F-finite modules

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Abstract

This is the note for the talk on F-modules in the winter semester, 2018. We follow the section 3 in the Lyubeznik's paper [Ly]

We fix R to be a Noetherian regular ring of characteristic $p > 0$. In this note, we will prove the finite length property and the existence of minimal root for F-finite modules, under some condition on R.

1 Finite length

Here is our first theorem

Theorem 1.1 ([Ly], 3.2)**.** *Assume R is furthermore finitely generated over a Noetherian regular ring of characteristic p. Let $\mathscr{M}$ be a F-finite module over R. Then it is of finite length in the category $F_R - mod$.*

We separate the proof into several parts:

Proof.

Step 1 We first reduce to the case that $R = R_1[x_1, \dots, x_n]$ is a polynomial ring over a Noetherian regular ring R_1.

Proposition 1.2 ([Ly], 3.1)**.** *Let B be a finitely generated regular ring of characteristic p such that $B \to R$ is surjective with the kernel I. Let $Z = \mathrm{Spec}(R)$ be the closed subscheme of $X = \mathrm{Spec}(B)$. Then there is an one-to-one correspondence between $F_B - mod$ with support in Z and $F_R - mod$, such that the F_B-finite modules with support in Z corresponds to F_R-finite modules.*

Remark 1.3. For the proposition given above, there is a naive way of working from a F_B-modules with support in Z to a F_R-modules, by taking the submodule killed by I directly, which is exactly $\mathrm{Hom}_B(R, \mathscr{M})$. It is a F_R-module, since there exists a B-linear map from $\mathrm{Hom}_B(R, \mathscr{M})$ to $F_B(\mathrm{Hom}_B(R, \mathscr{M})$ for the exactness of F_B. And by quotient $B' \to R' = B/I$, we get a R-linear map $\mathrm{Hom}_B(R, \mathscr{M}) \to F_R(\mathrm{Hom}_B(R, \mathscr{M}))$, whose limit is a F_R-module. Besides, this construction carries through F_R-finiteness (which is obvious once we know the proof above, since the difference is just a twist of ω). And under the assumption that R and B are regular, the R-linear map before taking the limit is already isomorphic.

However, to find out the opposite direction, namely given a F_R-module $\mathscr{N}$ we need to find a F_B-module $\mathscr{M}$ such that

$$\mathscr{N} = \mathrm{Hom}_B(R, \mathscr{M}).$$

This is not very clear: say if we start with $B = k[x_i]$ and $R = B/f$ for an irreducible polynomial f such that R is regular. We choose $\mathscr{N}$ equal R itself. Then we need an B-module such that $B/f = \mathrm{Hom}_B(B/f, \mathscr{M}) = \mathscr{M}[f]$. If we look at $F_B(R)$, which is isomorphic to $B/I^{[p]} = B/f^p$, then to give a B-linear map from R to $F_B(R)$, we need to specify an element in $B/I^{[p]}$ that is killed by I, which is equivalent to the annihilator of I in $B/I^{[p]}$. So this suggest us to look at the structure of annihilator and think about possible construction, making everything is B-linear.

Proof. The tricky part is to construct the correspondence, where we need the use of the annihilator of I in $B/I^{[p]}$.

We first give the following claim:

Claim 1.4. Let $r = codim_X(Z)$. Then we have

$$Ann_{B/I^{[p]}}(I) = \omega^{\otimes p-1},$$

where ω is the invertible sheaf $\wedge^r I/I^2$ over Z.

Since R and B are regular, Z is of complete intersection and the ideal I/I^2 is invertible over Z, locally of rank r. Assume I is locally generated by $f_1, \dots, f_r$ in B. Then the annihilator of I in $B/I^{[p]}$ is locally generated by $\prod_{i=1}^r f_i^{p-1}$, such that the base-changing coefficient over $B/I^{[p]}$ is $\det^{p-1}$, which is exactly the gluing datum for ω^{p-1}. Thus we get the above Claim.

Based on this, we now give the construction as follows:

$$\left\{F_B - mod\ \mathscr{M}\ with\ supp(\mathscr{M}) \subseteq Z\right\} \Longleftrightarrow \left\{F_R - mod\ \mathscr{N}\right\};$$
$$\mathscr{M} \longmapsto \omega \otimes_R \operatorname{Hom}_B(R, \mathscr{M});$$
$$\varinjlim F_B^n(\omega^{-1} \otimes_R \mathscr{N}) \longleftarrow \mathscr{N}.$$

Here we only check the F_B and F_R structure. Assume $\mathscr{M}$ is a F_B-module with support in Z. Then $\operatorname{Hom}_B(R, \mathscr{M})$ is the submodule of $\mathscr{M}$ killed by I, and we get

$$\begin{aligned}
\omega \otimes_R \operatorname{Hom}_B(R, \mathscr{M}) &\to \omega \otimes_R \operatorname{Hom}_B(R, F_B(\mathscr{M})) \\
&= \omega \otimes \operatorname{Hom}_B(R, \operatorname{Hom}_B(F_B(R), F_B(\mathscr{M}))) \\
&= \omega \otimes \operatorname{Hom}_B(R, F_B(\operatorname{Hom}_B(R, \mathscr{M}))) \\
&= \omega \otimes (\omega^{p-1} \otimes \operatorname{Hom}_B(R, \mathscr{M})) \\
&= \omega^p \otimes_R \operatorname{Hom}_B(R, \mathscr{M}) \\
&= F_R(\omega \otimes_R \operatorname{Hom}_B(R, \mathscr{M})).
\end{aligned}$$

On the other hand, for a F_R-module $\mathscr{N}$, we have

$$\begin{aligned}
\omega^{-1} \otimes_R \mathscr{N} &\to \omega^{p-1} \otimes_R \omega^{-p} \otimes F_R(\mathscr{N}) \\
&= \omega^{p-1} \otimes_R F_R(\omega^{-1} \otimes \mathscr{N}) \\
&= \omega^{p-1} \otimes_B B' \otimes_B (\omega^{-1} \otimes \mathscr{N}) \\
&\to B/I^{[p]} \otimes_B B' \otimes_B (\omega^{-1} \otimes \mathscr{N}) \\
&= F_B(\omega^{-1} \otimes \mathscr{N}).
\end{aligned}$$

□

Step 2 Then we reduce to the case that R is local complete. Consider the ring $R_2[t_0, \dots, t_n]$, and the multiplication subset consisting of elements not in $(m, t_0, \dots, t_n)$. We then define an embedding of R into $\widetilde{R} = (S^{-1}R_2)[t_0^{-1}]$, mapping x_i onto t_i/t_0. Then we make the following claim:

Claim 1.5. The map $R \to \widetilde{R}$ is faithfully flat.

Granting the Claim, by the faithful flatness, a sequence of F_R-modules satisfies d.c.c. if and only if its pullback along the map satisfies d.c.c. So by the fact that $\widetilde{R}$ is a localization of the local ring $S^{-1}R_2$, we could assume R itself is local. And since R is noetherian, where the completion is also faithfully flat, by taking the pullback we could assume R is a complete regular local ring.

The claim is not hard.

Step 3 At last, we consider the modules over the local complete ring. We denote by M to be a root of $\mathscr{M}$ over R. Let $N_1 \supset N_2 \supset \cdots$ be a descending sequence of submodules of M such that

$$N_i = M \cap F(N_i),$$

which corresponds to F-submodule of $\mathscr{M}$ (see [Ly] 2.6). We let N be $\cap_i N_i$. Then the goal is to prove that there exists some i such that $N = N_i$.

We first note that by

$$M \cap F(N) = M \cap F(\cap_i N_i) \subseteq M \cap F(N_i) = N_i,$$

we have $M \cap F(N) \subseteq \cap_i N_i = N$. And the condition becomes as follows:

$$\begin{cases} N = \cap N_i; \\ N_i \subseteq M \cap F(N_i); \\ M \cap F(N) \subseteq N. \end{cases}$$

Then we use the following observation derived from Nakayama's Lemma, together with Artin-Rees Lemma

Lemma 1.6 ([Ly], 3.3). *Let $\{N_i,\ i \in I\}$ be a collection of submodules of M over a complete Noetherian local ring R, such that M is finitely generated, and the collection is closed under finite intersection. Then for each $s \in \mathbb{N}$, there exists some i such that*

$$N_i \subset N + m^s M.$$

By the assumption and the lemma, for each $s > 0$, there exists some i such that

$$N_i = M \cap F(N_i) \subseteq M \cap F(N + m^s M) \subseteq M \cap (F(N) + m^{ps} F(M)),$$

Due to the Artin-Rees Lemma, there exists some $r > 0$, such that for any $t > r$, we have

$$M \cap (F(N) + m^t F(M)) \subseteq M \cap F(N) + m^{t-r} M.$$

So if there is no N_i equaling to N, we could then pick a s large enough, together with N_i such that

$$\begin{cases} N_i \subseteq N + m^s M; \\ N_i \nsubseteq N + m^{s+1} M. \end{cases}$$

Then from the long inclusion of N_i above and the Artin-Rees Lemma, we know that

$$N_i \subseteq M \cap (F(N) + m^{ps} F(M)) \subseteq M \cap (F) + m^{ps-r} M,$$

which contradicts to our assumption when $s + 1 \leq ps - r$, i.e. s is large enough. So

□

2 Minimal root

In this section, we prove the existence of the minimal root. Here is our statement:

Theorem 2.1 ([Ly], 3.5)**.** *Let R be a complete regular local ring of characteristics p. Then any F-finite module $\mathscr{M}$ has a minimal root, in the sense that any root of $\mathscr{M}$ contains it as a submodule.*

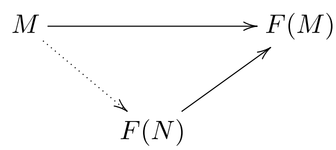
Proof.

Step 1 We first try to construct the one. Define θ to be the structure map $\theta : \mathscr{M} \to F(\mathscr{M})$, and let M be a root of $\mathscr{M}$. Recall that for a submodule N of M, it is a root of some F-submodule of $\mathscr{M}$ if and only if

$$N \subseteq \theta^{-1}(F(N)),$$

equivalently it means the map $N \to M \to F(M)$ factors through $F(N) \to F(M)$.

We then observe that under the map $\theta : M \to F(M)$, it may be possible to replace M by a submodule N such that



Besides, once the above condition holds, by taking the inclusion we naturally get

$$N \to F(N)$$

compatible with θ such that N is a smaller root than M. And since we want a minimal object, we want N to be as small as possible. Our strategy here is iterating the above process so that we could filter M by a series of sub-roots, and show that it will stabilizes to some step so that any root of $\mathscr{M}$ contains it.

Then to make the submodule N we get above as small as possible, we need the following claim:

Claim 2.2. The collection of submodules N of M such that

$$\theta : M \subseteq F(N)$$

has a unique minimal object, given by the intersection of all of those modules.

Granting the Claim, we proceed as follows: Let M_1 be the minimal module given in the Claim, and inductively let M_i be the minimal submodule of M such that

$$M_{i-1} \subseteq F(M_i).$$

Then we get a descending sequence

$$M_1 \supseteq M_2 \supseteq \cdots .$$

We will show that the intersection is what we want in the following.

Step 2 We then show that for each root $N \subseteq M$, there exists some i such that N contains M_i.Note that by definition of how we choose those M_j, in order for $M_i \subseteq N$, it is equivalent to the condition that

$$M_{i-1} \subset F(N)$$

under the structure map θ. And by proceed it to the bottom, it is equivalent to the condition that

$$M = M_0 \subset F^i(N).$$

But note that since N is a root of $\mathscr{M}$, we have

$$\mathscr{M} = \varinjlim_i F^i(N) = \bigcup_i F^i(N).$$

So by the noetherian condition of M, we see there exists some i such that

$$M \subseteq F^i(N),$$

and thus $M_i \subseteq N$.

Step 3 From the above discussion, if we show that the sequence M_i stabilizes, then since every root $N \subseteq M$ of $\mathscr{M}$ contains some M_i, the stabilized object will be contained in all of those roots, and we are done. (Here we note that the intersection of two roots is also a root, which follows from the exactness of F. So it suffices to talk about roots inside M.)

To show the stability, we use the thing we have before. Let N be the intersection of M_i. Then we note that by construction above, we have

$$M_i \subseteq M_{i-1} \subseteq F(M_i),\ M_i \subseteq M_0 = M,$$

i.e. $M_i \subseteq M \cap F(M_i)$. On the other hand, by the noetherian condition, for each $s \in \mathbb{N}$, there exists some i such that

$$M_i \subseteq N + m^s M,$$

and by taking the F functor, we have

$$N \subset \cap_i F(M_i) \subseteq \cap_s F(N + m^s M) = F(N),$$

i.e. $N \subseteq F(N)$, and N is a root of F-submodule $\mathscr{N}$ of $\mathscr{M}$.

Now we need the induction on the length $\ell(\mathscr{M})$. When $\ell(\mathscr{M}) = 1$, $\mathscr{M}$ is simple, either $\mathscr{N} = 0$ or $\mathscr{N} = \mathscr{M}$. If it is 0, then N itself is trivial, satisfies

$$M \cap F(N) \subseteq N = 0.$$

Thus by the last step discussion of the proof of the first theorem 1.1 (cf [Ly], 3.4), we see there exists some i such that $M_i = N = 0$, a contradiction to the construction. Hence $\mathscr{N} = \mathscr{M}$, and N is the minimal root of $\mathscr{M}$.

Next we assume the result is true for lower lengths. We take the exact sequence associated with some simple F-submodules of $\mathscr{M}$, get

$$0 \longrightarrow \mathscr{M}' \longrightarrow \mathscr{M} \longrightarrow \mathscr{M}'' \longrightarrow 0.$$

Then the intersection with $\mathscr{M}'$ (image in $\mathscr{M}''$) of M_i are roots of $\mathscr{M}'$ ($\mathscr{M}''$). By the same discussion as above, $\mathscr{M}' \cap M_i$ stabilizes. So by the induction and quotient by $\mathscr{M}'$, we see the image of M_i in $\mathscr{M}''$ also stabilizes. Hence the original sequence stops at some i, and we get the result.

□

Remark 2.3. I came up with a slightly different but more natural proof. The idea is to take the intersection of all of the root of $\mathscr{M}$, and show that it is also a root, of $\mathscr{M}$.

Let $\mathcal{S}$ be the collection of all roots of $\mathscr{M}$, and let N be the intersection of all of the roots. Then we apply the Zorn's lemma to $\mathcal{S}$. Here for each decreasing sequence under inclusions, we use the discussion given in the Step 3 of the original proof, to show that it stabilizes at some finite step. Then we get a minimal element N', which is also a root of $\mathscr{M}$.

At last, note that since $N' \cap M \subseteq M$ is also a root of $\mathscr{M}$ for any given root M (because of the exactness of F, and the exactness of the following

$$0 \longrightarrow N' \cap M \longrightarrow N' \oplus M \longrightarrow \mathscr{M}.$$

), by the choice of N', we see $N' \cap M = N'$, and $N' = N$.

Remark 2.4. The above proof is almost correct, except for the formalism. Namely the use of inclusions in the Step 1 and Step 2 are not precise, since by passing to the image of some module in $F^i(N)$, the compatible way is to use the map given by the pullback of θ along several powers of F.

References

[Ly] G. Lyubeznik. F-modules: applications to local cohomology and D-modules in characteristic p>0