Note on finiteness properties of D-modules

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Abstract

This short note is intended for the talk given by the author on November 29, 2017, about the section 2 of Lyubeznik's paper [Ly].

Notation

Let k be a field of characteristic 0. At the beginning, we fix the ring R to be the ring of power series $R := k[[x_1, \dots, x_n]]$, and let $D_k(R)$ be its k-linear differential module, defined as

$$D = D_k(R) := R\langle \partial_1, \dots, \partial_n \rangle.$$

Let m be the maximal ideal of R, and denote X to be $\mathrm{Spec}(R)$.

Before we jump into our main theorem, we first collect some basic results of D and holonomic D-modules.

Fact 0.1. As a ring, $D_k(R)$ is both left and right Noetherian. As an upshot, any finitely generated D-module is Noetherian.

Fact 0.2. We have the following facts about holonomic D-modules (R is always the power series ring)

(i) The ring R itself is a holonomic D-module.

(ii) For any $f \in R\backslash\{0\}$ and any holonomic D-module M, the localization M_f is also holonomic.

(iii) The holonomic D-modules form an abelian subcategory of D-modules, which is closed under submodules, quotient modules, and extensions. (of course, in the sense of D-modules)

(iv) For $Y = Y_1 - Y_2$ be a locally closed subset of X. Then $T(M) := H^i_Y(M)$ is also holonomic.

(v) A holonomic D-module is semisimple.

(vi) A simple holonomic M has only one associated prime ideal. Say $Ass(M) = \mathfrak{p}$, and $M_0 = \{x \in M \mid \mathfrak{p} \cdot x = 0\}$. Then there exists some nonzero $h \in R/\mathfrak{p}$, such that $(M_0)_h$ is finitely generated over $(R/\mathfrak{p})_h$. (here we need the Noetherian's Normalization)

Main Theorem and the proof

Here is our main theorem in this note.

Theorem 0.3. *Let k be a field of characteristic 0, R be the ring of formal power series of n variables, and $D = D_k(R)$ be its k-linear differential module. Assume M is a D-module. Then we have*

(a) If $\dim_R(M) = 0$, then M is the direct sum of D/Dm,

(b) *The injective dimension of M is upper bounded by* $\dim_R(M)$*, i.e.*

$$\text{inj.dim}_R(M) \leq \dim_R(M).$$

(c) *If M is finitely generated over D, then the set of the associated primes of M is finite.*

(d) *If M is holonomic, then all the Bass numbers of M are finite. (Here the Bass number is defined as*

$$\mu_i(\mathfrak{p}, M) = \text{length}_{k(\mathfrak{p})} \operatorname{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}),$$

where $k(\mathfrak{p}) = \operatorname{Frac}(R/\mathfrak{p})$ *is the fraction field of* $R/\mathfrak{p}$*.)*

The proof

Now we give the proof of the Main theorem. We first give a description of the injective hull of the residue field k, in terms of D:

Lemma 0.4. *The injective hull $E_R(k)$ of k is isomorphic to $D_k(R)/D_k(R) \cdot m$.*

Proof. The only nontrival fact we need is as follows:

Fact 0.5. For a local Gorenstein ring R of dimension n with residue field k, we have

$$E_R(k) = \mathrm{H}^n_m(R).$$

Based on the fact, we then need to compute the local cohomology. But note that since m is generated by $x_1, \ldots, x_n$, and the local cohomology is computed by (due to the l.e.s associated to sections of $Y \to X \leftarrow U = X \backslash Y$)

$$0 \to \oplus M_{x_i} \to \oplus M_{x_i x_j} \to \cdots \to M_{x_1 \cdots x_n} \to 0,$$

whose n-the cohomology is a k space generated by $x_1^{r_1} \cdots x_n^{r_n}$, where $r_i \leq -1$, and whose R-module structure is given by

$$x_1^{s_1} \cdots x_n^{s_n} \cdot x_1^{r_1} \cdots x_n^{r_n} = \begin{cases} x_1^{r_1+s_1} \cdots x_n^{s_n+r_n}, \ r_i + s_i \leq -1' \\ 0, \ otherwise. \end{cases}$$

So we only need to give a isomorphism between $E_R(k)$ and D/Dm, compatible with R action.

Here after some computation, D/Dm is generated as a k-space by $\partial_1^{u_1} \cdots \partial_n^{u_n}$ for $u_i \in \mathbb{N}$, such that

$$x_i^t \cdot \partial_i^n = \begin{cases} (-1)^t \binom{n}{t} t! \cdot \partial_i^{n-t}, \ t \leq n; \\ 0, \ t > n. \end{cases}$$

So based on this, we could give an isomorphism $D/Dm \to H^n_m(R)$, sending

$$\partial_1^{u_1} \cdots \partial_n^{u_n} \longmapsto \prod_i (-1)^{u_i} (u_i)! \cdot x_i^{-u_i - 1}.$$

□

Part (a)

By definition, since X has exactly one closed point of dimension 0, $\dim_R(M) = 0$ leads to

$$supp(M) = \{m\} = Ass(M),$$

and for each element x in M, we have

$$Ann(x) \ is \ m - primary.$$

Now consider the submodule M_0 of M (the socle, defined as the sum of minimal R-submodules of M) given by

$$M_0 = \{x \in M \mid m \cdot x = 0\},$$

which is a k-subspace of M such that the action of R on it factors through $R \to k$. Then the D-submodule $D \cdot M_0$ generated by M_0 is a direct sum of D/Dm (sum is clear; to see it is a direct sum, given any linear relation $\sum \xi_i \cdot e_i = 0$, by taking the dual of the highest degree monomial of $\{\xi_i,\ i\}$ and multiplies on it, we get a nontrivial linear relation of e_i.) By the Lemma above, since R is Noetherian, the module $D \cdot M_0$ is also R-injective, so the embedding $D \cdot M_0 \to M$ splits as

$$M = D \cdot M_0 \oplus N,$$

where N is also a D-module. But note that since $N \subset M$, $Ass(N) \subseteq Ass(M) = \{m\}$, so N is also supported at m. If $N \neq 0$, by taking the socle of N, we get something contradicting to the definition of M_0. Hence $N = 0$, and $M = \oplus D/Dm$.

Part (b)

We divide our proof into two parts.

- **Induction on** $\dim_R(M)$**.**

  When $d = \dim_R(M) = 0$, this is Part (a). Assume the result is true for every $\dim_R(N) < d$. Let φ be the collection of closed subsets of X which are of dimension $< d$, and let $M' = \Gamma_\varphi(M) = H^0_\varphi(M)$ be the submodule of M whose support is contained in φ. By what we know about local cohomology, M' is a sub-D-module of M, and we have a short exact sequence of D-modules

  $$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

  where primes in $Ass(M'')$ are all of dimension d.

  Now by the description of inj.dim in terms of vanishing of Ext (see [SP, Tag 0A5R]), it suffices to assume $M = M''$, whose associated primes are of dimension d. Then we have a morphism of D-modules (by the Fact at the beginning)

  $$M \longrightarrow \oplus_{\mathfrak{p} \in Ass(M)} M_\mathfrak{p},$$

  which is injective since the irreducible components (minimal elements) of $Ass(M)$ coincide with those of $Supp(M)$. Besides, the cokernel, which is also a D-module, has dimension strictly smaller than d, since there are no embedded associated ideals (prove by contradiction and definition). So by induction, we only need to show that for each $M_\mathfrak{p}$, its injective dimension is $\leq d$. In fact, they are injective.

- **Injectivity of** $M_\mathfrak{p}$

  Now assume M has a single associated prime ideal $\mathfrak{p}$, which is of height h. To show that $M_\mathfrak{p}$ is injective, we first make the following claim:

  Claim 0.6. The action of $D_k(R)$ on $M_\mathfrak{p}$ can be extended to the action of the $D_{k'}(R^\wedge_\mathfrak{p})$, where k' is a coefficient field of $R^\wedge_\mathfrak{p}$ (maximal subfields contained in $R^\wedge_\mathfrak{p}$, which is isomorphic to the residue field $k(\mathfrak{p}) = \mathrm{Frac}(R/\mathfrak{p})$ of $R^\wedge_\mathfrak{p}$, but may not be unique in characteristic 0).

  Due to the Cohn's structure theorem, $R^\wedge_\mathfrak{p}$, which is the completion of the regular ring $R_\mathfrak{p}$, is isomorphic to $k'[[z_1, \ldots, z_h]]$. Granting this, since the support of $M_\mathfrak{p}$ is $\mathfrak{p}$, the unique maximal ideal of the completion $R^\wedge_\mathfrak{p}$, by Part (a) the 0-dimensional $D_{k'}(R^\wedge_\mathfrak{p})$-module $M_\mathfrak{p}$ is isomorphic to the direct sum of $E_{R^\wedge_\mathfrak{p}}(R^\wedge_\mathfrak{p}/\mathfrak{p}R^\wedge_\mathfrak{p}) = E_R(R/\mathfrak{p})$, which is injective.

Sketch of the Claim. By the Noether's Normalization of power series ring, $R/\mathfrak{p}$ is finite over some subring $S = k[[x_{h+1}, \ldots, x_n]]$, after a possible of change of variables. Then the S-linear relative differential $\Omega_{R/S}$ is free of R-module generated by $dx_1, \ldots, dx_h$. We then use the correspondence of $\Omega_{R/S}$ to $Der_S(R)$ (which is contained in $Der_k(R)$) to give the map of $\Omega_{R/S}$ on $M_{\mathfrak{p}}$. And by the nilpotence of $\mathfrak{p}$ on $M_{\mathfrak{p}}$, we could extend them to the completion.

□

Part (c)

Though by the Fact at the beginning, since M is a finitely generated D-module, it is Noetherian over R, which leads to the finiteness of associated primes, we prove as if we do not know that.

We construct a filtration of D-submodules of M such that each quotient has exactly one associated prime ideal. Let $\mathfrak{p} \in Ass(M)$ be a maximal element. Take $M_1 = \Gamma_{V(\mathfrak{p})}(M)$ to be the subsets supported inside $V(\mathfrak{p})$. Then since it is a local cohomology of M, it is then a D-submodule. Besides, by looking at $Ann(x)$ for $x \in M_1 \subseteq M$ and the maximality of $\mathfrak{p}$ in $Ass(M)$, we see $Ass(M_1) = \{\mathfrak{p}\}$. So by repeating the same construction on the quotient D-module M/M_1, we get a filtration of M, which stops after finite steps by the Noetherian of M. And for each $x \in M$, by taking the factor M_i/M_{i-1} such that $x \neq 0$ in this quotient, we see $Ann(x) \subseteq Ass(M_i/M_{i-1})$. Thus we get the finiteness.

Part (d)

Here we need the following Lemma

Lemma 0.7. *Let $\mathfrak{p}$ be a prime in R, M is an R-module such that $\mathrm{H}^i_{\mathfrak{p}}(M)$ are injective for all i. Then we have*

$$\mu_i(\mathfrak{p}, M) = \mu_0(\mathfrak{p}, \mathrm{H}^i_{\mathfrak{p}}(M)).$$

Since M is holonomic, by the Fact at the beginning, we may assume M is simple with only one associated prime $\mathfrak{p}$. Then for each i, since $\mathrm{H}^i_{\mathfrak{p}}(M)$ is a D-module whose support is $V(\mathfrak{p})$, by the proof of the Part (b), we see $\mathrm{H}^i_{\mathfrak{p}}(M)_{\mathfrak{p}}$ is injective over R. Thus by the Lemma, we only need to prove that $\mu_0(\mathfrak{p}, \mathrm{H}^i_{\mathfrak{p}}(M))$ is finite. Here we note that $\mathrm{H}^i_{\mathfrak{p}}(M)$ is also a holonomic D-module, by our Fact.

Now it is enough to show that for a holonomic D-module N such that $supp(N) \subseteq V(\mathfrak{p})$, $\mathrm{H}^i_{\mathfrak{p}}(M)$ is finite. And still by a short exact sequence statement for $\mathrm{Hom}_{R_{\mathfrak{p}}}(k(\mathfrak{p}), -)$, it suffices to show when N is simple, with exactly one associated prime $\mathfrak{q}$. If $\mathfrak{q} \neq \mathfrak{p}$, then $N_{\mathfrak{p}} = 0$, by definition of associated primes. And if $\mathfrak{q} = \mathfrak{p}$, then we take the socle $N_0 = \{x \in N \mid \mathfrak{p} \cdot x = 0\}$. Then we get

$$\mathrm{Hom}_{R_{\mathfrak{p}}}(k(\mathfrak{p}), N_{\mathfrak{p}}) = (N_0)_{\mathfrak{p}} = ((N_0)_h)_{\mathfrak{p}},$$

where h is chosen in $R/\mathfrak{p}$ such that $(N_0)_h$ is finitely generated over $(R/\mathfrak{p})_h$. So by taking the localization as above, we get a finite $k(\mathfrak{p})$-space.

Corollary

Here we collect some Corollarys in the paper, which we may need later.

Corollary 0.8. *Let R be a ring of formal power series over k of characteristic 0, and let G be an k-linear covariant functor from the category of sheaves of k-modules on $X = \mathrm{Spec}(R)$ to the category of k-vector spaces. Then we have $\mathrm{inj.dim}(G(R)) \leq \dim_R(G(R))$. In particular, if $\dim_R(G(R)) = 0$, then $G(R)$ is injective. Here, the special case we may need is when $G = \mathrm{H}^i_Y(-)$ and the composition of local cohomologies.*

Proof. We only need the fact that when G is an additive covariant functor, $G(M)$ is also a D-module. Then the result follows from the Part (b) in the Main theorem. □

Remark 0.9. The result is true for k of characteristic p.

Corollary 0.10. *Let R be a ring of formal power series over k of characteristic 0. Then the set of associated primes of $T(R)$ is finite and all the Bass numbers of $T(R)$ are finite.*

Remark 0.11 (General case)**.** The fact given at the beginning are valid when k is algebraically closed of characteristic 0 and R is a regular domain finitely generated k-algebra. Under this situation, the analogous results to our main theorem and the first corollary above also hold, and the proof is almost identical to our proof here.

References

[Ly] G. Lyubeznik. Finiteness properties of local cohomology modules. Invent Math, 113, 41-55 (1993).

[SP] Stack-project.