"Spreading out" and its applications

February 7, 2018

Haoyang Guo

Abstract

Spreading out is a standard technique in algebraic geometry. It allows us to prove statements of geometric nature, say those over a field of characteristic 0 or even complex numbers, to their analogous results in more general bases like characteristic p. Moreover, it arises many interesting questions in arithmetic geometry, like the study of integral models. In this talk, we will introduce the procedure of spreading out and its properties. As an application, we will use the technique to show the Ax's theorem, which roughly states that any injective endomorphism of an affine variety is bijective.

1 Spreading out and basic properties

In this section, we first introduce what is spreading out. Here most of the results come from follow the [EGA IV], section 8.8.

We first give a simple example to illustrate what we are talking about.

Example 1.1. (a) Let $f : \mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ be a $\mathbb{C}$-morphism of affine line. Then it is given by $\phi : \mathbb{C}[x] \to \mathbb{C}[x]$, with $\phi(x) = a_n x^n + \cdots + a_0$ being the polynomial with coefficients in $\mathbb{C}$. Now if we take the $\mathbb{Z}$ sub-algebra R of $\mathbb{C}$ generated by $a_0, \ldots, a_n$, then the ring is the quotient of $\mathbb{Z}[t_0, \ldots, t_n]$, which is a finitely generated (presented) Noetherian $\mathbb{Z}$-algebra. And note that by taking the morphism $\phi_0 : R[x] \to R[x]$, sending x onto $a_n x^n + \cdots a_0 \in R[x]$, then the $\mathbb{C}$-homomorphism is the base change of ϕ_0 along $R \to \mathbb{C}$. In other words, the morphism $f : \mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ is defined in a finitely generated $\mathbb{Z}$-subalgebra of $\mathbb{C}$, in the sense that f is the base change of $f_0 : \mathbb{A}^1_{S_0} \to \mathbb{A}^1_{S_0}$ along $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{R})$.

Moreover, if we take the collection of all f.g. $\mathbb{Z}$-subalgebras R_i of $\mathbb{C}$ containing R, then we have

$$\mathbb{C} = \bigcup_i R_i = \varinjlim_{i \in I} R_i.$$

And if we take f_i be the morphism $\mathbb{A}^1_{S_i} \to \mathbb{A}^1_{S_i}$ given by sending x to the same polynomial above, where $S_i = \mathrm{Spec}(R_i)$, then we have

$$f = \varprojlim_{i \in I} f_i,$$

in the sense that for each pair of i, j with inclusion $R_i \to R_j$, the morphism f_i over S_i is compatible with f_j over S_j, which makes it possible to define such an inverse limit.

Intuitively, this means that for whatever results hold geometrically (over $\mathbb{C}$), they should come from more general statements over a much smaller model.

(b) Let A be a commutative ring, and $\mathfrak{p}$ is a prime ideal of A, then

$$A_{\mathfrak{p}} = \varinjlim_{f \notin \mathfrak{p}} A_f,$$

and in terms of schemes we have

$$S = \varprojlim_{f \notin \mathfrak{p}} \operatorname{Spec}(A_f),$$

which roughly means that a local phenomenon should also appear near that point.

Now we give the formal statement and properties of spreading out. Let S_0 be a scheme, I be an index set, with $\{S_i,\ i \in I\}$ being a projective system of S_0-schemes such that for each i, S_i is the spectrum of quasi-coherent $\mathcal{O}_{S_0}$-algebra. Then $u_{ij} : S_j \to S_i$ is affine for $j \geq i$, which is thus quasi-compact and separated. And under the situation, by taking the spectrum of $R := \varinjlim_{i \in I} R_i$, we get the $S = \operatorname{Spec}(R)$, which is the projective limit of $\{S_i,\ i \in I\}$ in the category of S_0-schemes. The reader could always assume S_0 and S_i are affine in the whole article to make life easier.

Suppose there are two collections of schemes $\{X_i,\ I\}$, $\{Y_i,\ I\}$, such that X_i and Y_i are S_i-schemes, with

$$X_j = X_i \times_{S_i} S_j,\ Y_J = Y_i \times_{S_i} S_j,\ j \geq i.$$

Then we could take the fiber product to get $X := X_i \times_{S_i} S$ and $Y := Y_i \times_{S_i} S$, which are projective limits of those two collections in the category of S_0-schemes. Besides, by the compatibility of two systems, there is a canonical map given as

$$\operatorname{Hom}_{S_i}(X_i, Y_i) \longrightarrow \operatorname{Hom}_S(X, Y),$$

which is given by pullback along $S \to S_i$ and compatible with respect to $i \in I$. So we have the map from the direct limit:

$$e : \varinjlim_{i \in I} \operatorname{Hom}_{S_i}(X_i, Y_i) \longrightarrow \operatorname{Hom}_S(X, Y).$$

Here is our first result:

Theorem 1.2 ([EGA IV], 8.8.2). *(a) Suppose X_i are quasi-compact, and Y_i are locally of finite type over S_i, then the map e given above is injective. If furthermore X_i are quasi-separated, and Y_i are locally of finite presentation over S_i, then the map*

$$e : \varinjlim_{i \in I} \operatorname{Hom}_{S_i}(X_i, Y_i) \longrightarrow \operatorname{Hom}_S(X, Y)$$

is bijective.

(b) Suppose S_0 is quasi-compact and quasi-separated (which is true when S_i are affine). Then for any X of finitely presentation over S, there exists some $i \in I$ together with a finitely presented S_i-scheme X_i, such that we have an S-isomorphism

$$X \to X_i \times_{S_i} S.$$

Let's give some propositions of this result. If you read carefully about Hartshorne, you must noticed that he mostly restrict his discussion for noetherian cases. But many statements actually true in more general framework, say locally of finite presentation. In other words, we could "eliminate the noetherian hypothesis":

Proposition 1.3 ([EGA IV], 8.9.1). *Suppose A is a ring, X is an A-scheme, then TFAE:*

(a) X is finitely presented over A;

(b) There exists a noetherian ring A_0 and a f.g. scheme X_0 over A_0, together with a homomorphism $A_0 \to A$, such that

$$X \cong X_0 \times_{A_0} A.$$

(c) The A_0 in the Part (b) above can be improved by a f.g. $\mathbb{Z}$-algebra.

Another proposition is about "generic flatness"

Proposition 1.4 ([EGA IV], 8.9.4)**.** *Suppose Y is an integral scheme, $u : X \to Y$ is finite type and locally of finitely presented. Let $\mathcal{F}$ be a quasi-coherent and finitely presented $\mathcal{O}_X$-module (see [EGA I] Chap 0, section 5 for definitions). Then there exists an non-empty open subset $U \subset Y$ such that $\mathcal{F}|_{u^{-1}(U)}$ is flat over U.*

Now here is our Theorem of Spreading out, which essentially tells us what spreading out can do for us:

Theorem 1.5 ([EGA IV], 8.10.5)**.** *Suppose S_0 is quasi-compact, X_i and Y_i are finitely presented over S_i, together with an S_i-morphism $f_i : X_i \to Y_i$. Denote by f to be the induced S-morphism $X \to Y$. Let **P** be any of the following properties:*

(i) *isomorphism;*

(ii) *monomorphism;*

(iii) *immersion/closed immersion/open immersion;*

(iv) *separated;*

(v) *surjective;*

(vi) *radial (universally injective);*

(vii) *affine;*

(viii) *quasi-affine;*

(ix) *finite;*

(x) *quasi-finite;*

(xi) *proper.*

Then f satisfies P if and only if there exists some $j \geq i$, such that f_j satisfies P. Moreover, if S_0 is quasi-separated, the statement is true for the following additional properties

(xii) *projective;*

(xiii) *quasi-projective.*

Here is a one direct application of the property (xi), which is known as finitely presented version of the Chow's Lemma:

Proposition 1.6 ([EGA IV], 8.10.5.1)**.** *Suppose A is an ring, X and Y are two A-schemes of finitely presentation, together with a separated A-morphism $f : X \to Y$. Then there exists some A-scheme X' with the following commutative diagram*

$$\begin{array}{ccc} X' & & \\ {\scriptstyle g}\downarrow & \searrow^{h} & \\ X & \xrightarrow[f]{} & Y, \end{array}$$

where h is quasi-projective, and g is surjective and projective.

There are also some direct applications to quasi-finite morphisms, which we refer the reader to the [EGA IV], 8.11.

2 Application: Ax-Grothendieck's Theorem

Here we give a standard application of spreading out, which is known as Ax-Grothendieck's theorem, about injectivity and bijectivity of endomorphisms.

Theorem 2.1 (Ax-Grothendieck, [EGA IV] 10.4.11)**.** *Let k be an algebraically closed field, and X is finite type over k. Then any universally injective k-endormorphism of X is bijective.*

Lemma 2.2. *Let k be an algebraically closed field, and $f : X \to Y$ be a k-morphism between two k-schemes of finite type. Then f is injective if and only if it is universally injective.*

Corollary 2.3. *Any injective morphism between two algebraic varieties over $k = \overline{k}$ is bijective.*

Remark 2.4.

1. If f above is moreover monomorphism, (i.e. it is unramified), then it is in fact an automorphism. See [EGA IV], 17.9.6.
2. This is in fact a very geometric statement, which somehow can be regarded as a generalization of the following theorem in complex analysis:

   For an entire function on $\mathbb{C}$, it is surjective if it is injective.

Now we prove the Ax-Grothendieck's theorem.

Proof. Take $S_0 = \mathrm{Spec}(\mathbb{Z})$. Consider the inductive system of all finitely generated $\mathbb{Z}$ subalgebras R_i of k, indexed by $i \in I$ such that $i \leq j$ if and only if $R_i \subseteq R_j$. Then $S_i = \mathrm{Spec}(R_i)$ becomes a projective system in the categories of S_0-schemes such that $S = \mathrm{Spec}(k)$ is its limit. Since S_0 is quasi-separated and quasi-compact, by the First Theorem 1.2, (b), there exists some X_i finitely presented over S_i, such that

$$X = X_i \times_{S_i} S.$$

Together with Part (a) and by changing a larger i, we may assume that f is induced by an S_i-endomorphism

$$f_i : X_i \longrightarrow X_i.$$

Besides, by the Lemma 2.2 above, $f : X \to X$ is radicial, so from the Theorem of spreading out 1.5, (vi), we could replace i by a larger one such that f_i is also universally injective.

Now again by the Theorem of spreading out 1.5, (v), to show that f is surjective, it suffices to show that f_i is surjective. And we make the following claim:

Claim 2.5 (Claim 1)**.** The map f_i is surjective if and only if it is surjective on closed points.

Since the surjectivity is a local question, we may assume that $X_i = \mathrm{Spec}(A_i)$ (hence X) is affine. And due to the Claim, we only need to show that for each closed point $x = \mathfrak{m} \subset \mathrm{Max}(A_i)$, its preimage is nonempty.

Recall that right now $X_i = \mathrm{Spec}(A_i)$ is a finitely presented affine scheme over $S_i = \mathrm{Spec}(R_i)$, where R_i is a f.g. $\mathbb{Z}$-algebra. Then we can make the following claim:

Claim 2.6 (Claim 2)**.** The intersection $\mathfrak{n}$ of the maximal ideal $\mathfrak{m}$ with R_i is also maximal, which concludes a prime integer in $\mathbb{Z}$.

Granting the second Claim, the point x is over a closed point $s = \mathfrak{n} \in S_i$, so to show that x is contained in the image of $f_i : X_i \to X_i$, it suffices to show that the morphism $f_{i,s} : X_{i,s} \to X_{i,s}$ over the special point $s \in S_i$ is surjective. And since $\mathfrak{n}$ contains some integer in $\mathbb{Z}$, the residue field at s_i is a finite field $\mathbb{F}_q$.

Now we are almost done: By our choice of i, the map f_i is universally injective, which by definition satisfies that any base change of f_i is injective. In particular, by taking the closed fiber at s, the morphism $f_{i,s}$ is also injective. But note that since $X_{i,s}$ is a finitely generated scheme over $\mathbb{F}_q$, its number of $\mathbb{F}_q$ points is finite. In this way, the injectivity (which leads to the injectivity on closed points) implies the surjectivity on closed points. Hence back the first Claim above, we see f_i thus f is surjective.

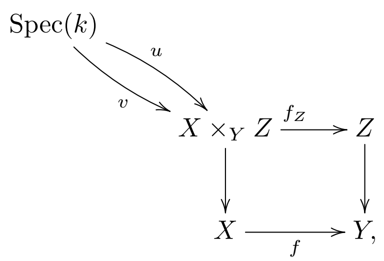
□

Now we finish those Lemmas and Claims.

Proof of the Lemma. We first show that for $f : X \to Y$ between two schemes over $k = \overline{k}$, it is injective if and only if injective on closed points. Assume the latter is true; and we assume both X and Y are reduced. If there exists two points $x_1, x_2 \in X$ mapped onto the same $y \in Y$, then by replacing Y by the closure of y, and X by the preimage of $\overline{y}$, we may assume that x_i are generic points of two irreducible components that are mapped the generic point of the integral variety Y. Then we may choose an open disconnected subset $U = U_1 \coprod U_2$ of X, with x_i lying in different connected components U_i Now by the generic flatness of f (1.4), by making $U = U_1 \coprod U_2$ smaller, we may assume $f|_U$ is flat, thus open. Based on this, since the image $f(U_1) \cap f(U_2)$ is a nonempty open neighborhood of the generic point $y \in Y$, by the Nullstellensatz over $k = \overline{k}$, there exist some closed point $y' \in f(U_1) \cap f(U_2)$; in other words, the preimage of y' has at least two closed points lying on two disjoint components, contradicting to our assumption that f is injective on closed points.

Now we prove the universal injectivity. By definition, it suffices to show that for any $\pi : Z \to Y$, the pullback f_Z of f along π is injective. Here we notice that by the composition $Z \to Y \to \mathrm{Spec}(k)$, Z is endowed with a k-structure and becomes a k-scheme, which is also true for the fiber product $X \times_Y Z$. So by what we just showed before, we only need to check the injectivity on k-points.

We look at the following diagram



where u, v represent two different closed points such that $f_Z \circ u = f_Z \circ v$, mapped onto a single closed point in Z; denoted by z. Then by composing with $\pi : Z \to Y$ and using the commutative diagram, since f it injective, we see the image of u, v along $X \times_Y Z \to X$ is also the same, which is also a closed point in X; denoted by x. In this way, the image of u, v in $X \times YZ$ is lying on $\mathrm{Spec}(k(x) \otimes_k (y) k(z)) = \mathrm{Spec}(k)$, which is also a single closed point. In this way, f_Z is injective on closed points, which is then injective on all the other points. Hence we finish the proof. □

Proof of the Claim 1. It is enough to show that for a S_i-morphism between two f.g. schemes $f : X \to Y$, it is surjective if surjective on closed points. Let $y \in Y$ be any point. By replacing Y by the reduced closure of y, and replacing X by the preimage of $\overline{y}$, we may assume Y is integral and y is the generic point. Then y is contained in the image of f is equivalent to say that f is dominant. Besides, by looking at affine open subsets of generic points, we could assume $X = \mathrm{Spec}(B)$, $Y = \mathrm{Spec}(A)$. Then it suffices to show that $A \to B$ is injective.

If not , $A \to B$ factors through $A \to A/I \to B$ for some nonzero ideal I. But note that since A is finitely generated over R_i, it is in fact f.g. over $\mathbb{Z}$, where the latter is a Jacobson ring. Thus A itself is a Jacobson ring. Notice that for Jacobson ring A, its nil-radical coincides with Jacobson radical, where the former is nilpotent. So since A is a domain, its Jacobson radical, which is the intersection of all maximal ideal, is 0. Thus for the non-trivial ideal I, there must exists some maximal ideal $\mathfrak{m}$ in A which does not contains I. However, this contradicts the assumption that f is surjective on closed points, since $A \to A/I \to B$ means that the image of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is lying over $V(I)$, which does not contain the point $\mathfrak{m}$. In this way, we are done. □

Proof of the Claim 2. We first reduce to the case that $\mathfrak{m}$ is a maximal ideal of $A_i = \mathbb{Z}[x_1, \ldots, x_n]$. Then if $\mathfrak{m} \cap \mathbb{Z} = (0)$, the map $\mathbb{Z} \to A_i/\mathfrak{m}$ factors through $\mathbb{Z} \to \mathbb{Q}$, where the latter then becomes a f.g. $\mathbb{Q}$-algebra and a field. So the image of x_i in $A_i/\mathfrak{m}$ are all algebraic over $\mathbb{Q}$, where we could extract their denominators to assume that they are integral over $\mathbb{Z}[\frac{1}{N}]$ for a large N. Thus the algebra $A/_i/\mathfrak{m}$ becomes a f.g. $\mathbb{Z}[\frac{1}{N}]$-module containing $\mathbb{Z}$, which thus has a non-trivial ideal, contradicting to the assumption that it is a field. Hence $\mathfrak{m} \cap \mathbb{Z}$ is maximal.

□

References

[EGA I] A. Grothendieck. EGA I.

[EGA IV] A. Grothendieck. EGA IV.