Riemann-Zariski spaces and Nagata's compactification

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Abstract

This is the note for the talk given in Student Algebraic Geometry Seminar in University of Michigan, at October 25, 2018. We follow Temkin [Tem11] and introduce the basics about the relative Riemann-Zariski spaces. An as an application, we sketch the proof for Nagata's compactification.

1 Definitions and examples

Throughout the article, every scheme is assumed to be quasi-compact and quasi-separated. We first define the relative Riemann-Zariski space for a morphism $f : Y \to X$ of schemes.

Definition 1.1. (a) A Y-modification of X is defined as a scheme X' by the following commutative diagram

$$\begin{array}{ccc} & & Y \\ & \swarrow^{f'} & \downarrow \\ X' & \xrightarrow[g]{} & X, \end{array}$$

such that g is proper and f' is dominant.

(b) The Riemann-Zariski space $\mathcal{X} = RZ_Y(X)$ for the morphism $f : Y \to X$ is defined as the limit of all Y-modification X' of X, in the category of locally ringed space.

Here we note that the underlying topological space of $RZ_Y(X)$ is $\varprojlim X'$ with the inverse limit topology, and the (structure) sheaf of rings is $\mathcal{O}_\mathcal{X} = \varinjlim \mathcal{O}_{X'}$.

We first look at some examples.

Example 1.2. (a) Let K/k be a field extension, such that k is algebraically closed. We let $Y = \mathrm{Spec}(K)$ and $X = \mathrm{Spec}(k)$. Then $RZ_Y(X)$ is given by the limit of all the integral proper k-schemes whose function field is contained in K. In particular, when K/k is finitely generated, the transcendental dimension is then finite. So any such Y-modification of X has dimension no larger than $tr.\dim_k(K)$.

We look at the following special cases:

- When $tr.\dim_k(K)$ is 0, i.e. K is equal to k. Then $RZ_Y(X)$ is exactly the scheme $\mathrm{Spec}(k)$, since the only possible integral k scheme of dim 0 is $\mathrm{Spec}(k)$ itself, by the assumption that $k = \overline{k}$.
- When $tr.\dim_k(K) = 1$, we claim that $RZ_Y(X)$ is the nonsingular projective model of K over k, which is a scheme. By dimension restriction, any Y-modification X' of X is either of dimension 0, which is X itself, or of dimension 1. When $\dim_k(X')$ is of dimension 1, its function field coincides with K. So any such X' is a proper (projective) k-model of K. But note that in the category of proper k-models of K, there exists an initial object $\mathcal{X}$ given by the nonsingular projective model of K. Thus the inverse limit is equal to the nonsingular projective k-schem $\mathcal{X}$, by the surjectivity and the properness for $\mathcal{X} \to X'$.

- When $tr.\dim_k(K) = 2$, Zariski gave a complete classification of points in $RZ_Y(X)$, by describing all possible valuations of K.

Recall here that to give a projective model of a function field of dimension one, it is equivalent to find out all of the valuation (rings) of K containing k. It turns out (later) that Riemann-Zariski spaces can be characterized totally by valuation theory.

(b) Another two examples are related to the non-archimedean geometry.

- Let $Y = \mathrm{Spec}(\mathbb{Q}_p)$ and $X = \mathrm{Spec}(\mathbb{Z}_p)$. Assume X' is an Y-modification of X. Then since $f : Y \to X$ is dominant, the proper morphism $g' : X' \to X$ is also dominant. In particular, the preimage of the generic point of X is a proper $\mathbb{Q}_p$-scheme dominated by $\mathbb{Q}_p$, which must be $\mathbb{Q}_p$ itself. So X' is an integral proper $\mathbb{Z}_p$-scheme with $\mathbb{Q}_p$ being its function field. We take any open affine subset $\mathrm{Spec}(A)$ of X', then A is an ring between $\mathbb{Z}_p$ and $\mathbb{Q}_p$, which can only be either of them. But notice that by the properness X' cannot be $\mathrm{Spec}(\mathbb{Q}_p)$. So X' must equal to $\mathrm{Spec}(\mathbb{Z}_p)$, and $RZ_Y(X) = \mathrm{Spec}(\mathbb{Z}_p)$.
- Consider $Y = \mathbb{A}^1_{\mathbb{Q}_p}$ and $X = \mathbb{A}^1_{\mathbb{Z}_p}$. Then any Y-modification X' of X is an integral proper X-scheme.
  How do we construct such schemes? Assume X' has an affine open subset $U = \mathrm{Spec}(A)$ such that $Y \to X'$ maps inside of U. Then by the dominance of f' and f, we get the composition of injections
  $$\mathbb{Z}_p[T] \longrightarrow A \longrightarrow \mathbb{Q}_p[T].$$
  By the definition of properness, A is finite type over $\mathbb{Z}_p[T]$. So any such affine open subset is given by a finite type $\mathbb{Z}_p[T]$-subalgebra inside of $\mathbb{Q}_p[T]$.
  On the other hand, assume we are given a finite type $\mathbb{Z}_p[T]$-subalgebra A in $\mathbb{Q}_p[T]$. Then we have the surjection $\mathbb{Z}_p[T][x_1, \ldots, x_n] \to A$, with kernel generated by $(f_1(x_i), \ldots, f_m(x_i))$. Now we homogenize those functions by replacing each x_i as $\frac{X_i}{X_0}$ and multiply each f_j by $X_0^{\deg(f_j)}$. We get a homogeneous ring B such that $B_{(X_0)} = A$. Now we take the projective spectrum $\mathrm{Proj}(B)$ and let $X' = \mathrm{Proj}(B)$. Then it is easy to check that X' is an Y-modification of X. (As an basic example, assuming $A = \mathbb{Z}_p[T][x_1]/(x_1 - p^n T)$ for $n \in \mathbb{Z}$, we have $B = \mathbb{Z}_p[T][X_0, X_1]/(X_1 - p^n T X_0)$.)

Remark 1.3. The above construction using homogenization is in called Y-blowup of X, which is very useful for studying the Riemann-Zariski space when $f : Y \to X$ is affine: every Y-modification of X can be refined by a Y-blowup of X.

2 Valuation spectrum and $RZ_Y(X)$

In this section, we begin to study the geometry of the Riemann-Zariski space. From the example above, especially examples about classical RZ spaces, the thing we get at the end become the collection of valuations of K with some topology. So it is quite natural to ask if there exists a general way of describing $RZ_Y(X)$ using valuation theory. It turns out that the answer is true, and we are going to sketch how to give such an identification.

We first define the following two topological spaces. Again assume $f : Y \to X$ is a morphism of (qcqs) schemes.

Definition 2.1. By an Y-valuation over X, we mean the following commutative diagram:

$$\begin{array}{ccc} y & \longrightarrow & Y \\ \downarrow & & \downarrow f \\ \mathrm{Spec}(R) & \longrightarrow & X, \end{array}$$

where y is a point in Y, R is a valuation ring of the residue field $k(y)$, and $y \to \mathrm{Spec}(R)$ is the canonical map. We denote it by the pair (y, R) in short.

We define $\mathrm{Spa}(Y, X)$ to be the collection of isomorphism classes of all Y-valuations over X, such that the topology is generated by image of the form $\mathrm{Spa}(\overline{Y}, \overline{X})$ for

$$\begin{array}{ccc} \overline{Y} & \longrightarrow & Y \\ \downarrow & & \downarrow f \\ \overline{X} & \longrightarrow & X, \end{array}$$

where $\overline{Y}$ is open in Y, and $\overline{X}$ is finite type and separated over X. Here we note that $\mathrm{Spa}(\overline{Y}, \overline{X})$ is in fact a subset of $\mathrm{Spa}(Y, X)$, by the valuation criterion.

We then define $\mathrm{Val}_Y(X)$ to be the subset of $\mathrm{Spa}(Y, X)$ consisting of all of those minimal (y, R), in the sense that $y \to \mathrm{Spec}(R) \times_X Y$ is a closed immersion. Here we let $\mathrm{Val}_Y(X)$ have the induced topology.

Now we are going to connect valuation spectra with our Riemann-Zariski spaces.

We first note that by the valuation criterion again, in fact there exists a map from the valuation spectrum to the Riemann-Zariski spaces: Let (y, R) be any Y-valuation over X. Then for each Y-modification X' of X, by the properness of $X' \to X$, the map $\mathrm{Spec}(R) \to X$ factors through a unique map from $\mathrm{Spec}(R)$ to X'. In this way, by taking the inverse limit, (y, R) maps to an point $\mathbf{x} \to RZ_Y(X)$, so we get a map

$$\psi : \mathrm{Spa}(Y, X) \longrightarrow RZ_Y(X).$$

In fact, by restricting to the subset $\mathrm{Val}_Y(X)$, we get an homeomorphism:

Theorem 2.2. *Assume $f : Y \to X$ is an affine morphism. Then the induced map*

$$\psi : \mathrm{Val}_Y(X) \longrightarrow RZ_Y(X)$$

is a homeomorphism.

Surjection and continuity We first show that ψ is surjective. In fact, the map ψ admits a section.

Proposition 2.3. *Assume f is an affine morphism. Then there exists a section $\lambda : RZ_Y(X) \to \mathrm{Spa}(Y, X)$ whose images are in $\mathrm{Val}_Y(X)$.*

Proof. Let $\mathbf{x} = (x_i)$ be a point in $RZ_Y(X)$, where x_i is a point in the Y-modification X_i of X. We denote by $f_i : Y \to X_i$ to be the corresponding dominant map. To construct such a section, we look at the inverse of each x_i in Y, and take the limit. Consider the localization $\mathcal{O}_{Y, f_i^{-1}(x_i)}$ of $\mathcal{O}_Y$ at $f_i^{-1}(x_i)$. The morphism $f_i : Y \to X_i$ induces a morphism of rings $\mathcal{O}_{X_i, x_i} \to \mathcal{O}_{Y, f_i^{-1}(x_i)}$, and by passing to the limit we get a map of rings

$$\mathcal{O}_{\mathcal{X}, \mathbf{x}} \longrightarrow \varinjlim_i \mathcal{O}_{Y, f_i^{-1}(x_i)} = B_\infty.$$

The pair $(B_\infty, \mathcal{O}_{\mathcal{X}, \mathbf{x}}$ is not in general a field with its valuation ring; but it is in fact a semi-valuation, in the sense that B_∞ is a local ring with maximal ideal $\mathfrak{m}$, such that $\mathcal{O}_{\mathcal{X}, \mathbf{x}}$ contains $\mathfrak{m}$ and $\mathcal{O}_{\mathcal{X}, \mathbf{x}}/\mathfrak{m}$ is a valuation ring of $B_\infty/\mathfrak{m}$. Besides, by using the construction for Y-blowup of X by the affineness of f, the limit $\varprojlim \mathrm{Spec}(\mathcal{O}_{X_i, x_i}) \times_{X_i} Y = \bigcap \mathrm{Spec}(\mathcal{O}_{X_i, x_i})$ is the unique point y in Y. So from this, we produce a valuation $(y, \mathcal{O}_{\mathcal{X}, \mathbf{x}}/\mathfrak{m})$. At last, to show the minimal condition, i.e. the map

$$y \longrightarrow \mathrm{Spec}(\mathcal{O}_{\mathcal{X}, \mathbf{x}}) \times_X Y$$

is a closed immersion, it suffices to notice that the local ring $\mathrm{Spec}(\mathcal{O}_{Y,y})$ is given by $\varprojlim \mathrm{Spec}(\mathcal{O}_{X_i, x_i}) \times_{X_i} Y$, while the product $\mathrm{Spec}(\mathcal{O}_{\mathcal{X}, \mathbf{x}}) \times_X Y$ is given by $\varprojlim \mathrm{Spec}(\mathcal{O}_{X_i, x_i}) \times_X Y$. So by the separatedness of $X_i \to X$, we get the result.

□

And to show the continuity, it is suffices to check that $\mathrm{Spa}(Y, X) \to RZ_Y(X) \to X_i$ is continuous, which is straightforward.

Openness and injection To show the openness and injection, we will need to look in more detail about a special types of Y-modification of X we mentioned before, the Y-blowup of X. The following is the basic property for Y-blowups of X:

Lemma 2.4. *Assuming f is affine. Then the following is true:*

(a) The family of Y-blowups of X is filtered. Then X_j is also a Y-blowup of X_i.

(b) Assume X' is open in X and $Y' = X' \times_X Y$. Then any Y'-blowup of X' can be extended to a Y-blowup of X.

Then we can prove the following fact

Lemma 2.5. *Given a quasi-compact open subset $\mathcal{U} \subset \mathrm{Val}_Y(X)$, there exists a Y-modification X' of X and an open subset $U \subset X'$, such that $\mathcal{U}$ is the preimage of U in $\mathrm{Val}_Y(X)$.*

By those properties above, it suffices to assume $Y = \mathrm{Spec}(B)$, $X = \mathrm{Spec}(A)$, and $\mathcal{U} = \mathrm{Val}_Y(X) \cap \mathrm{Spa}(B_b, A[\frac{a_1}{b}, \ldots, \frac{a_n}{b}])$. Then we get the Y-blowup by taking the homogenization. As an upshot, we get the openness and the injection of ψ:

Corollary 2.6. *The map $\psi : \mathrm{Val}_Y(X) \to RZ_Y(X)$ is open and injective.*

Here the injectivity is given by pick an $\mathcal{U}$ that contains $\mathbf{y}_1$ but not $\mathbf{y}_2$, and look at the U as the Lemma above.

Another byproduct is the following property for the Y-blowup

Corollary 2.7. *Any Y-modification of X can be dominated by a Y-blowup of X.*

Quasi-compactness For the quasi-compactness of Val_Y, we will need to look in detail about the relation between $\mathrm{Val}_Y(X)$ and $\mathrm{Spa}(Y, X)$. Recall that in a topological space, a point y is a specialization of η if y is contained in the closure $\overline{\{\eta\}}$. Then we actually have the following relation, which illustrate why points in $\mathrm{Val}_Y(X)$ are called "minimal":

Lemma 2.8. *Assume f is separated. Then any point $y \in \mathrm{Spa}(Y, X)$ admits a (unique minimal horizontal) specialization in $\mathrm{Val}_Y(X)$,.*

Another thing we need is the quasi-compactness of $\mathrm{Spa}(Y, X)$, which is showed by Huber.

Fact 2.9. Assume f is separated. Then $\mathrm{Spa}(Y, X)$ is isomorphic to the adic space defined by Huber. In particular, it is quasi-compact.

Granting those two facts, we can get the quasi-compactness of ψ as follows. Let U_j be any open covering of $\mathrm{Val}_Y(X)$. Since the topology on $\mathrm{Val}_Y(X)$ is defined as the induced topology from $\mathrm{Spa}(Y, X)$, we can find open subsets $\overline{U}_j$ in $\mathrm{Spa}(Y, X)$ restricting to U_j. Then every point in $\mathrm{Spa}(Y, X)$ has a specialization in $\mathrm{Val}_Y(X)$, $\overline{U}_j$ is an open covering of $\mathrm{Spa}(Y, X)$. So by the quasi-compactness of $\mathrm{Spa}(Y, X)$, we are done.

3 Compactification

Again, we assume f is separated. In this section, we deal with the Nagata's compactifaction, together with the homeomorphism between $\mathrm{Val}_Y(X)$ and $RZ_Y(X)$ under this generality.

We first note that the homeomorphism $\psi : \mathrm{Val}_Y(X) \to RZ_Y(X)$ is true when f is decomposable: i.e. it can be written as a composition of an affine morphism followed by a proper morphism: when f is factored as

$$Y \xrightarrow{g} Z \xrightarrow{h} X,$$

where h is proper, by definition we have $RZ_Y(Z) \cong RZ_Y(X)$.

It then turns out by using the Y-blowup of X, we can actually "glue" affine morphisms, up to the identity of RZ spaces:

Theorem 3.1. *Assume $f : Y \to X$ is a separated morphism between two qcqs schemes as before. Then we have*

1. *There exists a topological base for $\mathrm{Val}_Y(X)$ by affinoid subdomains, which are defined as $\mathrm{Val}_{Y'}(X')$ for open affine subscheme Y' in Y, and affine scheme X' that is finite type and separeted over X, satisfying an additional condition.*

2. *The finite union of affinoid subdomains is an open subset of $\mathrm{Val}_Y(X)$ that is of the form $\mathrm{Val}_{Y'}(X')$, where Y' is open in Y and X' is separated over X, such that $Y' \to X'$ is affine.*

In particular, by the quasi-compactness of $\mathrm{Val}_Y(X)$, there exists the following

$$\begin{array}{ccc} Y' & \xrightarrow{i} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow[h]{} & X. \end{array}$$

where g is affine and i is open immersion, such that $\mathrm{Val}_{Y'}(X') \to \mathrm{Val}_Y(X)$ is a homeomorphism. Here it is clear that Y' must be the whole Y. So by using a stronger version of valuation criterion, we get the morphism $h : X' \to X$ is proper. As an upshot, f is a composition of affine and proper map. In this way,assuming f is finite type, then since a finite type affine morphism is quasi-projective, we get the following

Corollary 3.2. *Assume f is seperated and finite type between two qcqs schemes. Then it can be factored as a compostion of open immersion and a proper map.*

And note that by the homeomorphism for ψ when assuming f is affine, we get

Corollary 3.3. *Assume f is seperated between two qcqs schemes. Then the map $\psi : \mathrm{Val}_Y(X) \to RZ_Y(X)$ is a homeomorphism.*

Remark 3.4. It is worthwhile to mention that in order to show the Nagata's compactification, it suffices to study $\mathrm{Val}_Y(X)$ directly, without introducing $RZ_Y(X)$ and prove the homeomorphism of ψ. This is because the main ingredients for proving the compactification are the following:

- Basic topological properties of $\mathrm{Val}_Y(X)$ ([Tem11], 3.1), and the topological base of $\mathrm{Val}_Y(X)$ by affinoid subdomains ([Tem11], 3.3.4).
- Properties of Y-blowups of X when f is affine ([Tem11], 3.4).
- Strong valuation criterion ([Tem11], 3.2).
- Patching of the affineness ([Tem11], 3.5.1).

All of those above does not need the limit interpretation of $\mathrm{Val}_Y(X)$.

However, we still need to notice that without the use of $RZ_Y(X) = \varprojlim_i X_i$, we will need to define the map $\eta : Y \to \mathcal{X}$, together with the meromorphic functions $\eta_* \mathcal{O}_Y$ in another way, instead of doing as [Tem11], 3.5.2. And it is still very interesting to see how a pro-scheme, defined as an inverse limit of schemes, can be described in terms of valuations.

References

[Tem11] M. Temkin. Relative Riemann-Zariski spaces. Israel J. Math. 185 (2011), 142.