)$ to all commutative algebras A over k, defined as extending the Postnikov filtration of $\mathrm{HH}(P/k)$ for smooth k-algebras P. This induces the *HKR spectral sequence* as below

$$E_2^{s,t} = \mathrm{H}^s(L \wedge^t \mathbb{L}_{A/k}) \Longrightarrow \mathrm{H}^{s+t}(\mathrm{HH}(A/k)).$$

A natural question is when this spectral sequence degenerates.

2.1 General results

Degeneracy In the characteristic zero, the spectral sequence degenerates canonically in the derived category. Precisely, we have the following:

Proposition 2.1.1. *Let k be a field of characteristic zero. Then for any k-algebra A, there exists a canonical quasi-isomorphism between the Hochschild homology $\mathrm{HH}(A/k)$ and the derived wedge algebras*

$$\bigoplus_{i \in \mathbb{N}} \mathrm{Sym}^i_A(\Omega^1_{A/k}[1]) \cong \bigoplus_{i \in \mathbb{N}} L \wedge^i \mathbb{L}_{A/k}[i].$$

In particular, the HKR spectral sequence for $\mathrm{HH}(A/k)$ degenerates.

The idea of the proof is to use the S^1-action and introduce the Adams operator on $\mathrm{HH}(A/k)$, which induces a decomposition of the Hochschild homology into different eigenspaces in a canonical way.

Proof. We first assume A is smooth over k.

Recall that the Hochschild homology $\mathrm{HH}(A/k)$ is equivalent to the tensor product $A \otimes S^1$, that comes with a natural action by the circle S^1. We consider the endomorphism $S^1 \to S^1$ given by $z \mapsto z^2$. This induces an endomorphism $\psi_2 : \mathrm{HH}(A/k) \to \mathrm{HH}(A/k)$, called *Adams operator*. Moreover, this allows us to regard $\mathrm{HH}(A/k)$ to be a natural object in the (∞) category of connective commutative ring in $D(k[k])$, where t acts on HH by the Adams operator.

Then we notice that the first Hochschild homology $\pi_1\mathrm{HH}(A/k)$ is generated by $\mathrm{H}_1(S^1, \mathbb{Z})$ under the following map

$$\pi_0\mathrm{HH}(A/k) \otimes_k \mathrm{H}_1(S^1, \mathbb{Z}) \longrightarrow \pi_1\mathrm{HH}(A/k),$$

where the generator of $\mathrm{H}_1(S^1, \mathbb{Z})$ is the k-derivative operator. Furthermore, by applying the Adams operator onto $\pi_1\mathrm{HH}(A/k)$, we see the first Hochschild homology is the eigenspace of value 2. By the multiplicativity, since $\pi_*\mathrm{HH}(A/k)$ is generated by $\pi_1\mathrm{HH}(A/k)$, we see $\pi_i\mathrm{HH}(A/k)$ is the eigenspace of ψ_2 of eigenvalue 2^i.

At last, notice the assumption that k is of characteristic zero. In this way, since $\pi_i\mathrm{HH}(A/k)$ is supported over $t - 2^i$ in $\mathrm{Spec}(k[t]) = \mathbb{A}^1_k$, we get

$$\mathrm{HH}(A/k) = \bigoplus_{i \in \mathbb{N}} \pi_i\mathrm{HH}(A/k)[i].$$

So the results follows by a standard left Kan extension argument to the general case.

□

Remark 2.1.2. The proof above also implies that when k is a field of characteristic $p > 0$, either assume A is smooth of dimension $\leq p - 1$, or only consider the truncated object $\mathrm{HH}(A/k)/\mathrm{Fil}^i_{HKR}$, we will still get the decomposition (notice that we may change 2 to some other prime numbers if necessary).

Nondegeneracy In the positive characteristic case, the degeneracy fails in general. This was first observed by Antieau-Bhatt-Mathew in [ABM]. Their main result is the following.

Theorem 2.1.3 ([ABM], Theorem 1.1)**.** *Let k be a perfect field of characteristic $p > 0$. Then there exists a smooth projective variety X of dimension $2p$ over k such that the HKR spectral sequence of* $\mathrm{HH}(X/k)$ *does not degenerate.*

We now follow [ABM] and explain how to construct such an example.

We assume k to be a field of characteristic $p > 2$ throughout the rest of this subsection. To simplify the notation, for a scheme/syntomic stack X over k, we denote by $\mathbb{L}^i_X$ to be the i-th derived wedge product $L\wedge^i \mathbb{L}_{X/k}$ of its cotangent complex over k. Following [ABM], we use $E(x)$ to denote the k-exterior algebra generated by the variable x, and $P(y)$ the k-polynomial generated by the variable y.

The idea is essentially about studying the Hodge cohomology and Hochschild cohomology of certain stacks in positive characteristic. It turns out that even for the classifying stack $B\mu_p$ and $B\alpha_p$, their HKR spectral sequences already do not degenerate. In this way, by a technique of approximating the stack by smooth projective varieties, we could eventually get examples in the scheme case.

Remark 2.1.4 (Non-degeneracy for $B\mu_p$)**.** The non-degeneracy of Hochschild homology already appears when we consider the classyfying stack $B\mu_p$.

On the one hand, pointed to the author by Bhatt, the Hochschild homology $\mathrm{HH}(\mathscr{X}/k)$ of a syntomic stack $\mathscr{X}$ admits a natural map from the *Hochschild homology* $\mathrm{HH}(\mathscr{D}(\mathscr{X}/k)$ *of the derived (∞)-category of quasi-coherent sheaves over* $\mathscr{X}$, which exhibits $\mathrm{HH}(\mathscr{X}/k)$ as the *Atiyah-Sagal completion* of $\mathrm{HH}(\mathscr{D}(\mathscr{X})/k)$. In the case of schemes $\mathscr{X}$, the two Hochschild homology coincide. When $\mathscr{X} = B\mathbb{G}_m$ or $B\mu_p$, as the categories of quasi-coherent sheaves over $B\mathbb{G}_m$ and $B\mu_p$ are equivalent to the categories of $\mathbb{Z}$-graded and $\mathbb{Z}/p$-graded k-vector spaces, their Hochschild homology are the same as the Hochschild homology of disjoint union of $\mathbb{Z}$ and p copies of single points separately, which are thus living in the cohomological degree zero.

On the other hand, by a computation similar to the Theorem 2.2.1.(i) below using the co-Lie complex, the Hodge-cohomology of $B\mu_p$ has higher terms. So we get the non-degeneracy for $B\mu_p$, assuming the above fact about the Hochschild homology of (stable ∞) categories.

2.2 Cohomology of $B\alpha_p$

We first give the computational results for the Hodge cohomology, de Rham cohomology and Hochschild cohomology of $B\alpha_p$.

To compute cohomology of $B\alpha_p$, we will need to make use of the canonical action by $\mathbb{G}_m$. Recall that there exists a natural action of $\mathbb{G}_m$ on the affine line $\mathbb{A}^1 = \mathrm{Spec}(k[t])$ with t being of weight one, which leaves the subgroup α_p invariant and gives an action of $\mathbb{G}_m$ on α_p. This induces an action on $B\alpha_p$ and its Hodge, de Rham and Hochschild cohomology. In particular, by the functoriality, the differential maps in various spectral sequences of $B\alpha_p$ we consider will preserve the weight of the $\mathbb{G}_m$-action, and we get a natural grading on cohomology of $B\alpha_p$.

Now we state the calculation for $B\alpha_p$ as below.

Theorem 2.2.1 ([ABM], 4.10-4.13)**.** *(i) The Hodge cohomology ring of $B\alpha_p$ is isomorphic to the ring $E(\alpha)\otimes P(\beta)\otimes E(u)\otimes P(v)$, where $\alpha \in \mathrm{H}^1(B\alpha_p, \mathcal{O})$, $\beta \in \mathrm{H}^2(B\alpha_p, \mathcal{O})$, $u \in \mathrm{H}^0(B\alpha_p, \mathbb{L}_{B\alpha_p})$, and $v \in \mathrm{H}^1(B\alpha_p, \mathbb{L}_{B\alpha_p})$. Moreover, the weights of α, β, u, v are $1, p, p, 1$ separately.*

(ii) The de Rham cohomology ring of $B\alpha_p$ is isomorphic to $E(u') \otimes P(\beta')$, where α' is of cohomological degree 1 and weight p, and β' is of degree 2 and weight p. In particular, both the Hodge-de Rham and the conjugate spectral sequence do not degenerate.

(iii) The Hochschild cohomology ring of $B\alpha_p$ is isomorphic to $E(u) \otimes P(\beta) \otimes k[v]/v^p$, where α is of cohomological degree 1 weight 1, β of degree 2 and weight p, u is of degree 0 and weight 1. In particular, the HKR spectral sequence of $B\alpha_p$ does not degenerate.

Computation

(i) We first consider how to compute $\mathrm{H}^*(B\alpha_p, \mathcal{O})$.

For a finite flat commutative group scheme G, there exists a natural equivalence of the categories

$$\mathrm{Coh}(BG) \cong \mathrm{coMod}_k^f(\mathcal{O}_G) = (\mathrm{Mod}_k^f(\mathcal{O}_{G^\vee}))^{\mathrm{op}}.$$

Moreover, the structure sheaf $\mathcal{O}_{B\alpha_p}$ under this equivalence is transformed into the trivial representation of G over k. When $G = \alpha_p$, its Cartier dual is equal to itself, so the derived global section of $\mathcal{O}_{B\alpha_p}$ is computed by the extension group

$$\mathrm{H}^*(B\alpha_p, \mathcal{O}) = \mathrm{Ext}^*_{\mathcal{O}_{\alpha_p^\vee}}(k,k) = \mathrm{Ext}^*_{k[t]/t^p}(k,k).$$

In this way, by the standard resolution of the residue field k over $k[t]/t^p$, we get the isomorphism of graded rings

$$\mathrm{H}^*(B\alpha_p, \mathcal{O}) \cong E(\alpha) \otimes P(\beta),$$

where α is of cohomological degree 1 and weight 1, while β is of cohomological degree 2 and weight p.

To compute the Hodge cohomology, we use a result of Totaro to relate it to the cohomology of co-Lie complex. Here we recall that the *co-Lie complex* $\mathrm{coLie}(G) \in D(Rep_k(G)) = D(BG)$ for a k-group scheme G is defined as the G-representation $i^*\mathbb{L}_G$, where $i : \mathrm{Spec}(k) \to G$ is the identity map for the group scheme G. When $G = \alpha_p$, by computing the cotangent complex for the $\mathbb{G}_m$-equivariant triple $\alpha_p \to \mathbb{A}^1 \to \mathrm{Spec}(k)$, the co-Lie complex of α_p is then isomorphic to the following k-complex

$$\mathrm{coLie}(\alpha_p) = i^*(\ (t^p)/(t^{2p}) \xrightarrow{d} k[t]/t^p \cdot dt\ ) \cong \mathcal{O}_{B\alpha_p} u[1] \bigoplus \mathcal{O}_{B\alpha_p} v[0].$$

Here u is of cohomological degree (-1) and weight p, and v is of cohomological degree 0 and weight 1. And the splitting of $\mathrm{coLie}(\alpha_p)$ in the derived category follows from the observation that the obstruction class of it, which corresponds to an element in $\mathrm{Ext}^2(\mathcal{O}_{B\alpha_p}, \mathcal{O}_{B\alpha_p})$ of weight $p-1$, does not exist by the previous computation.

Now we are able to compute the Hodge cohomology of a classifying stack. The technique we use is the following general fact by Totaro:

Fact 2.2.2. [Tot18, Theorem 3.1] Let G be a flat affine group scheme over k. Then there exists a natural multiplicative graded quasi-isomorphism in the derived category $\mathcal{D}(k)$

$$R\Gamma(BG, \bigoplus_{i\in\mathbb{N}} \mathbb{L}^i_{BG}) \cong R\Gamma(G, \bigoplus_{i\in\mathbb{N}} \mathrm{Sym}^i(\mathrm{coLie}(G))[-i]).$$

Here the right site is the cohomology of G-representation.

When $G = \alpha_p$, by the previous computation, the symmetric algebra on the right side above can be written as

$$\bigoplus_{i\in\mathbb{N}} \mathrm{Sym}^i(\mathrm{coLie}(\alpha_p))[-i] \cong E(u)\otimes P(v)\otimes_k \mathcal{O}_{B\alpha_p} \in D(B\alpha_p).$$

In particular, applying the above fact to $G = \alpha_p$, we get

$$\begin{aligned} \mathrm{H}^*(B\alpha_p, \mathbb{L}^*_{B\alpha_p}) &\cong \mathrm{H}^*(B\alpha_p, \mathcal{O}_{B\alpha_p})\otimes_k \cong E(u)\otimes P(v) \\ &\cong E(\alpha)\otimes P(\beta)\otimes E(u)\otimes P(v). \end{aligned}$$

So we are done.

(ii) We then consider the de Rham cohomology.

Recall the *Hodge filtration* is a $\mathbb{N}$-indexed, complete descending filtration on the derived de Rham complex, which induces the following E_1-spectral sequence

$$E_1^{s,t} = \mathrm{H}^t(B\alpha_p, \mathbb{L}^s_{B\alpha_p}) \Longrightarrow \mathrm{H}^{s+t}_{\mathrm{dR}}(B\alpha_p/k).$$

There exists another $\mathbb{N}$-indexed, increasing exhaustive filtration on the derived de Rham complex, which exists only for positive characteristic and is called the *conjugate filtration*, which induces an E_2-spectral sequence

$$E_2^{s,t} = \mathrm{H}^s(B\alpha_p^{(1)}, \mathbb{L}^t_{B\alpha_p^{(1)}}) \Longrightarrow \mathrm{H}^{s+t}_{\mathrm{dR}}(B\alpha_p/k).$$

Here $B\alpha_p^{(1)}$ is the Frobenius twist of $B\alpha_p$, and is abstractly (and k-linearly through the Frobenius) isomorphic to $B\alpha_p$. In particular, by the finite dimensionality of the Hodge cohomology, the degeneracy of one of the above spectral sequences would imply the another.

Now as $B\alpha_p^{(1)}$ is twisted by the Frobenius, the $\mathbb{G}_m$ action on the abutment of the conjugate spectral sequence all has weights divisible by p.[2] In particular, the conjugate spectral sequence implies that the weights of de Rham cohomology of $B\alpha_p$ are all p-divisible. On the other hand, in the Hodge-de Rham spectral sequence, both the cohomological classes α and v are of weight 1. So by the $\mathbb{G}_m$-equivariance of the Hodge-de Rham spectral sequence, after computing a

[2]To see this, we need to notice that the only possible $\mathbb{G}_m$-action on $X^{(1)}$ for $X = \mathring{\mathbb{A}}^1$ that is compatible with the Cartier isomorphism is the p-th power action, which can be seen from the following diagram of $\mathrm{Spec}(k)$-schemes

$$\begin{array}{ccc} X^{(1)}\times\mathbb{G}_m & \longrightarrow & X^{(1)} \\ \uparrow{\scriptstyle (c,id)} & & \uparrow{\scriptstyle c} \\ X\times\mathbb{G}_m & \longrightarrow & X. \end{array}$$

finite amount of images of v and α under differential maps, we see α has to map onto (up to a unit) the element v. Similarly, the classes β and u will be kept. As an upshot, we get

$$\mathrm{H}^*_{\mathrm{dR}}(B\alpha_p/k) \cong P(\beta') \otimes E(u'),$$

where β' is of cohomological degree 1 and weight p, and u' is of degree 2 and weight p.

(iii) At last, we compute the Hochschild homology.

We first notice that by the construction, there exists a natural map from $R\Gamma(B\alpha_p, \mathcal{O})$ to $R\Gamma(B\alpha_p, \mathrm{HH})$ that admits a section by choosing a projection from the sphere S^1 to a point.[3] In particular, the class α and β coming from the cohomology of the structure sheaf $\mathcal{O}_{B\alpha_p}$ will leave invariant. So the cohomology algebra $\mathrm{H}^*(\mathrm{HH}(B\alpha_p/k))$ is a graded algebra over $\mathrm{H}^*(B\alpha_p, \mathcal{O}) = E(\alpha) \otimes P(\beta)$.

It is left to consider the class u and v. We first notice that as the Frobenius morphism of α_p factors though $\mathrm{Spec}(k)$, the induced endomorphism on the Hochschild homology kills both u and v. On the other hand, pointed by [ABM, Page 10, footnote], the induced morphism on the zero-th Hochschild homology $\mathrm{H}^0(\mathrm{HH}(B\alpha_p/k))$ coincides with the p-th power map. This implies that the element v^p, which is in the abutment $E_2^{p,-p}$ and is of weight p, will be killed in the Hochschild homology. Note that by looking at the weights of lower terms, the only weight p elements that could map onto the element v^p in the spectral sequence is u (up to a unit) via the differential d_p. In this way, in the E_{p+1}-page of the HKR spectral sequence, the only possiblely alive elements are permanent sub-algebra $E(\alpha) \otimes P(\beta)$ together with $c, c^2, \ldots, c^{p-1}$, which by degree reason will leave invariant. So we are done.

2.3 Approximations by schemes

In this subsection, we introduce the technique of approximating the classifying stack by schemes. We will use this to finish the proof of the Theorem 2.1.3, showing the existence of schematic examples where HKR spectral sequences fails to degenerate.

We first introduce the notion of the Hodge d-equivalence.

Definition 2.3.1. *Let $\mathscr{X} \to \mathscr{Y}$ be a map of syntomic stacks over k. We call it is a* Hodge d-equivalence *if the cofiber for the induced morphism $R\Gamma(\mathscr{Y}, \mathbb{L}^s_{\mathscr{Y}}) \to R\Gamma(\mathscr{X}, \mathbb{L}^s_{\mathscr{X}})$ lives in $\mathcal{D}(k)^{\geq d-s}$, for each $s \in \mathbb{N}$.*

Note that by applying the conjugate spectral sequence, the Hodge d-equivalence implies that the cofiber of $R\Gamma_{\mathrm{dR}}(\mathscr{Y}) \to R\Gamma_{\mathrm{dR}}(\mathscr{X})$ lives in $\mathcal{D}(k)^{\geq d}$. (Here you might be wondering why don't we use the Hodge-de Rham spectral sequence instead, which is left as an exercise.)

The condition is satisfied for example for a complete intersection of in $\mathbb{P}^n$, and is equivariant under the group quotient. Precisely we have the following results.

Proposition 2.3.2 ([ABM], Proposition 5.3)**.** *Let X be a d-dimensional complete intersection in the projective space $\mathbb{P}^n_k$. Then $X \to \mathbb{P}^n_k$ is a Hodge d-equivalence.*

Here we note that the Proposition 2.3.2 should be a singular generalization of the classical weak Lefschetz theorem, proved for example in SGA7.

Proposition 2.3.3 ([ABM], Proposition 5.10)**.** *Let $\mathscr{X} \to \mathscr{Y}$ be a Hodge d-equivalence of syntomic k-schemes that is equivariant under an action by an affine k-group G of finite type. Then the induced quotient map $[\mathscr{X}/G] \to [\mathscr{Y}/G]$ is also a Hodge d-equivalence.*

Assuming the above, we can give a proof for the Theorem 2.1.3.

Theorem 2.3.4. *Let G be an affine k-group scheme which is either finite or geometrically reductive. Then for any $d \geq 0$, there exists a smooth projective k-schemes X of dimension d together with a map $X \to BG$, such that the induced natural map $\mathrm{H}^s(BG, \mathbb{L}^t_{BG}) \to \mathrm{H}^s(X, \mathbb{L}^t_X)$ is injective for $s + t \leq d$.*

[3]This could also be seen from the definition $\mathrm{HH}(A/k) = A \underset{A\otimes_k A}{\otimes^L} A$.

Proof. We first assume G is finite. Then by a Bertini type result, we can find a finite dimensional representation V of G together with a d-dimensional complete intersection $X \subset \mathbb{P}(V)$, such that X is G-stable and G acts freely on X, with the quotient $X/G = [X/G]$ being smooth and projective. This provides us with the following diagram

$$\begin{array}{ccc} X/G & \xrightarrow{f} & [\mathbb{P}(V)/G] \\ & & \downarrow{\scriptstyle g} \\ & & BG, \end{array}$$

where f is a Hodge d-equivalence by Proposition 2.3.2 and Proposition 2.3.3, and $g : [\mathbb{P}(V)/G] \to BG$ is a projective bundle morphism that induces an injection on Hodge cohomology. So we are done in this case.

We then consider the case of geometrically reductive group. (Where do we need "geometrically"?) Denote by G_r to be the Frobenius kernel of G. Then the tower of closed immersions $\{G_r\}$ produces the following chain of co-Lie complexes

$$\mathrm{coLie}(G) \longrightarrow \cdots \longrightarrow \mathrm{coLie}(G_r) \longrightarrow \mathrm{coLie}(G_{r-1}) \longrightarrow \cdots \longrightarrow \mathrm{coLie}(G_1).$$

By the assumption of the geometric reductiveness of G, each transition map $\mathrm{coLie}(G_r) \to \mathrm{coLie}(G_{r-1})$ is an isomorphisms on its zero-th cohomology and is zero on its (-1)-th cohomology. So for each i, we get the quasi-isomorphisms

$$R\lim R\Gamma(BG_r, \mathrm{Sym}^i \mathrm{coLie}(G_r)) \cong R\lim R\Gamma(BG_r, \mathrm{H}^0(\mathrm{Sym}^i \mathrm{coLie}(G_r))) \cong R\lim R\Gamma(BG_r, \mathrm{Sym}^i \mathrm{coLie}(G)).$$

At last, to finish the proof of this case, we recall the following general fact about finite dimensional representation of G and its Frobenius kernels.

Fact 2.3.5. [Jan87, Corollary II4.12] Let G be a reductive group over a perfect field k of positive characteristic, and let G_r be its r-th Frobenius kernel. Then for a finite dimensional representation V of G over k, we have

$$\mathrm{H}^i(G, V) \cong \varprojlim_r \mathrm{H}^i(G_r, V), \ \ i \in \mathbb{N}.$$

In this way, by the finite dimensionality of $\mathrm{H}^i(G, \mathrm{Sym}^i \mathrm{coLie}(G))$ and $\mathrm{H}^i(G_r, \mathrm{Sym}^i \mathrm{coLie}(G))$, we get

$$R\Gamma(BG, \mathrm{Sym}^i \mathrm{coLie}(G)) \cong \lim R\Gamma(BG_r, \mathrm{Sym}^i \mathrm{coLie}(G_r)).$$

Note that by the Fact 2.2.2 we have

$$\mathrm{H}^s(BG, \mathbb{L}^t_{BG}) \cong \varprojlim_r \mathrm{H}^s(BG_r, \mathbb{L}^t_{BG_r}).$$

Hence the result reduces to the case of finite group schemes. □

Proof of Theorem 2.1.3. We apply the Theorem 2.3.4 to $G = \alpha_p$ and $d = 2p$, and let $Y = X/G$ be the quotient scheme. So the result follows from the injectivity of Hodge-cohomology $\mathrm{H}^s(B\alpha_p, \mathbb{L}^t_{B\alpha_p}) \to \mathrm{H}^s(Y, \mathbb{L}^t_Y)$ for $s + t \leq 2p$, and the observation that the class $v \in \mathrm{H}^1(B\alpha_p, \mathbb{L}^p_{B\alpha_p})$ is the image of α under the differential $d_p : \mathrm{H}^0(B\alpha_p, \mathbb{L}_{B\alpha_p}) \to \mathrm{H}^p(B\alpha_p, \mathbb{L}^p_{B\alpha_p})$. So we are done. □

3 THH and its basic properties

Now we introduce the topological Hochschild homology for ring spectra, generalizing the Hochschild homology for rings.

3.1 Construction

Recall that a (connective) *spectrum* $X = X_\bullet$ is a sequence of spaces (Kan complexes) indexed by $\mathbb{N}$, together with maps $\epsilon_n : X_n \to \Omega X_{n+1}$ from X_n to the loop space of X_{n+1}, such that each ϵ_n is a weak equivalence. The collection of spectra Sp forms an infinity category. Let Sp be the infinity category of spectra, and let Alg(Sp) (resp. CAlg(Sp)) be the monoidal (resp. symmetric monoidal) infinity category of (resp. commutative) ring spectra, or the $\mathbb{E}_1$ (resp. $\mathbb{E}_\infty$) algebras in Sp. Then for a map of objects $k \to A$ in Alg(Sp), we can define the *Hochschild homology of spectra* to be

$$\mathrm{HH}(A/k) := |\, A \leftleftarrows A \otimes_k A \,\substack{\leftarrow\\ \leftarrow \\ \leftarrow}\, A \otimes_k A \otimes_k A \quad \cdots |\, .$$

When k is the sphere spectrum $\mathbb{S}$ (the unit object in Alg(Sp)), we have

$$\mathrm{HH}(A/\mathbb{S}) := |\, A \leftleftarrows A \otimes_{\mathbb{S}} A \,\substack{\leftarrow\\ \leftarrow \\ \leftarrow}\, A \otimes_{\mathbb{S}} A \otimes_{\mathbb{S}} A \quad \cdots |\, .$$

In this case, we call $\mathrm{HH}(A/\mathbb{S})$ the *topological Hochschild spectrum* of A, and denote it by $\mathrm{THH}(A)$.

Here we note that similar to the case of the Hochschild complex, the $\mathrm{THH}(A)$ can be defined as the product

$$A \otimes_{\mathbb{S}} S^1,$$

where the object $A \otimes_{\mathbb{S}} S^1$ satisfies the required adjunction condition of mapping spaces:

$$\mathrm{Map}_{\mathrm{CAlg(Sp)}}(A \otimes_{\mathbb{S}} S^1, B) \cong \mathrm{Map}_{Space}(S^1, \mathrm{Map}_{\mathrm{CAlg(Sp)}}(A, B)),$$

where B is an E_∞-ring spectrum. In particular, the $\mathrm{THH}(A)$ is equipped with a natural S^1-action. Moreover, the topolocical Hochschild spectrum is connective, namely it has no cohomology of positive degrees.

Example 3.1.1. Let k be an ordinary commutative ring, and let A be a k-algebra. We abuse A and k to denote the Eilenberg-Maclane spectrum of A and k separately, which comes with a natural structure of ring spectra. Then the Hochschild spectrum of A/k is the quasi-isomorphic to the Eilenberg-Maclane spectrum of the Hochschild complex of A/k.

3.2 Relation to HH

Recall that the sphere spectrum $\mathbb{S}$ is connective such that $\pi_0(\mathbb{S})$ is the ring of integers $\mathbb{Z}$. This provides us with a map of commutative ring spectra $\mathbb{S} \to \mathbb{Z}$ (where the latter is the Eilenberg-Maclane spectrum of $\mathbb{Z}$), together with a natural restriction (forgetful) functor from monoidal ∞-category of $\mathbb{Z}$-algebras to that of $\mathbb{S}$-algebras. Here the restriction functor can be extended to the symmetric monoidal ∞-category of commutative ring objects in the derived infinity category $\mathcal{D}(\mathbb{Z})$ and $\mathcal{D}(\mathbb{S})$. In particular, for any ordinary ring A, the restriction map allows us to regard it as a $\mathbb{S}$-algebra and compute the topological Hochschild complex $\mathrm{THH}(A) = \mathrm{HH}(A/\mathbb{S})$. Besides, the functoriality for $\mathbb{S} \to \mathbb{Z}$ induces a map

$$\mathrm{THH}(A) \longrightarrow \mathrm{HH}(A/\mathbb{Z}).$$

In fact, the Proposition 1.2.2 of transitivity can be extended to the ring spectra, and we have the following quasi-isomorphism

$$\mathrm{THH}(A) \otimes_{\mathrm{THH}(\mathbb{Z})} \mathbb{Z} \longrightarrow \mathrm{HH}(A/\mathbb{Z}).$$

Here we note that by taking the p-completion of spectra, we can get a p-adic completed version of the base change formula.[4]

We can then take the homology to get the map $\alpha_n : \mathrm{THH}_n(A) \longrightarrow \mathrm{HH}_n(A/Z)$ for $n \in \mathbb{N}$. Then we have the following two basic facts about their homologies:

[4] We want to mention that though the Hochschild homology $\mathrm{HH}(A/\mathbb{Z})$ can be regraded as either an E_∞-spectrum, or a commutative ring object in the derived ∞-category of abelian groups, the p-adic completions in those two ∞-categories are compatible. This can be seen by the given tensor product formula and the following commutative

(i) The map α_n is an isomorphism for $n \leq 2$.

(ii) For each α_n, its kernel and the cokernel are killed by integers that are depending on n.

Here the observations is induced from the following fact of the stable homotopy groups of spheres:

Fact 3.2.1. The homotopy groups of sphere spectrum $\mathbb{S}$ are given by

$$\pi_n(\mathbb{S}) = \begin{cases} 0, \ n < 0; \\ \mathbb{Z}, \ n = 0; \\ n - th \ stable \ homotopy \ group \ of \ spheres, \ n > 0. \end{cases}$$

In particular, $\pi_n(\mathbb{S})$ are torsion for $n \geq 1$.

Remark 3.2.2. The philosophy of considering the restriction from $\mathbb{Z}$ to $\mathbb{S}$ has more general analogues: the inclusion $\mathbb{R} \to \mathbb{C}$ allows us to give a complex manifold the complex strucutre, and the inclusion $\mathbb{Q} \to K$ for a finite Galois extension $K/\mathbb{Q}$ gives us the Galois structure.

4 THH and its brothers of perfectoid ring

In this section, we compute the p-completed topological Hochschild homology of a perfectoid ring, following the Section 6 of [BMS2]. The main idea is to use the computation of $\mathrm{THH}(\mathbb{F}_p)$ by Bökstedt and the formal properties of THH to deduce the general cases.

4.1 THH of a perfectoid ring

We first compute the p-completed topological Hochschild homology $\mathrm{THH}(R; \mathbb{Z}_p)$ of a perfectoid ring R.

For a ring R, we denote by $\mathrm{THH}(R; \mathbb{Z}_p)$ and $\mathrm{HH}(R; \mathbb{Z}_p)$ to be the p-completion of $\mathrm{THH}(R)$ and $\mathrm{HH}(R)$ separately, regarded as E_∞-rings.

The first main result is the following.

Theorem 4.1.1 (Theorem 6.1, [BMS2])**.** *Let R be a perfectoid ring. Then the graded algebra $\pi_*\mathrm{THH}(R; \mathbb{Z}_p)$ is isomorphic to a polynomial ring $R[u]$ over R, where $u \in \pi_2\mathrm{THH}(R; \mathbb{Z}_p) \cong \pi_2\mathrm{HH}(R; \mathbb{Z}_p) \cong \ker(\theta)/\ker(\theta)^2$ is a generator of degree 2 over R.*

Proof.

Step 1 We first notice that the p-completed Hochschild homology of a perfectoid ring R is naturally described as follows

$$\pi_i\mathrm{HH}(R; \mathbb{Z}_p) \cong \begin{cases} R, \ 2|i \geq 0; \\ 0, \ else. \end{cases}$$

To see this, we recall from the HKR filtration in the Theorem 1.2.3 that $\mathrm{HH}(R; \mathbb{Z}_p)$ admits a decreasing $\mathbb{N}$-filtration such that i-th graded piece is quasi-isomorphic to the derived p-completion of the $(L \wedge^i \mathbb{L}_{R/\mathbb{Z}})[i]$. To compute this cotangent complex, we use the natural triple

$$\mathbb{Z} \longrightarrow \mathrm{A_{inf}}(R) \to R,$$

diagram

$$\begin{array}{ccc} \mathrm{Sp} & \xrightarrow{p-completion} & \mathrm{Sp}_p^\wedge \\ \downarrow{\scriptstyle \otimes_{\mathrm{THH}(\mathbb{Z})}\mathbb{Z}} & & \downarrow{\scriptstyle \otimes_{\mathrm{THH}(\mathbb{Z};\mathbb{Z}_p)}\mathbb{Z}_p} \\ \mathcal{D}(\mathbb{Z}) & \xrightarrow[p-completion]{} & \mathcal{D}_{\mathrm{comp}}(\mathbb{Z}_p) \end{array}.$$

where the latter is the map θ that is a complete intersection with $\ker(\theta)/\ker(\theta)^2$ generated by an element u. The ring $\mathrm{A}_{\mathrm{inf}}(R)/p$ is perfect, so we get

$$(L\wedge^i \mathbb{L}_{R/\mathbb{Z}})^\wedge \cong L\wedge^i (R[1]) \cong R[i],$$

thus the i-th graded piece of $\mathrm{HH}(R;\mathbb{Z}_p)$ is isomorphic to $R[2i]$.

We want to warn the reader that though each homotopy group of $\mathrm{HH}(R;\mathbb{Z}_p)$ looks identical to that of $\mathrm{THH}(R;\mathbb{Z}_p)$ (which we will prove soon), the essential difference of them is the ring structure of $\mathrm{HH}(R;\mathbb{Z}_p)$ and $\mathrm{THH}(R;\mathbb{Z}_p)$: the latter is a polynomial ring, while the first one is divided algebra over R.[5]

Step 2 Next thing we show is the base change formula. Namely, for a map of perfectoid rings $R\to R'$, there exists a natural quasi-isomorphism

$$\mathrm{THH}(R;\mathbb{Z}_p)\otimes_R^L R' \longrightarrow \mathrm{THH}(R';\mathbb{Z}_p).$$

To show this, we consider the base change $-\otimes^L_{\mathrm{THH}(\mathbb{Z})}\mathbb{Z}$. After applying this functor, we get

$$\mathrm{HH}(R;\mathbb{Z}_p)\otimes_R^L R' \longrightarrow \mathrm{HH}(R';\mathbb{Z}_p),$$

which is a quasi-isomorphism by the Step 1. Moreover, by induction and the fact that $\pi_i\mathrm{THH}(\mathbb{Z})$ is finite abelian, we can show that the truncated map below is a quasi-isomorphism

$$(\mathrm{THH}(R;\mathbb{Z}_p)\otimes_R^L R')\otimes_{\mathrm{THH}(\mathbb{Z})}\tau_{\leq n}\mathrm{THH}(\mathbb{Z}) \longrightarrow \mathrm{THH}(R';\mathbb{Z}_p)\otimes_{\mathrm{THH}(\mathbb{Z})}\tau_{\leq n}\mathrm{THH}(\mathbb{Z}),\ \forall n\in\mathbb{N}.$$

The rest then follows from the equality

$$\begin{aligned}
\mathrm{THH}(R;\mathbb{Z}_p)\otimes_R^L R' &= \lim_n \tau_{\leq n}(\mathrm{THH}(R;\mathbb{Z}_p)\otimes_R^L R')\\
&\cong \lim_n \big((\mathrm{THH}(R;\mathbb{Z}_p)\otimes_R^L R')\otimes_{\mathrm{THH}(\mathbb{Z})}\tau_{\leq n}\mathrm{THH}(\mathbb{Z})\big)\\
&\cong \lim_n (\mathrm{THH}(R';\mathbb{Z}_p)\otimes_{\mathrm{THH}(\mathbb{Z})}\tau_{\leq n}\mathrm{THH}(\mathbb{Z}))\\
&\cong \lim_n \tau_{\leq n}(\mathrm{THH}(R';\mathbb{Z}_p)\otimes_{\mathrm{THH}(\mathbb{Z})}\mathrm{THH}(\mathbb{Z}))\\
&= \mathrm{THH}(R';\mathbb{Z}_p)\otimes_{\mathrm{THH}(\mathbb{Z})}\mathrm{THH}(\mathbb{Z}).
\end{aligned}$$

The only non-trivial fact we are going to need is the following, which was due to Bökstedt [Bök85]:

Fact 4.1.2. The graded algebra $\pi_*\mathrm{THH}(\mathbb{F}_p)$ is isomorphic to the polynomial ring of one variable in degree 2 over $\mathbb{F}_p$.

This together with the base change formula implies that for any perfect algebra R in characteristic p, the natural map $\mathbb{F}_p\to R$ induces the natural quasi-isomorphism

$$\mathrm{THH}(R;\mathbb{Z}_p) = \mathrm{THH}(\mathbb{F}_p;\mathbb{Z}_p)\otimes^L_{\mathbb{F}_p} R \cong R[u].$$

Step 3 To prepare for the case where R is of mixed characteristic, we want to show that $\mathrm{THH}(R;\mathbb{Z}_p)$ is pseudo-coherent (i.e. it is quasi-isomorphic to a bounded to the right complex of finite free R-modules).

To see this, we use the specialization from THH to HH (Paragraph 3.2)

$$\mathrm{THH}(R;\mathbb{Z}_p)\otimes^L_{\mathrm{THH}(\mathbb{Z})}\mathbb{Z} \cong \mathrm{HH}(R;\mathbb{Z}_p).$$

[5] By a result of Illusie (cf. [Ill72]), for a flat R-module M, there exists an isomorphism of graded rings

$$\pi_* L\wedge^*_R (M[1]) \cong \Gamma^*_R M.$$

From the Step 1, we know that $\mathrm{HH}(R;\mathbb{Z}_p)$ has no negative homology. Moreover, as shown in the Lemma 2.5 in [BMS2], each $\pi_i\mathrm{THH}(\mathbb{Z})$ is a finite group. This together with the Tor spectral sequence shows that each $\mathrm{THH}(R;\mathbb{Z}_p)\otimes_{\mathrm{THH}(\mathbb{Z})}\tau_{\leq n}\mathrm{THH}(\mathbb{Z})$ is pseudo-coherent by induction. So the claim follows by taking the limit of the following equality

$$\tau_{\leq n}\mathrm{THH}(R;\mathbb{Z}_p)\cong \mathrm{THH}(R;\mathbb{Z}_p)\otimes_{\mathrm{THH}(\mathbb{Z})}\tau_{\leq n}\mathrm{THH}(\mathbb{Z}).$$

Here the equality follows from the observation that the fiber $\mathrm{THH}(R;\mathbb{Z}_p)\otimes_{\mathrm{THH}(\mathbb{Z})}\tau_{>n}\mathrm{THH}(\mathbb{Z})$ of the map above is the homotopical twist by degree n of the tensor product of two connective $\mathrm{THH}(\mathbb{Z})$-modules (which is also connective). Note that the pseudo-coherence of $\mathrm{THH}(R;\mathbb{Z}_p)$ implies that each homotopy group of it is a finitely generated R-module.

Step 4 Now we are ready to show the quasi-isomorphism for the general case, where R is of mixed characteristic. We prove this by induction on $\tau_{\leq n}\mathrm{THH}(R;\mathbb{Z}_p)$. Notice that as $\tau_{\leq 2}\mathrm{THH}=\tau_{\leq 2}\mathrm{HH}$, the case for $n=2$ follows from the computation in the Step 1.

Assume the isomorphism $\tau_{\leq n}\mathrm{THH}(R;\mathbb{Z}_p)\cong\tau_{\leq n}R[u]$. By the pseudo-coherence of $\mathrm{THH}(R;\mathbb{Z}_p)$, the $n+1$-th homotopy of $\mathrm{THH}(R;\mathbb{Z}_p)$ is a finitely generated R-module, which we denote by M. By the multiplicativity, we have a natural map of graded rings

$$\tau_{\leq n+1}R[u]\longrightarrow\tau_{\leq n+1}\mathrm{THH}(R;\mathbb{Z}_p)$$

which is an isomorphism on the quotient ring of truncation $\tau_{\leq n}$. This induces a natural map of R-modules

$$M':=\begin{cases} R\cdot u^{\frac{n+1}{2}},\ 2|n+1; \\ 0,\ \textit{else}. \end{cases}\longrightarrow M,$$

Here the image of M' in M is the natural map, where the existence of u^i in the homotopy $\pi_{2i}\mathrm{THH}(R;\mathbb{Z}_p)$ follows from the ring property of $\pi_*\mathrm{THH}(R;\mathbb{Z}_p)$. So to finish the induction process, it suffices to show that the map of R-modules $M'\to M$ is an isomorphism.

To do this, we consider the two specializations: over p and away from p. As p is in the radical ideal of the ring R, away from p we have

$$\pi_{n+1}\mathrm{THH}(R;\mathbb{Z}_p)\otimes\mathbb{Q}=\pi_{n+1}\mathrm{HH}(R;\mathbb{Z}_p)\otimes\mathbb{Q}\cong R\otimes\mathbb{Q},$$

where we use the fact that $\mathbb{Q}=\mathrm{THH}(\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q}$ in the Paragraph 3.2.

For the locus over p, consider the surjection $R\to\overline{R}:=\varinjlim_{x\mapsto x^p}R/pR$, where the ring $\overline{R}$ is a perfect algebra in characteristic p. Note that by the induction hypothesis and the Tor spectral sequence of E_∞-rings (cf. [DAGIII] 4.2.13), we natural map below is a quasi-isomorphism

$$M\otimes_R\overline{R}=\pi_{n+1}\mathrm{THH}(R;\mathbb{Z}_p)\otimes_R\overline{R}\longrightarrow\pi_{n+1}\mathrm{THH}(\overline{R};\mathbb{Z}_p).$$

Here the induction is used to show the vanishing of higher Tor-groups in the spectral sequence, as each $\tau_j\mathrm{THH}(R;\mathbb{Z}_p)$ is free over R for $j\leq n$. So we can base change the map $M'\to M$ along $R\to\overline{R}$, then $M'\otimes_R\overline{R}\to M\otimes_R\overline{R}$ is an isomorphism by the base change formula (along $\mathbb{F}_p\to\overline{R}$) in the Step 2:

$$\begin{array}{ccc} M'\otimes\overline{R} & \longrightarrow & M\otimes\overline{R}\cong\pi_{n+1}\mathrm{THH}(\overline{R};\mathbb{Z}_p) \\ & \nwarrow & \uparrow \\ & & \pi_{n+1}\mathrm{THH}(\mathbb{F}_p)=\mathbb{F}_p\cdot u^{\frac{n+1}{2}} \end{array}$$

Here the above is the case when $2|n+1$; for the case when $2\nmid n+1$ both sides above are zero so the equality follows trivially.

When $2 \nmid n+1$, as $M \otimes \overline{R}$ is zero, by the finitely generatedness of M and the Nakayama's lemma (as the kernel of $R \to \overline{R}$ is in the radical ideal of R), we know M is also zero. When $2|n+1$, the map

$$M' \to M$$

is surjective, which follows again from the Nakayama's lemma. As M' is free of rank one over R, to show the injectivity, it suffices to show that $M \otimes \kappa$ is of dimension at least one for any point κ of $\mathrm{Spec}(R)$ over p. But note that the subset of points of $\mathrm{Spec}(R)$ that are lying over p is exactly $\mathrm{Spec}(\overline{R})$, where the base change $M \otimes_R \overline{R}$ is isomorphic to $\overline{R}$. So we are done.

□

4.2 TC^- and TP of a perfectoid ring

We then compute the (topological) negative cyclic homology and periodic homology of a perfectoid ring R.

As we mentioned in the last subsection, the topological Hochschild homology $\mathrm{THH}(A)$ and its p-completion $\mathrm{THH}(A;\mathbb{Z}_p)$ admits a S^1-action. This allows us to define the homotopical (co)invariants and the Tate construction of $\mathrm{THH}(A)$ under S^1, namely:

$$\mathrm{TC}^-(A) = \mathrm{THH}(A)^{hS^1}, \qquad \mathrm{TP}(A) = \mathrm{THH}(A)^{tS^1} := \mathrm{cofib}(\mathrm{Nm} : \mathrm{THH}(A)_{hS^1}[1] \to \mathrm{THH}(A)^{hS^1}).$$

Here $\mathrm{Nm} : \mathrm{THH}(A)_{hS^1}[1] \to \mathrm{THH}(A)^{hS^1}$ is the norm map from the homotopical S^1-coinvariants to the homopotical S^1-invariants. Here each of them is an E_∞-ring spectra.

Moreover, following [NS18], we could define a Frobenius action on those homology theories. Let C_p be the cyclic subgroup of order p in the S^1. Then there is a natural Frobenius map of E_∞-ring spectra

$$\varphi : \mathrm{THH}(A) \longrightarrow \mathrm{THH}(A)^{tC_p},$$

which is equivariant under the S^1-action. Here the S^1-action on the target is given through $S^1/C_p \cong C_p$.

The same construction holds when we replace everything above by their p-completion.

Remark 4.2.1. We want to mention that the existence of the Frobenius map φ makes use of the universal property of the sphere spectrum $\mathbb{S}$ in the ∞-category of E_∞ rings. In particular, the Frobenius map does not exist in general over the Hochschild homology. We refer the reader to the [NS18, III 1.9] for detailed discussion.

Now we specify to the case for a given perfectoid ring $A = R$. Consider the natural map of E_∞ ring spectra

$$\begin{array}{ccc} \mathrm{TC}^-(R;\mathbb{Z}_p) & \xrightarrow{\varphi^{hS^1}} & (\mathrm{THH}(R;\mathbb{Z}_p)^{tC_p})^{hS^1} \cong \mathrm{TP}(R;\mathbb{Z}_p) \\ \downarrow & & \downarrow \\ \mathrm{THH}(R;\mathbb{Z}_p) & \xrightarrow[\varphi]{} & \mathrm{THH}(R;\mathbb{Z}_p)^{tC_p}. \end{array} \qquad (*)$$

Here the identity $(\mathrm{THH}(R;\mathbb{Z}_p)^{tC_p})^{hS^1} \cong \mathrm{TP}(R;\mathbb{Z}_p)$ is given by [NS18, Lemma II.4.2]. Our goal is to identify the homotopy of those ring spectra and indicating how the classical Frobenius appears among them. Precisely, we have the following.

Theorem 4.2.2 (Theorem 6.2, [BMS2]). *Let R be the perfectoid ring as above. Then by applying the homotopy functor π_* to the diagram $(*)$, we get the following commutative diagram*

$$\begin{array}{ccc} \mathrm{A}_{\mathrm{inf}}(R)[u,v]/(uv-\xi) & \xrightarrow[\varphi-linear]{u\mapsto\sigma,\, v\mapsto\varphi(\xi)\sigma^{-1}} & \mathrm{A}_{\mathrm{inf}}(R)[\sigma,\sigma^{-1}] \\ \theta \Big\downarrow u\mapsto u,\, v\mapsto 0 & & \widetilde{\theta} \Big\downarrow \sigma\mapsto\sigma \\ R[u] & \xrightarrow[R-linear]{u\mapsto\sigma} & R[\sigma,\sigma^{-1}]. \end{array}$$

Here $\widetilde{\theta}$ *is the map* $\theta \circ \varphi^{-1} : \mathrm{A}_{\mathrm{inf}}(R) \to R$*, the element* ξ *generates* $\ker(\theta)$ *and is of degree zero,* u *and* σ *are of degree* 2*, and* v *is in the degree* -2*.*

Before the proof, we want to mention that the prototype of this results, which are special case for $R = \mathbb{F}_p$, are shown in [NS18, IV.4]. We will make use of these in the following proof for the general case.

Proof.

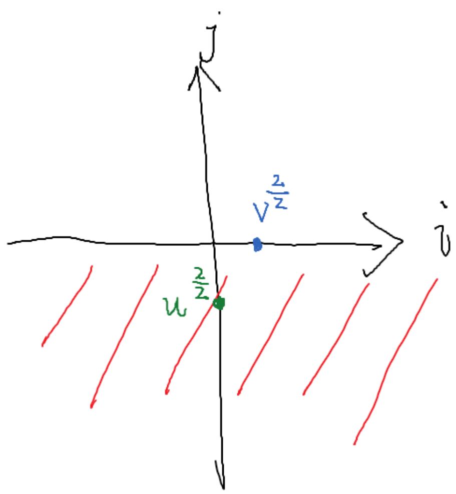
$\mathrm{TP}(R;\mathbb{Z}_p)$ We first compute $\pi_0\mathrm{TP}(R;\mathbb{Z}_p)$. To do so, consider the multiplicative Tate spectral sequence

$$E_2^{i,j} = \pi_{-i}(\pi_{-j}\mathrm{THH}(R;\mathbb{Z}_p))^{tS^1}) \Longrightarrow \pi_{-i-j}\mathrm{THH}(R;\mathbb{Z}_p)^{tS^1} = \pi_{-i-j}\mathrm{TP}(R;\mathbb{Z}_p),$$

where the formation $E_2^{i,j}$ on the left is the i-th Tate cohomology of the S^1-action on the $(-j)$-th homotopy group of $\mathrm{THH}(R;\mathbb{Z}_p)$. Here we note that as BS^1 is homotopy equivalent to the $\mathbb{CP}^\infty$(check this!), so its cohomology ring $\mathrm{H}^*(BS^1,\mathbb{Z})$ is isomorphic to $\mathrm{H}^*(\mathbb{CP}^\infty,\mathbb{Z}) = \mathbb{Z}[v]$ for v being in the degree 2. The Tate cohomology of BS^1 is isomorphic to $\mathbb{Z}[v^{\pm 1}]$ for v living in degree 2. Moreover, by the universal coefficient theorem and the Theorem 4.1.1, we know the Tate cohomology of BS^1 with coefficient being R is isomorphic to $R[v^{\pm 1}]$. So we get

$$E_2^{i,j} = \begin{cases} Rv^{\frac{-i}{2}}u^{\frac{-j}{2}}, & 2|i,j \text{ and } j \leq 0; \\ 0, & \text{otherwise.} \end{cases}$$

The spectral sequence can be drawn as follows:



This implies that $\mathrm{TP}(R;\mathbb{Z}_p)$ lives in even degrees and $F(R) := \pi_0\mathrm{TP}(R;\mathbb{Z}_p)$ admits a multiplicative complete descending filtration $\mathrm{Fil}^i F(R) \subset F(R)$, such that $\mathrm{gr}^i F(R) \cong \pi_{2i}\mathrm{THH}(R;\mathbb{Z}_p) \cong R$ in degree $i \geq 0$ and vanishes otherwise. In particular, the map surjection $F(R) \to \mathrm{gr}^0 F(R) = R$ is a p-adically complete pro-thickening.

Now by the universal property of $\mathrm{A}_{\mathrm{inf}}(R) \to R$, there exists a unique morphism $\mathrm{A}_{\mathrm{inf}}(R) \to F(R)$ lifts the surjections onto R, such that $\ker(\theta) \subset \mathrm{Fil}^1 F(R)$. As both $\mathrm{A}_{\mathrm{inf}}(R)$ and $F(R)$ are filtered complete, to show they are isomorphic it suffices to show this for the graded pieces $\mathrm{gr}^i\mathrm{A}_{\mathrm{inf}}(R) \to \mathrm{gr}^i F(R) = \pi_{2i}\mathrm{THH}(R;\mathbb{Z}_p)$. Similar to the proof of the Theorem 4.1.1, we show this by base change to $\mathbb{Q}$ and to every point over p, where in the latter case we are taking the tensor product $- \otimes_R \kappa$, for κ being a perfect field in characteristic p. Then the rest follows from the functoriality of $\mathbb{F}_p \to \kappa$, and the computation of the special case when $R = \mathbb{F}_p$ as in

[NS18, Corollary IV4.8], by matching up the generators as follows:

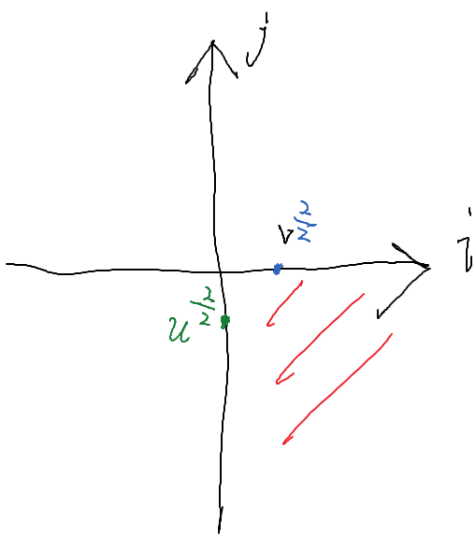
$$\begin{array}{ccc} \mathrm{gr}^i \mathrm{A}_{\mathrm{inf}}(R) & \longrightarrow & \mathrm{gr}^i \mathrm{A}_{\mathrm{inf}}(R) \\ \uparrow & & \uparrow \\ \mathrm{gr}^i \mathbb{Z}_p & \xrightarrow[\sim]{} & \mathrm{gr}^i F(\mathbb{F}_p) = \pi_{2i}\mathrm{THH}(\mathbb{F}_p;\mathbb{Z}_p) \end{array}$$

Moreover, the multiplicativity of the Tate spectral sequence implies that $\mathrm{TP}(R;\mathbb{Z}_p)$ is 2-periodic. This allows us to identify $\pi_*\mathrm{TP}(R;\mathbb{Z}_p)$ to the graded ring $\mathrm{A}_{\mathrm{inf}}(R)[\sigma^{\pm 1}]$, where σ generates the second homotopy group.

$\mathrm{TC}^-(R;\mathbb{Z}_p)$ To compute the negative topological Hochschild homotopy $\mathrm{TC}^-(R;\mathbb{Z}_p) = \mathrm{THH}(R;\mathbb{Z}_p)^{hS^1}$, we consider the homotopy fixed point spectral sequence

$$E_2^{i,j} = \mathrm{H}^i(BS^1, \pi_{-j}\mathrm{THH}(R;\mathbb{Z}_p)) \Longrightarrow \pi_{-i-j}\mathrm{TC}^-(R;\mathbb{Z}_p).$$

This admits a canonical map (not the Frobenius map) to the Tate spectral sequence $\pi_{-i}(\pi_{-j}\mathrm{THH}(R;\mathbb{Z}_p))^{tS^1})$, identifying the terms for $i \geq 1$ since $\pi_{-i}(-)^{tS^1} = \mathrm{H}^i(BS^1, -)$ for $i \geq 1$. The picture of this spectral sequence is the following:



In particular, as the spectral sequences are multiplicative, we see $\pi_*\mathrm{TC}^-(R;\mathbb{Z}_p)$ is also living in even degrees, and the generators of degree 2 and -2 multiply to a generator of $\mathrm{Fil}^1 F(R) = \ker(\theta) \subset \mathrm{A}_{\mathrm{inf}}(R)$ (this is seen as $v^{\frac{2}{2}} \cdot u^{\frac{-2}{2}}$ falls into the Fil^1 of $\pi_0\mathrm{TP}(R;\mathbb{Z}_p)$). In this way, we can find an isomorphism $\pi_*\mathrm{TC}^-(R;\mathbb{Z}_p) \cong \mathrm{A}_{\mathrm{inf}}(R)[u,v]/(uv-\xi)$, where u is of degree 2 and v is of degree -2, such that under the canonical map to $\pi_*\mathrm{TP}(R;\mathbb{Z}_p)$ we have

$$u \longmapsto \xi \cdot \sigma, \quad v \longmapsto \sigma^{-1}.$$

The map φ^{hS^1} Now we consider the Frobenius map φ^{hS^1}. We first compute the induced map on π_0. To do so, we apply the functor π_0 to an extended diagram of $(*)$, and get

$$\begin{array}{ccccc} \mathrm{A}_{\mathrm{inf}}(R) = \pi_0\mathrm{TC}^-(R;\mathbb{Z}_p) & \longrightarrow & \mathrm{A}_{\mathrm{inf}}(R) = \pi_0\mathrm{TP}(R;\mathbb{Z}_p) & \longrightarrow & \pi_0 R^{tS^1} = R \\ \downarrow{\scriptstyle\theta} & & \downarrow & & \downarrow \\ R = \pi_0\mathrm{THH}(R;\mathbb{Z}_p) & \longrightarrow & \pi_0\mathrm{THH}(R;\mathbb{Z}_p)^{tC_p} & \longrightarrow & \pi_0 R^{tC_p} = R/pR, \end{array}$$

Here the right bottom map is induced by applying $(-)^{tC_p}$ at the zero-th homotopical projection

$$\mathrm{THH}(R;\mathbb{Z}_p) \longrightarrow R,$$

where R has trivial S^1-action. Moreover, the top right horizontal map is the $\theta : \mathrm{A}_{\mathrm{inf}}(R) \to R$, which is induced from the universal property of $\mathrm{THH}(R;\mathbb{Z}_p)$. Here we use the equality $R^{tC_p} = R/pR$ for an ordinary commutative ring R (cf. [NS18, Example IV.1.2]), and the right vertical map is the mod p reduction.

The composition of the bottom line above gives the usual absolute Frobenius

$$\varphi : R \longrightarrow R/pR;\ x \longmapsto x^p.$$

So the above induces the following commutative diagram

$$\begin{array}{ccc} \mathrm{A}_{\mathrm{inf}}(R) & \xrightarrow{\pi_0\varphi^{hS^1}} & \mathrm{A}_{\mathrm{inf}}(R) \\ {\scriptstyle\theta}\downarrow & & \downarrow \\ R & \xrightarrow[\varphi]{} & R/pR. \end{array}$$

In this way, by the universal property of the $\mathrm{A}_{\mathrm{inf}}(R)$ among all p-complete pro-nilpotent thickening of R/p, we see $\pi_0\varphi^{hS^1}$ is the Frobenius map of $\mathrm{A}_{\mathrm{inf}}(R)$.

To get $\pi_{2i}\varphi^{hS^1} : \mathrm{A}_{\mathrm{inf}}(R)\cdot u \to \mathrm{A}_{\mathrm{inf}}(R)\cdot\sigma$ for $i \geq 1$, we use the same strategy before and prove this by specializing to perfect fields of characteristic p and $\mathbb{Q}$. Then the proof reduces to $\mathbb{F}_p$, which is proved in [NS18, Proposition IV.4.9].

$\mathrm{THH}(R;\mathbb{Z}_p)^{hC_p}$ At last, we compute the homotopy groups of $\mathrm{THH}(R;\mathbb{Z}_p)^{hC_p}$. In fact, the claimed result in the statement will follow from the Tor spectral sequence as in [DAGIII, 4.2.13] and the following claim:

Claim 4.2.3. The diagram $(*)$ is a pushout diagram of E_∞-rings.

To see this, we prove a stronger result: for a S^1-equivariant $\mathrm{THH}(R;\mathbb{Z}_p)$-module M (which we apply to $M = \mathrm{THH}(R;\mathbb{Z}_p)^{tC_p}$, we have a natural quasi-isomorphism

$$M^{hS^1} \otimes_{\mathrm{TC}^-(R;\mathbb{Z}_p)} \mathrm{THH}(R;\mathbb{Z}_p) \longrightarrow M.$$

To show this, we first notice that from the computations above, we know $\mathrm{THH}(R;\mathbb{Z}_p) = \mathrm{TC}^-(R;\mathbb{Z}_p)/v$ is quasi-isomorphic to the perfect $\mathrm{TC}^-(R;\mathbb{Z}_p)$-complex $\mathrm{TC}^-(R;\mathbb{Z}_p) \xrightarrow{\cdot v} \mathrm{TC}^-(R;\mathbb{Z}_p)$. As the tensor product with a perfect complex commutes with limits (This follows from the truncation and induction, on any finite length complex of finite free $\mathrm{TC}^-(R;\mathbb{Z}_p)$-modules), we may truncate the map above and do a shift to assume M is coconnective. Then as $M^{hS^1} \otimes_{\mathrm{TC}^-(R;\mathbb{Z}_p)} \mathrm{THH}(R;\mathbb{Z}_p)$ commutes with colimits of coconnective modules M,[6] we may then truncate above to assume M concentrates at degree zero. So we are left to prove that for a $\mathrm{THH}(R;\mathbb{Z}_p)$-module M of degree zero, we have

$$M^{hS^1}/v \cong M,$$

which can be proved as in [NS18, Lemma IV.4.12]. So we are done.

□

[6]The statement is similar to the Step 2 and 3 in the proof of the Theorem 4.1.1: we truncate the coconnective M as $\tau_{\geq -n}M \to M \to \tau_{<-n}M$ for $M \in \mathbb{N}$, then notice that $(\tau_{<-n}M)^{hS^1}$ is still living in the homological degree $< -n$. So we get $(\tau_{\geq -n}M)^{hS^1} \cong \tau_{\geq -n}M^{hS^1}$.

Remark 4.2.4. The proof above also shows that the canonical map $\mathrm{TC}^-(R;\mathbb{Z}_p) \to \mathrm{TP}(R;\mathbb{Z}_p)$ induces the following on homotopy rings

$$\mathrm{A_{inf}}(R)[u,v]/(uv-\xi) \xrightarrow[\mathrm{A_{inf}}(R)-linear]{u\mapsto \xi\cdot\sigma,\ v\mapsto \sigma^{-1}} \mathrm{A_{inf}}(R)[\sigma^{\pm 1}]\ .$$

Remark 4.2.5 (Case for $R=\mathbb{F}_p$)**.** Let us say a little bit about the computation of $\pi_0\mathrm{TP}(\mathbb{F}_p;\mathbb{Z}_p) = \pi_0\mathrm{TC}^-(\mathbb{F}_p;\mathbb{Z}_p) = \mathbb{Z}_p$ that we skipped above.[7]

By the Step 2 in the Theorem 4.2.2 above, the ring $\pi_0\mathrm{TC}^-(\mathbb{F}_p;\mathbb{Z}_p)$ is complete under the multiplicative $\mathbb{N}$-indexed descending filtration induced from the homotopy fixed point spectral sequence, where each graded piece is isomorphic to $\mathbb{F}_p$. So by the multiplicity, to show the ring $\pi_0\mathrm{TC}^-(\mathbb{F}_p;\mathbb{Z}_p)$ is isomorphic to the ring $\mathbb{Z}_p$, it suffices to truncate it and consider the following extension problem

$$0 \longrightarrow \mathrm{gr}^1=\mathbb{F}_p \longrightarrow \pi_0\mathrm{TC}^-(\mathbb{F}_p;\mathbb{Z}_p)/\mathrm{Fil}^2 \longrightarrow \mathrm{gr}^0=\mathbb{F}_p \longrightarrow 0.$$

Here we recall that up to isomorphisms there are only two classes of extensions of $\mathbb{F}_p$ by $\mathbb{F}_p$, where one is the direct sum and the another is $\mathbb{Z}_p$. Thus the goal is to show that the above exact sequence is the extension sequence for $\mathbb{Z}_p$.

We then notice that the above extension problem only involves $\tau_{\le 2}\mathrm{THH}(\mathbb{F}_p;\mathbb{Z}_p) \cong \tau_{\le 2}\mathrm{HH}(\mathbb{F}_p/\mathbb{Z};\mathbb{Z}_p)$ (c.f Subsection 3.2) and $\tau_{\le 2}BS^1 = \tau_{\le 2}\mathbb{CP}^\infty = \mathbb{CP}^1$ in the homotopy fixed point spectral sequence, so we may replace the middle one by $\mathrm{H}^0(\mathbb{CP}^1, \tau_{\le 2}\mathrm{HH}(\mathbb{F}_p/\mathbb{Z};\mathbb{Z}_p))$ via the following projection morphisms

$$R\Gamma(BS^1,\mathrm{THH}(\mathbb{F}_p)) \longrightarrow R\Gamma(BS^1,\tau_{\le 2}\mathrm{THH}(\mathbb{F}_p;\mathbb{Z}_p)) = R\Gamma(BS^1,\tau_{\le 2}\mathrm{HH}(\mathbb{F}_p/\mathbb{Z};\mathbb{Z}_p)) \longrightarrow R\Gamma(\mathbb{CP}^1,\tau_{\le 2}\mathrm{HH}(\mathbb{F}_p;\mathbb{Z}_p)).$$

At last, we observe that the cohomology $\mathrm{H}^0(\mathbb{CP}^1,\tau_{\le 2}\mathrm{HH}(\mathbb{F}_p/\mathbb{Z};\mathbb{Z}_p))$, which is the truncated piece for the filtration of $\pi_0\mathrm{HC}^-(\mathbb{F}_p/\mathbb{Z};\mathbb{F}_p)$ given by the homotopy fixed point sequence, is isomorphic to $\pi_0\mathrm{HC}^-(\mathbb{F}_p/\mathbb{Z};\mathbb{Z}_p)/\mathrm{Fil}^2 \cong \mathrm{dR}^\wedge_{\mathbb{F}_p/\mathbb{Z}}/\mathrm{Fil}^2$ (c.f [BMS2, Proposition 5.15]). Thus comparing with the extension problem of $\pi_0\mathrm{HC}^-(\mathbb{F}_p/\mathbb{Z};\mathbb{Z}_p)/\mathrm{Fil}^2$ as below, we are done:

$$0 \longrightarrow \mathrm{gr}^1=\mathbb{L}_{\mathbb{F}_p/\mathbb{Z}_p}[-1]=\mathbb{F}_p \longrightarrow \mathrm{dR}^\wedge_{\mathbb{F}_p/\mathbb{Z}}/\mathrm{Fil}^2=\mathbb{Z}/p^2 \longrightarrow \mathrm{gr}^0=\mathbb{F}_p \longrightarrow 0.$$

5 THH of quasi-regular semi-prefectoid rings

In this section, we compute the $\mathrm{THH}(A;\mathbb{Z}_p)$ for a quasi-regular semi-perfectoid (QRSP) R-algebra A, where R is a fixed perfectoid algebra.

5.1 General facts for THH over a perfectoid ring

In the first subsection, we provide some general results of THH over the perfectoid ring, following the section 6.3 in [BMS2].

Theorem 5.1.1 (Theorem 6.7, [BMS2])**.** *Let A be a R-algebra. Then there exists a S^1-equivariant cofiber sequence of* $\mathrm{THH}(A;\mathbb{Z}_p)$*-module spectra*

$$\mathrm{THH}(A;\mathbb{Z}_p)[2] \xrightarrow{u} \mathrm{THH}(A;\mathbb{Z}_p) \longrightarrow \mathrm{HH}(A/R;\mathbb{Z}_p).$$

By passing to the homotopy fixed points (resp. Tate constructions) of the above sequence, we get the similar cofiber sequences for TC^- *and* HC^- *(resp.* TP *and* HP*), with the maps on the left being the multiplication-by-u (resp. by $\xi\cdot u$).*

Proof. By the relative version of the base change formula for THH and HH in the Paragraph 3.2, we have

$$\mathrm{THH}(A;\mathbb{Z}_p)\otimes_{\mathrm{THH}(R);\mathbb{Z}_p} R \cong \mathrm{HH}(A/R;\mathbb{Z}_p).$$

[7] The author thanks Shizhang Li for explaining this in the BMS2 reading seminar at Michigan.

Then note that by the Theorem 4.1.1 we know the ordinary ring R, as a $\mathrm{THH}(R;\mathbb{Z}_p)$-module, can be resolved by the sequence

$$\mathrm{THH}(R;\mathbb{Z}_p)[2] \xrightarrow{u} \mathrm{THH}(R;\mathbb{Z}_p)\ .$$

Note that everything above are S^1-equivariant, where the action on R is trivial. Thus the result follows by the tensor product and applying the p-adic completion. □

The next result describe the homotopy ring of $\mathrm{THH}(A;\mathbb{Z}_p)$ for a quasi-smooth algebra A over a perfectoid ring R (cf. [BMS2, Definition 4.10]). In particular, we will see how the topological Hochschild homology encodes the deformation, namely the algebraic information, of the ring A (over a perfectoid base).

Theorem 5.1.2 (Hesselholt; Theorem 6.9, [BMS2])**.** *For any R-algebra A, there exists a natural map of graded $A \otimes_R \pi_* \mathrm{THH}(R;\mathbb{Z}_p)$-algebras*

$$\widehat{\Omega}^*_{A/R} \otimes_R \pi_* \mathrm{THH}(R;\mathbb{Z}_p) \longrightarrow \pi_* \mathrm{THH}(A;\mathbb{Z}_p).$$

Moreover, when A/R is quasi-smooth (for instance when A is the p-adic completion of a smooth R-algebra), the above map is an isomorphism.

To construct the map above, it suffices to construct a R-linear map from the p-complete de Rham complex $\widehat{\Omega}^*_{A/R}$ to the graded algebra $\pi_*\mathrm{THH}(A;R)$. The universal property of the de Rham complex $\Omega^*_{A/\mathbb{Z}}$ among all cdga allows us to produce a map from $\Omega^*_{A/\mathbb{Z}}$ to $\pi_*\mathrm{HH}(A/\mathbb{Z})$, where the latter is equipped with the Connes differential induced by the S^1-action (Proposition 1.3.1). So using the identification of $\pi_{\leq 2}$ between $\mathrm{HH}(A/\mathbb{Z};\mathbb{Z}_p)$ and $\mathrm{THH}(A;\mathbb{Z}_p)$ (c.f Paragraph 3.2), we get

$$\widehat{\Omega}^1_{A/R} \cong \widehat{\Omega}^1_{A/\mathbb{Z}} \longrightarrow \pi_1 \mathrm{HH}(A/\mathbb{Z};\mathbb{Z}_p) \cong \pi_1 \mathrm{THH}(A;\mathbb{Z}_p).$$

Thus we have a natural anti-commutative homomorphism of graded algebras

$$\widehat{\Omega}^*_{A/R} \longrightarrow \pi_* \mathrm{THH}(A;\mathbb{Z}_p).$$

Proof. The map above is canonical, and we are left to show the isomorphism for quasi-smooth R-algebra A. Note that by the assumption of quasi-smoothness, the p-complete Hochschild homology is given by the p-completed de Rham complex

$$\pi_*\mathrm{HH}(A/R;\mathbb{Z}_p) \cong \widehat{\Omega}^*_{A/R}.$$

Moreover, the map $\widehat{\Omega}^*_{A/R} \to \pi_*\mathrm{THH}(A;\mathbb{Z}_p)$ above produces a natural section to the projection

$$\pi_*\mathrm{THH}(A;\mathbb{Z}_p) \longrightarrow \pi_*\mathrm{HH}(A/R;\mathbb{Z}_p)$$

which is given by applying π_* at the map in the Theorem 5.1.1. So by induction, the cofiber sequence in the Theorem 5.1.1 breaks into the following short exact sequence

$$0 \longrightarrow \pi_{i-2}\mathrm{THH}(A;\mathbb{Z}_p) \longrightarrow \pi_i\mathrm{THH}(A;\mathbb{Z}_p) \longrightarrow \pi_*\mathrm{HH}(A/R;\mathbb{Z}_p) \longrightarrow 0.$$

In this way, starting from $i = 0$, it follows from the induction again that the map of algebras below is an isomorphism

$$\widehat{\Omega}^*_{A/R} \otimes_R \pi_* \mathrm{THH}(R;\mathbb{Z}_p) \longrightarrow \pi_* \mathrm{THH}(A;\mathbb{Z}_p).$$

So we are done. □

A quick upshot is the computation of each homotopy group of $\mathrm{THH}(A;\mathbb{Z}_p)$.

Corollary 5.1.3 ([BMS2], Corollary 6.10). *Let R be a perfectoid ring. Then the functor* $\mathrm{THH}(-;\mathbb{Z}_p)$ *from the category of p-complete R-algebras admits a complete descending multiplicative $\mathbb{N}$-indexed filtration such that the n-th graded piece is naturally isomorphic to the following direct sum:*

$$\bigoplus_{\substack{0\le i\le n\\ n-i\ even}} L\wedge^i \hat{\mathbb{L}}_{-/R}[n].$$

Proof. This follows from the left Kan extension of the Postnikov filtration for quasi-smooth R-algebras in the Theorem 5.1.2, where we use the fact that THH commutes with sifted colimits of R-algebras, and the left Kan extension of the n-th Postnikov filtration is still n-connective. □

5.2 THH of QRSP rings

Now we compute the THH of quasi-regular semi-perfectoid ring, following the Section 7.1 in [BMS2].

First we recall the definition of the quasi-regular semi-prefectoid rings.

Definition 5.2.1. *A commutative ring A is called* quasi-syntomic *if is satisfies the following two conditions*

- *The ring A is classically p-complete with bounded p-torsion.*
- *The cotangent complex $\mathbb{L}_{A/\mathbb{Z}}$ is of p-complete Tor amplitude $[-1,0]$; namely the object $\mathbb{L}_{A/\mathbb{Z}}\otimes^L_{\mathbb{Z}}\mathbb{Z}/p\in D(\mathbb{Z}/p)$ is of Tor amplitude $[-1,0]$.*

The ring A is called quasi-regular semi-perfectoid *if it is quasi-syntomic and admits a surjection from the a perfectoid ring.*

Here we notice that the condition of A being QRSP implies that A/p has surjective Frobenius.

Now we can state the first result about $\mathrm{THH}(A;\mathbb{Z}_p)$ for A being QRSP.

Theorem 5.2.2 ([BMS2], Theorem 7.1). *Let R be a perfectoid ring, and let A be a QRSP algebra over R. Denote by M to be the ordinary A-module $\pi_1\widehat{\mathbb{L}}_{A/R}$, which is p-completely flat over A (i.e. M is of p-complete Tor amplitude $[0,0]$).*

(i) The $\mathrm{THH}(A;\mathbb{Z}_p)$ *is concentrated in homological non-negative even degrees.*

(ii) The multiplication-by-u-map for $u\in\pi_2\mathrm{THH}(R;\mathbb{Z}_p)$ is injective from $\pi_{2i-2}\mathrm{THH}(A;\mathbb{Z}_p)$ to $\pi_{2i}\mathrm{THH}(A;\mathbb{Z}_p)$. This produces a natural finite increasing filtration on $\pi_{2i}\mathrm{THH}(A;\mathbb{Z}_p)$ by the following sequence

$$\pi_{2i}\mathrm{THH}(A;\mathbb{Z}_p)\supset u\cdot\pi_{2i-2}\mathrm{THH}(A;\mathbb{Z}_p)\supset u^2\cdot\pi_{2i-4}\mathrm{THH}(A;\mathbb{Z}_p)\supset\cdots\supset u^i\cdot\pi_0\mathrm{THH}(A;\mathbb{Z}_p),$$

where the j-th graded piece is $u^{i-j}\cdot\Gamma^j_A(M)$. In particular, the homotopy group $\pi_{2i}\mathrm{THH}(A;\mathbb{Z}_p)$ is p-completely flat over A.

Proof. The first item can be deduced directly from the Corollary 5.1.3, as each $L\wedge^i\hat{\mathbb{L}}_{A/\mathbb{Z}}$ quasi-isomorphic to the $\Gamma^i_A M[i]$ in the cohomological degree i.

For the second item, we use the distinguished triangle in the Theorem 5.1.1, to get

$$\mathrm{THH}(A;\mathbb{Z}_p)[2]\xrightarrow{u}\mathrm{THH}(A;\mathbb{Z}_p)\longrightarrow\mathrm{HH}(A/R;\mathbb{Z}_p).$$

Applying the HKR spectral sequence (Theorem 1.2.3), we know it degenerates at the E_2-page with $\mathrm{HH}(A/R;\mathbb{Z}_p)$ lives in the even degrees, such that

$$\pi_{2i}\mathrm{HH}(A/R;\mathbb{Z}_p)\cong\pi_i(L\wedge^i\widehat{\mathbb{L}}_{A/R})=\Gamma^i_A M.$$

So the above distinguished triangle with the vanishing of $\mathrm{THH}(A/R;\mathbb{Z}_p)$ at odd degrees produces the following short exact sequence

$$0\longrightarrow\pi_{2i-2}\mathrm{THH}(A;\mathbb{Z}_p)[2]\xrightarrow{u}\pi_{2i}\mathrm{THH}(A;\mathbb{Z}_p)\longrightarrow\pi_{2i}\mathrm{HH}(A/R;\mathbb{Z}_p)=\Gamma^i_A M\longrightarrow 0.$$

So we are done.

□

The next result concerns the TC^- and TP of a QRSP ring over a perfectoid base.

Theorem 5.2.3 ([BMS2], Theorem 7.2)**.** *Let R be a perfectoid ring, and A a QRSP algebra over R.*

(i) The homotopy fixed point spectral sequence for $\mathrm{TC}^-(A;\mathbb{Z}_p)$ (resp. Tate spectra sequence for $\mathrm{TP}(A;\mathbb{Z}_p)$) degenerates and $\mathrm{TC}^-(A;\mathbb{Z}_p)$ (resp. $\mathrm{TP}(A;\mathbb{Z}_P)$) lives in homological even degrees. Moreover, the canonical map $\pi_*\mathrm{TC}^-(A;\mathbb{Z}_p) \to \pi_*\mathrm{TP}(A;\mathbb{Z}_p)$ is injective, and is an isomorphism for all non-positive degrees.

(ii) The homotopy fixed point spectral sequence for $\mathrm{TC}^-(A;\mathbb{Z}_p)$ induces a complete descending $\mathbb{N}$-indexed Nygaard *filtration $\mathcal{N}^{\geq\star}\widehat{\Delta}_A$ over $\widehat{\Delta}_A := \pi_0\mathrm{TC}^-(A;\mathbb{Z}_p)$. The graded piece $\mathcal{N}^i\widehat{\Delta}_A$ is naturally isomorphic to the $\pi_{2i}\mathrm{THH}(A;\mathbb{Z}_p)$.*

Here the Nygaard filtration $\mathcal{N}^{\geq i}\widehat{\Delta}_A$ can be identified with the image of the injective multiplication-by-v^i-map for $v^i \in \pi_{-2i}\mathrm{THH}(R;\mathbb{Z}_p)$ in $\widehat{\Delta}_A = \pi_0\mathrm{TC}^-(A;\mathbb{Z}_p)$:

$$\pi_{2i}\mathrm{TC}^-(A;\mathbb{Z}_p) \xrightarrow{\cdot v^i} \pi_0\mathrm{TC}^-(A;\mathbb{Z}_p)$$

(iii) The cyclotomic Frobenius φ^{hS^1} from TC^- to TP induces an endomorphism

$$\varphi : \widehat{\Delta}_A \longrightarrow \widehat{\Delta}_A.$$

Moreover, the image of $\mathcal{N}^{\geq i}\widehat{\Delta}_A$ is contained in $\varphi(\xi)^i\widehat{\Delta}_A$.

(iv) There is a natural isomorphism of R-algebras

$$\widehat{\Delta}_A/\xi \to \widehat{\mathrm{dR}}^{\mathrm{an}}_{A/R},$$

where $\widehat{\mathrm{dR}}^{\mathrm{an}}_{A/R}$ is the (Hodge-completed) p-completed derived de Rham complex of A over R, which is an ordinary ring in this case. Moreover, $\widehat{\Delta}_A$ is ξ-torsion free.

Proof. The first two items are clear from the spectral sequences.

The item (iii) follows from the observation that $\mathrm{TC}^-(A;\mathbb{Z}_p)$ is a module over $\mathrm{TC}^-(R;\mathbb{Z}_p)$, and the effect of the multiplication by v^i maps on the homotopy fixed point spectral sequence.

At last, the quotient identity $\widehat{\Delta}_A/\xi$ is given by the applying π_0 at the distinguished triangle for TP as in Theorem 5.1.1

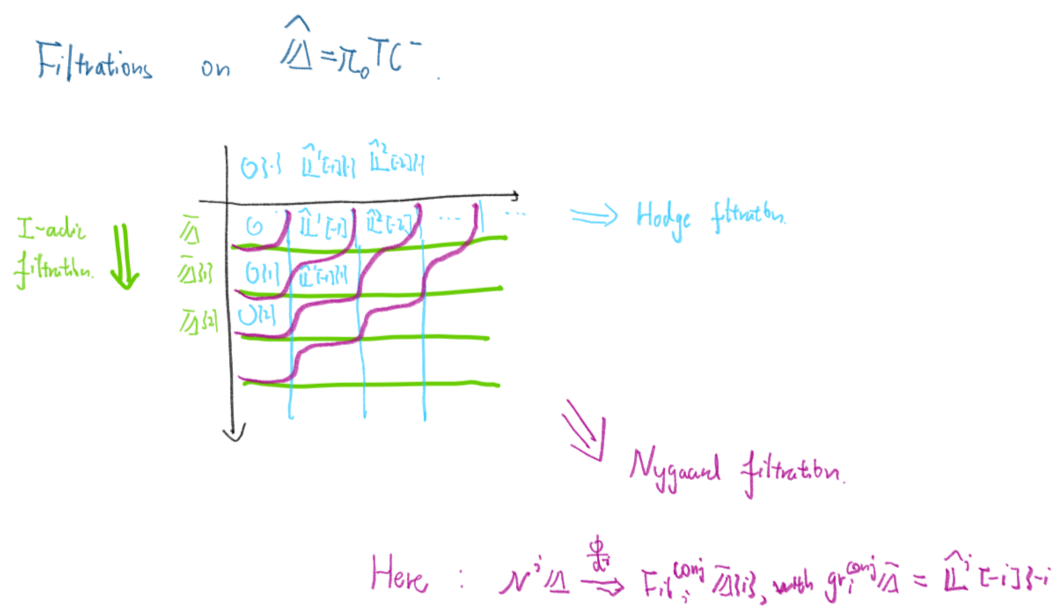
$$\mathrm{TP}(A;\mathbb{Z}_p)[2] \xrightarrow{\xi\cdot\sigma} \mathrm{TP}(A;\mathbb{Z}_p) \longrightarrow \mathrm{HP}^-(A/R;\mathbb{Z}_p).$$

Here we use the fact that for A being QRSP over R, we have a natural isomorphism

$$\pi_0\mathrm{HP}^-(A/R;\mathbb{Z}_p) \cong \widehat{\mathrm{dR}}^{\mathrm{an}}_{A/R},$$

as in [BMS2, Proposition 5.15]. The ξ-torsionfreeness follows from the vanishing of $\pi_1\mathrm{HP}(A/R;\mathbb{Z}_p)$ as $\pi_{\mathrm{odd}}\mathrm{HH}(A/R;\mathbb{Z}_p) = 0$ (by the HKR filtration). □

Here we want to remark that the ring $\widehat{\Delta}_A$ is in fact the (Nygaard completed) *prismatic cohomology of A over R*, which is developed in broader generalities in Bhatt-Scholze [BS]. We at the end of this subsection provide a picture illustrating various filtrations on the prismatic cohomology as below.



5.3 Special case in characteristic p

At last, we give an explicit description for various constructions of $\widehat{\Delta}_A$, for A being a QRSP ring in positive characteristic. We will not give any proof here, but refer the reader to [BMS2, Section 8] for details.

We fix the ring A to be a QRSP ring in characteristic p throughout the subsection.

The ring $\mathrm{A}_{\mathrm{cris}}(A)$ We first introduce the ring $\mathrm{A}_{\mathrm{cris}}(A)$ together with three filtrations on it.

Definition 5.3.1. *Let A be a QRSP ring in characteristic p be as above.*

(i) The ordinary ring $\mathrm{A}_{\mathrm{cris}}(A)$ is defined as the p-adic completion of the pd-envelope $D_{W(A^\flat)}(\ker\theta)$, where $A^\flat$ is the perfect ring given by the inverse limit perfection $\varprojlim_{x\mapsto x^p} A$, and the map $\theta : W(A^\flat) \to A$ is the canonical surjection. The ring $\mathrm{A}_{\mathrm{cris}}(A)$ is naturally equipped with the Frobenius endomorphism ϕ be the functoriality of the construction.[8]

(ii) The Nygaarad filtration $\mathcal{N}^{\geq i}\mathrm{A}_{\mathrm{cris}}(A)$ *is the descending filtration defined by the p-adic completed ideal of $\mathrm{A}_{\mathrm{cris}}(A)$*

$$\mathcal{N}^{\geq i}\mathrm{A}_{\mathrm{cris}}(A) := \{x \in \mathrm{A}_{\mathrm{cris}}(A) \mid \phi(x) \in p^i\mathrm{A}_{\mathrm{cris}}(A)\}^{\wedge}.$$

The i-th graded piece for the Nygaard filtration, which is an $\mathbb{F}_p$-vector space, is denoted as $\mathcal{N}^i\mathrm{A}_{\mathrm{cris}}(A)$, and the Nygaard completed ring is denoted as $\widehat{\mathrm{A}}_{\mathrm{cris}}(A)$.

[8] Precisely, the endomorphism ϕ on $\mathrm{A}_{\mathrm{cris}}(A)$ is given by the functoriality of the Witt ring functor the following commutative diagram of Frobenius

$$\begin{array}{ccc} A^\flat & \xrightarrow{x\mapsto x^p} & A^\flat \\ \downarrow & & \downarrow \\ A & \xrightarrow{x\mapsto x^p} & A, \end{array}$$

where the vertical maps are the natural projection maps from the $\mathbb{N}$-indexed limit onto the first entry.

(iii) *The* divided power filtration *of* $\mathrm{A_{cris}}(A)$ *is the descending filtration defined by the p-completed ideal generated by the subset*

$$\{x^{[j]} = \text{“}\frac{x^j}{j!}\text{”} \mid x \in \ker(\theta : W(A^\flat) \to A),\ j \geq i\}.$$

(iv) *The* conjugate filtration $\mathrm{Fil}_n^{\mathrm{conj}}\mathrm{A_{cris}}(A)$ *is the ascending filtration on* $\mathrm{A_{cris}}(A)$ *is defined by the* $W(A^\flat)$*-submodules of* $\mathrm{A_{cris}}(A)$ *generated (over* $W(A^\flat)$ *by* $a_1^{[m_1]}\cdots a_s^{[m_s]}$*, where* $a_i \in \ker(\theta : W(A^\flat) \to A)$ *and* $\sum_{1\leq i\leq s} m_i < (n+1)p$.

Here we want to remark that the conjugate filtration of $\mathrm{A_{cris}}(A)$ is multiplicative and exhaustive. Moreover, it can be showed that after mod p, the n-th conjugate filtration $\mathrm{Fil}_n^{\mathrm{conj}}\mathrm{A_{cris}}(A)/p$ is the $A^\flat$-submodule generated by elements of the form $a_1^{[pm_1]}\cdots a_s^{[pm_s]}$ for $a_i \in \ker(S^\flat \to S)$ and $\sum_{1\leq i\leq s} m_i \leq n$. See [BMS2, Proposition 8.11] for details.

The main results about the ring $\mathrm{A_{cris}}(A)$ is the following.

Theorem 5.3.2 ([BMS2], Theorem 8.14)**.** *Let A be a QRSP ring in characteristic p as before.*

(i) *The ring* $\mathrm{A_{cris}}(A)$ *is p-torsion free.*

(ii) *The divided Frobenius* $\phi_i = \frac{\phi}{p^i}$ *induces an injection*

$$\phi_i : \mathcal{N}^i\mathrm{A_{cris}}(A) \longrightarrow \mathrm{A_{cris}}(A)/p,$$

whose image is $\mathrm{Fil}_i^{\mathrm{conj}}(\mathrm{A_{cris}}(A))/p$*, for* $i \geq 0$.

(iii) *The images of the Nygaard filtration* $\mathcal{N}^{\geq i}\mathrm{A_{cris}}(A)$ *and the divided power filtration* $\mathrm{A_{cris}}(A)^{[\geq i]}$ *under the composition of the inclusion maps and the mod p reduction* $\mathrm{A_{cris}}(A) \to \mathrm{A_{cris}}(A)/p$ *coincide.*

(iv) *The induced endomorphism of* ϕ *on* $\mathrm{A_{cris}}(A)/p$ *is the absolute Frobeius of* $\mathrm{A_{cris}}(A)/p$*; namely it maps* x *to* x^p *in* $\mathrm{A_{cris}}(A)/p$.

We use the following example to illustrate the structure in a concrete way.

Example 5.3.3. Consider the $\mathbb{F}_p$-algebra $A = \mathbb{F}_p[T^{\frac{1}{p^\infty}}]/T$, which is QRSP. The inverse limit perfection $A^\flat$ is the perfectoid algebra $\mathbb{F}_p\langle T^{\flat\frac{1}{p^\infty}}\rangle$, which is given by the T-adic completion of $\mathbb{F}[T^{\flat\frac{1}{p^\infty}}]$. We denote by u to be the Teichmüller lift of the elemebt $T^\flat = (T, T^{\frac{1}{p}}, \cdots) \in A^\flat$. Then the Witt vector ring $W(A^\flat)$ is the (p,u)-adic completion of the ring $\mathbb{Z}_p[u^{\frac{1}{p^\infty}}]$. The canonical surjection map $\theta : W(A^\flat) \to A$ sends $u^n = [T^\flat]^n$ onto the element $T^n \in A$, and has kernel (p,u).

Then we consider the $\mathrm{A_{cris}}(A)$. By the Definition 5.3.1, we know the $\mathrm{A_{cris}}(A)$ is the p-adic completion of the divided power envelope

$$D_{W(A^\flat)}(p,u).$$

So by the basic properties of the divided power envelope over a p-adic ring, we get

$$\begin{aligned}\mathrm{A_{cris}}(A) &= (W(A^\flat)[\frac{u^i}{i!}, i \in \mathbb{N}])^\wedge \\ &= (\mathbb{Z}_p[u^{\frac{1}{p^\infty}};\ u^{[i]} = \frac{u^i}{i!},\ i \in \mathbb{N}])^\wedge.\end{aligned}$$

Here the completion is the p-adic completion, and the ring $\mathrm{A_{cris}}(A)$ is u-complete automatically as the element $u^i = u^{[i]}\cdot i!$ is p-adically convergent to zero when i goes to ∞ (as an exercise, check the statement is true!). By the description, it is clear that the ring $\mathrm{A_{cris}}(A)$ is p-torsion free. Moreover, the endomorphism ϕ, induced from the absolute Frobenius of $A^\flat$, sends u^n onto u^{np} for $n \in \mathbb{Z}[\frac{1}{p}]$ and $u^{[i]} = \frac{u^i}{i!}$ onto $\frac{u^{ip}}{i!} = u^{[ip]}\cdot\frac{(ip)!}{i!}$ for $i \in \mathbb{N}$, where the element $\frac{(ip)!}{i!}$ is equal to $p^i \cdot v$ for a unit

v in $\mathbb{Z}_p$. This makes it clear that the endomorphism coincides with the absolute Frobenius on the quotient ring $\mathrm{A}_{\mathrm{cris}}(A)/p$, as in the item (iv) of the Theorem 5.3.2.

Now we can compute the various filtrations on $\mathrm{A}_{\mathrm{cris}}(A)$. By the description above, we could write down explicitly a $\mathbb{Z}_p$-basis of the ring $\mathbb{Z}_p[u^{\frac{1}{p^\infty}};\ u^{[i]},\ i \in \mathbb{N}]$ together with their image under ϕ. In particular, we get the Nygaard filtration as the ideal below

$$\mathcal{N}^{\geq i}\mathrm{A}_{\mathrm{cris}}(A) = (p^i, p^{i-1}u^{[1]}, \ldots, u^{[i]}, u^{[i+1]}, \ldots)^\wedge,$$

whose image in $\mathrm{A}_{\mathrm{cris}}(A)/p$ under the canonical inclusion map $\mathcal{N}^{\geq i}\mathrm{A}_{\mathrm{cris}}(A) \to \mathrm{A}_{\mathrm{cris}}(A)$ is

$$(u^{[i]}, u^{[i+1]}, \ldots).$$

Notice that the i-th divided power filtration is the p-completed ideal

$$(\frac{p^a}{a!} \cdot u^{[b]},\ a + b \geq i)^\wedge.$$

Thus we see the i-th Nygaard filtration is contained in the i-th divided power filtration, and their images in $\mathrm{A}_{\mathrm{cris}}(A)/p$ coincide. Moreover, as the image of $u^{[j]}$ under the map ϕ is equal to $p^j \cdot v \cdot u^{[jp]}$ for $v \in \mathbb{Z}_p^\times$, the i-th Nygaard graded piece $\mathcal{N}^i\mathrm{A}_{\mathrm{cris}}(A)$ is the $A^\flat$-module generated by

$$\{p^i/p^{i+1}, p^{i-1}/p^i u^{[1]}, \ldots, u^{[i]}\}.$$

The image of $\mathcal{N}^i\mathrm{A}_{\mathrm{cris}}(A)$ under the divided Frobenius $\phi_i = \frac{\phi}{p^i}$ *mod* p is

$$(1, u^{[p]}, u^{[2p]}, \ldots, u^{[ip]}),$$

which is exactly the i-th conjugate filtration $\mathrm{Fil}_i^{\mathrm{conj}}\mathrm{A}_{\mathrm{cris}}(A)/p$. The injection of ϕ_i *mod* p is clear. Thus the structure of these three filtrations and their relations under ϕ do match up with the expectation in the Theorem 5.3.2.

The use of the ring $\mathrm{A}_{\mathrm{cris}}(A)$ is to give an explicit way to study the structure of the prismatic ring $\widehat{\mathbb{\Delta}}_A = \pi_0\mathrm{TC}^-(A;\mathbb{Z}_p)$. This is clear via the following result.

Theorem 5.3.4 (Theorem 8.17, [BMS2])**.** *Let A be a QRSP ring in characteristic p. Then there exists an isomorphism between $\widehat{\mathbb{\Delta}}_A$ and the Nygaard completed ring $\widehat{\mathrm{A}}_{\mathrm{cris}}(A)$, identifying their Nygaard filtrations and intertwining the cyclotomic Frobenius φ on $\widehat{\mathbb{\Delta}}_A$ with the endomorphism ϕ on $\widehat{\mathrm{A}_{\mathrm{cris}}}$. In particular, on the ring $\widehat{\mathbb{\Delta}}_A/p$ the cyclotomic Frobenius φ acts as $x \mapsto x^p$.*

Corollary 5.3.5. *The i-th Nygaard filtration on $\widehat{\mathbb{\Delta}}_A$ satisfies the following equality*

$$\mathcal{N}^{\geq i}\widehat{\mathbb{\Delta}}_A = \{x \in \widehat{\mathbb{\Delta}}_A \mid \varphi(x) \in p^i\widehat{\mathbb{\Delta}}_A\}.$$

Notice that the general inclusion in the last subsection (Theorem 5.2.3, (iii)) only implies the one side inclusion.

dRW complexes and crystalline cohomology The last paragraph is to relate the previous two theories, one is the prismatic cohomology $\widehat{\mathbb{\Delta}}_A$ using THH, and another is the explicit construction of $\mathrm{A}_{\mathrm{cris}}(A)$, to the classical theory of the de Rham-Witt complexes and the crystalline cohomology, in characteristic p.

For a smooth $\mathbb{F}_p$-algebra S or more generally a smooth algebraic variety X over $\mathbb{F}_p$, following Grothendieck's idea, Berthelot developed a cohomology theory $R\Gamma_{\mathrm{cris}}(X, \mathbb{Z}_p)$ for X in mixed characteristic coefficient (namely $\mathbb{Z}_p$ or integral p-adic rings), aiming to build a ℓ-adic cohomology for ℓ equal to p. When X is proper over $\mathbb{F}_p$, the crystalline cohomology behaves quite well and is a Weil cohomology theory in the technical sense. Moreover, when X admits an integral lift to a smooth proper scheme $\mathscr{X}$ over $\mathbb{Z}_p$, the crystalline cohomology is quasi-isomorphic to the de Rham cohomology of $\mathscr{X}$ over $\mathbb{Z}_p$; namely the cohomology of the de Rham complex $\Omega^\bullet_{\mathscr{X}/\mathbb{Z}_p}$.

In general, to compute the crystalline cohomology, Illusie constructed a naturally defined actual complex $\mathrm{W}\Omega^\bullet_X$ of sheaves of $\mathbb{Z}_p$-modules over X, called the *de Rham-Witt complex* (dRW complex in short), whose hypercohomology $R\Gamma(X, \mathrm{W}\Omega^\bullet_X)$ is naturally quasi-isomorphic to the crystalline cohomology of X over $\mathbb{Z}_p$. Moreover, the functoriality of $\mathrm{W}\Omega^\bullet_X$ endows itself with a Frobenius action, and the quotient $\mathrm{W}\Omega^\bullet_X/p$ is quasi-isomorphic to the de Rham complex of $X/\mathbb{F}_p$. Furthermore, the dRW complex bears a natural descending filtration of subcomplexes $\mathcal{N}^{\geq\star}\mathrm{W}\Omega^\bullet_X$, which was first introduced by Nygaard. The Nygaard filtration reduces to the Hodge filtration on $\Omega^\bullet_{X/\mathbb{F}_p}$ after mod p.

Following [BMS2, Construction 2.1], we could generalize the dRW complexes together with its Nygaard filtrations to all simplicial $\mathbb{F}_p$-algebra via the left Kan extension. In particular, given a QRSP ring A in characterstic p, we can associate it with the derived dRW complex $L\mathrm{W}\Omega^\bullet_A$ together with the derived Nygaard filtration $\mathcal{N}^{\geq i}L\mathrm{W}\Omega^\bullet_A$. The quotient $L\mathrm{W}\Omega^\bullet_A/p$ is naturally quasi-isomorphic to the de Rham complex $\mathrm{dR}_{A/\mathbb{F}_p} = L\Omega^\bullet_{A/\mathbb{F}_p}$, and the reduction of the Nygaard filtration is the algebraic Hodge filtration for $\mathrm{dR}_{A/\mathbb{F}_p}$.

It turns out that the derived dRW complexes is naturally identified with the ring $\mathrm{A}_{\mathrm{cris}}(A)$ for a QRSP ring A in characteristic p. Precisely we have the following.

Theorem 5.3.6. *Let A be a QRSP ring in characteristic p. Then there exists a functorial Frobenius-equivariant quasi-isomorphism between the ring $\mathrm{A}_{\mathrm{cris}}(A)$ and the derived dRW complex $L\mathrm{W}\Omega^\bullet_A$, identifying their Nygaard filtrations and the mod p conjugate filtrations.*

This together with the Theorem 5.3.4 produces a natural isomorphism between the Nygaard completed de Rham-Witt complex $\widehat{\mathrm{W}\Omega^\bullet}_A$ and the prismatic cohomology $\widehat{\mathbb{\Delta}}_A$.

Another quick upshot of the above theorem together with the Theorem 5.3.2 is the isomorphism between the mod p Nygaard completed ring $\widehat{\mathrm{A}}_{\mathrm{cris}}(A)/p$ and the derived de Rham complex $\mathrm{dR}_{A/\mathbb{F}_p}$.[9] Notice that since the derived de Rham complex satisfies the quasi-syntomic descent (c.f [BMS2, Section 4-5]), by the p-completeness of $\mathrm{W}\Omega^\bullet_A$ we get the quasi-syntomic descent for $\mathrm{A}_{\mathrm{cris}}(A)$ and $\mathrm{W}\Omega^\bullet_A$. By taking the Čech nerve associated to a quasi-syntomic covering $S \to A$ for A being QRSP, we can compute the usual crystalline cohomology of a smooth $\mathbb{F}_p$-algebra S via the following cochain complex

$$R\Gamma_{\mathrm{cris}}(S/\mathbb{Z}_p) \cong \mathrm{A}_{\mathrm{cris}}(\check{\mathrm{C}}\mathrm{ech}(S \to A)) = \mathrm{A}_{\mathrm{cris}}(A) \longrightarrow \mathrm{A}_{\mathrm{cris}}(A \otimes_S A) \longrightarrow \mathrm{A}_{\mathrm{cris}}(A \otimes_S A \otimes_A) \longrightarrow \cdots,$$

which is the totalization of the cosimplicial complex associated to the Čech nerve.

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[9] This could be reduced to the analogous statement for the cotangent complex via the conjugate filtration.

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