On the integral Hodge conjecture and integral Tate conjecture for 3-folds

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Abstract

This is the note for the talk on the integral Hodge conjecture for 3-folds, in the Preprint Seminar at the University of Michigan, on December 6, 2019. In this talk, we follow the paper [Tot19] by Burt Totaro and sketch the proof of the integral Hodge conjecture for smooth projective complex 3-folds X of Kodaira dimension zero with $h^0(K_X) > 0$.

1 Statement and examples

1.1 Conjecture and main results

We first state the statement of the main results.

Integral Hodge conjecture Let X be a smooth complex projective varieties. Recall that the Chow ring $\mathrm{CH}(X) = \oplus_{r=0}^{\dim(X)} \mathrm{CH}^r(X)$ of X is defined as the graded ring, where $\mathrm{CH}^r(X)$ is generated by finite $\mathbb{Z}$-linear combinations of closed subvarieties of codimension r in X. There exists a natural cycle map from $\mathrm{CH}^r(X)$ to the singular cohomology $\mathrm{H}^{2r}(X, \mathbb{Z})$ of $X(\mathbb{C})$.

On the other hand, the singular cohomology of complex coefficients is computed by a functorial E_1-spectral sequence

$$E_1^{i,j} = \mathrm{H}^j(X, \Omega_X^i) \Longrightarrow \mathrm{H}^{i+j}(X, \mathbb{C}).$$

This is called the *Hodge-de Rham spectral sequence*, and is degenerated at its E_1-page.

To relate the algebraic structure and the analytic structure together, we can take the $\mathrm{Hom}(-, \mathbb{C})$ at the integral singular cohomology, and get a map from the Chow group to the direct sum of Hodge cohomologies. It is a classical result of Hodge theory that the image of $\mathrm{CH}^i(X)$ is inside of the intersection

$$\mathrm{H}^{2i}(X, \mathbb{Z}) \cap \mathrm{H}^i(X, \Omega_X^i).$$

The image of a k-dimensional subvariety Z under the cycle map can be interpreted as taking the integration of a given multi-differential form over Z. Here the right side is defined as the intersection of the image of $\mathrm{H}^{2i}(X, \mathbb{Z})$ and $\mathrm{H}^i(X, \Omega_X^i)$ inside of $\mathrm{H}^{2i}(X, \mathbb{C})$, and elements there are called *Hodge classes*. We will use $\mathrm{H}^{i,j}(X)$ to abbreviate the Hodge cohomology $\mathrm{H}^j(X, \Omega_X^i)$.

The integral Hodge conjecture states that every Hodge class is algebraic. Precisely, we have:

Conjecture 1.1.1 (Integral Hodge conjecture)**.** *Let X be a smooth projective complex variety. Then the cycle map induces a surjection*

$$\mathrm{CH}^r(X) \longrightarrow \mathrm{H}^{2r}(X, \mathbb{Z}) \cap \mathrm{H}^{r,r}(X),$$

for every $0 \leq r \leq \dim(X)$.

If we replace the coefficient $\mathbb{Z}$ by the field $\mathbb{Q}$ of rational numbers, we get the *Hodge conjecture*.

The integral Hodge conjecture is true for the cases when $r = 1$, which is the so-called Lefschetz (1,1) theorem and can be proved using the exponential sequence (assuming we know the image falls into $\mathrm{H}^{1,1}$.) This in particular implies that the integral Hodge conjecture is true for $\dim(X) \leq 2$. For 3-folds, the Hodge conjecture is proved to be true by the hard Lefschets theorem relating the codimension 2 cycles and codimension 1 cycles. However, Kollár showed that the integral Hodge conjecture fails for some smooth hypersurfaces in $\mathbb{P}^4$. Voisin showed that the integral Hodge conjecture is true for Kodaira dimension $-\infty$. In the case when the Kodaira dimension is zero, Vosin showed that the integral Hodge conjecture holds for those 3-folds that have trivial canonical bundles and first Betti numbers are zero. Moreover, Grabowski proved the case of abelian 3-folds. See the beginning of the [Tot19] for references of these results.

In this article, we follow Totaro [Tot19] and give an improvement of Voisin and Grabowski's results, showing that the integral Hodge conjecture holds for 3-fold of Kodaira dimension zero with $h^0(X, K_X) > 0$. Precisely, we have:

Theorem 1.1.2 ([Tot19], 4.1)**.** *Let X be smooth projective complex variety of dimension 3, such that its Kodaira dimension is zero, and* $\dim \mathrm{H}^0(X, K_X) > 0$. *Then the integral Hodge conjecture holds for X.*

Integral Tate conjecture As an application, we follow the Section 6 in [Tot19] and proves the integral Tate conjecture for 3-folds in characteristic 0. Let k be a finitely generated field (over $\mathbb{Q}$ or $\mathbb{F}_p$), and let X be a smooth projective variety over k. Recall that there exists a natural cycle map from $\mathrm{CH}^i(X)$ to the ℓ-adic étale cohomology

$$\mathrm{CH}^i(X) \longrightarrow \mathrm{H}^{2i}(X_{k^s}, \mathbb{Z}_\ell(i)),$$

whose image is contained in the $\mathbb{Z}_\ell$-submodule of $\mathrm{Gal}(k^s/k)$-invariant elements. As the target is of $\mathbb{Z}_\ell$-linear, the map above factors through $\mathrm{CH}^i(X) \to \mathrm{CH}^i(X) \otimes \mathbb{Z}_\ell$. Then the integral Tate conjecture states as follows:

Conjecture 1.1.3 (Integral Tate conjecture)**.** *Let X be a smooth projective variety over the finitely generated field k. Then the cycle map*

$$\mathrm{CH}^i(X) \otimes \mathbb{Z}_\ell \longrightarrow \mathrm{H}^{2i}(X_{k^s}, \mathbb{Z}_\ell(i))^{\mathrm{Gal}(k^s/k)}$$

is surjective for any integer i.

We are going to use the Theorem 1.1.2 to show the following:

Theorem 1.1.4. *Let X be a smooth projective 3-fold over a finitely generated field k of characteristic 0. Assume either X is rational connected, or it has Kodaira dimension zero with $h^0(X, K_X) > 0$. Then the integral Tate conjecture holds for X.*

1.2 Examples

We give two examples where we now know the integral Hodge conjecture holds.

Example 1.2.1. Let S be a $K3$ surface, and let E be an elliptic curve. Assume G is a finite group acting on S *symplecticly*, namely its induced action on $\mathrm{H}^0(S, K_S) = \mathbb{C}$ is trivial. The examples of symplectic actions on S can be found in [Mu88]. Assume G is abelian and can be generated by two elements. Fix an embedding of G into E. Then we can define an action of G on the product $S \times E$, where the action on the second component is given by translation. This is a free action, and we can form the quotient $X = (S \times E)/G$. Then we claim that X is a smooth projective 3-folds with trivial canonical bundle K_X, and $\mathrm{H}^1(X, \mathcal{O}_X) \neq 0$, in particular it satisfies the condition in the Theorem 1.1.2. Thus the integral Hodge conjecture holds for X.

To show K_X is trivial, we notice that the product $S \times E$ has trivial canonical bundle, and its sections can be formed by taking the product of canonical forms of S and E separately. Moreover as the action of G on canonical forms of S and E are trivial, the action of G on $\mathrm{H}^0(S \times E, K_{S\times E})$ is also the identity, and any canonical form of X is G-invariant. In this way, by taking the étale descent for the covering $S \times E \to X$, we get a canonical form on X that vanishes nowhere. This shows that the line bundle K_X is trivial.

To show the nontriviality of $\mathrm{H}^1(X, \mathcal{O}_X)$, we use the projection map $X \to E/G$ onto an elliptic curve and notice that it has a section. So the $\mathrm{H}^1(X, \mathcal{O}_X) \neq 0$ follows from $\mathrm{H}^1(E/G, \mathcal{O}_{E/G}) = \mathbb{C}$.

Example 1.2.2. Another example is the quotient of abelian 3-folds.

Take A an abelian surface, E an elliptic curve, and G a finite abelian group of at most two generators. We embed G into A and E separately and consider the action of G on $A \times E$. Then the quotient $X := (A \times E)/G$ satisfies the condition of the Theorem 1.1.2. The proof is similar to the last example, and notice that as the translation leaves the canonical form of A invariant.

This can be generalized to more abelian group G of at most two generators, whose action on A is symplectc and faithful.

2 Sketch of the proof

In this section, we sketch the proof of the main theorem 1.1.2.

Proof of the Theorem 1.1.2. Let Y be the minimal model of X (whose existence was proved by Mori), which is a projective 3-fold that has terminal singularity.[1] It comes with a birational map $X \dashrightarrow Y$, and the canonical bundle K_Y of Y is nef (which means it is numerically effective). Moreover, by assumption, we have $h^0(Y, K_Y) > 0$, thus it is equal to one as the Kodaira dimension (the maximal dimension among images of the rational maps defined by pluricanonical forms) is equal to zero.

We first notice that by the Lefschetz (1,1) theorem, the integral Hodge conjecture for codimension-1 Hodge classes is true. So it is left to consider the codimension-2 case. In the following, we use Poincaré duality and consider the image of $\mathrm{CH}_1(X)$ into $\mathrm{H}_2(X, \mathbb{Z}) \cap \mathrm{H}_{1,1}(X)$. By a result of Voisin ([Voi07], Lemma 15), the integral Hodge conjecture for codimension-2 cycles are birational invariants. So by taking a resolution of singularity of the image for the rational map $X \dashrightarrow Y$, we may assume $X \to Y$ is a birational morphism, such that the exceptional locus $E = \coprod E_i$ in X is a disjoint union of snc divisors. Then since Y has terminal singularity, Y is nonsingular in codimension ≤ 2 and has isolated singularities of dimension zero ([Deb01] 7.17). So the map $X \to Y$ is given by contracting identifying each divisor E_i onto a point, and by the Excision Theorem in algebraic topology, we get a long exact sequence of homologies

$$\cdots \longrightarrow \oplus \mathrm{H}_2(E_i, \mathbb{Z}) \longrightarrow \mathrm{H}_2(X, \mathbb{Z}) \longrightarrow \mathrm{H}_2(Y, \mathbb{Z}) \longrightarrow \cdots .$$

Here we have the following observations.

Lemma 2.0.1 ([Tot19], Lemma 3.1)**.** *Let $X \to Y$ be a resolution of singularity of complex 3-folds with isolated rational singularities, such that the exceptional divisor D is snc. Then $\mathrm{H}_2(D, \mathbb{Z})$ is generated by algebraic 1-cycles on D.*

Another result is about the algebraicity of Hodge class on Y, which is the key observation of the article [Tot19].

Theorem 2.0.2 ([Tot19], Lemma 4.2, Proposition 5.3)**.** *Let Y be a terminal projective complex 3-fold with trivial canonical bundle. Then every element in $\mathrm{H}_2(Y, \mathbb{Z})$ whose image in $\mathrm{H}_2(Y, \mathbb{C})$ is contained in $\mathrm{H}_{1,1}(Y)$ is algebraic.*

We first deduce the main theorem 1.1.2 from the above two results. Since the singular locus of Y has only finite many points, every 1-cycle of Y can be lifted to a 1-cycle of X by taking the closure. In particular, the map of Chow group $\mathrm{CH}_1(X) \to \mathrm{CH}_1(Y)$ is surjective. Moreover, the

[1] Recall that a variety Y has terminal singularity if it is normal, and there exists a positive integer j such that jK_Y is a Cartier divisor, with the condition that any j-canonical form of Y_{reg} can be extended to a j-canonical form of Z with zeros at any exceptional divisor. Here Z is any resolution of singularity of Y. (cf. [Deb01], 7.2)

Lemma 2.0.1 gives the surjectivity of $\mathrm{CH}_1(E_i) \to \mathrm{H}_2(E_i,\mathbb{Z})$. So we could extend the above long exact sequence into the following bigger diagram

$$\begin{tikzcd}
 & \oplus\mathrm{CH}_1(E_i) \arrow[r] \arrow[d] & \mathrm{CH}_1(X) \arrow[r, two heads] \arrow[d] & \mathrm{CH}_1(Y) \arrow[d] & \\
\cdots \arrow[r] & \oplus\mathrm{H}_2(E_i,\mathbb{Z}) \arrow[r] & \mathrm{H}_2(X,\mathbb{Z}) \arrow[r] & \mathrm{H}_2(Y,\mathbb{Z}) \arrow[r] & \cdots .
\end{tikzcd}$$

Pick any Hodge class $u \in \mathrm{H}_{1,1}(X) \cap \mathrm{H}_2(X,\mathbb{Z})$. As the image $\overline{u}$ of u in $\mathrm{H}_2(Y,\mathbb{Z})$ is mapped into $\mathrm{H}_{1,1}(Y)$, by the Theorem 2.0.2 we know $\overline{u}$ is algebraic, coming from some element in $\mathrm{CH}_1(Y)$. By the surjectivity of $\mathrm{CH}_1(X) \to \mathrm{CH}_1(Y)$, we can pick an algebraic 1-cycle $\alpha \in \mathrm{CH}_1(X)$ whose image in $\mathrm{H}_2(Y)$ is $\overline{u}$. Then the element $u-\alpha$ comes from an element in $\oplus\mathrm{H}_2(E_i,\mathbb{Z})$, which by the Lemma 2.0.1 is algebraic. As an upshot, the image $u-\alpha$ is algebraic. In this way, since both α and $u-\alpha$ are algebraic, we see the Hodge class u comes from the Chow group.

□

The Lemma 2.0.1 is essentially about diagram chasing, with a use of the Hodge structure. We give the proof as follows.

Proof of the Lemma 2.0.1. Recall that under the assumption that Y has isolated rational singularities, by taking a contractible open neighborhood of singular points of Y and the definition of the rational singularity, we have

$$\mathrm{H}^i(D,\mathcal{O}) = 0, i > 0.$$

Now let D_i be irreducible components of D, and let $D_{i_0\ldots i_p}$ be the intersection $D_{i_0} \cap \cdots \cap D_{i_p}$. Then by the assumption of being snc, we have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_D \longrightarrow \oplus\mathcal{O}_{D_i} \longrightarrow \oplus\mathcal{O}_{D_{ij}} \longrightarrow \mathcal{O}_{D_{ijk}} \longrightarrow 0.$$

This could be regarded as the map of structure sheaves associated to a topological hypercovering of D coming from $\coprod D_i \to D$. In particular, we get an E_1 spectral sequence of cohomology

$$E_1^{p,q} = \oplus\mathrm{H}^q(D_{i_0\cdots i_p},\mathcal{O}) \Longrightarrow \mathrm{H}^{p+q}(D,\mathcal{O}).$$

On the other hand, by the generalized version of Mayer-Vietories sequence, we get an E^1-spectral sequence of homology

$$E^1_{p,q} = \oplus\mathrm{H}_q(D_{i_0\cdots i_p},\mathbb{Z}) \Longrightarrow \mathrm{H}_{p+q}(D,\mathbb{Z}).$$

By taking the functor $\mathrm{Hom}(-,\mathbb{C})$, the second functor produces an E_1-spectral sequence of $\mathbb{C}$-coefficient singular cohomology, convergent to $\mathrm{H}^{p+q}(D,\mathbb{C})$.

We explicitly write down the first E_1 spectral sequence of cohomology as follows:

$$\begin{tikzcd}
\oplus\mathrm{H}^2(D_i,\mathcal{O}) \arrow[r] & 0 & 0 & 0 \\
\oplus\mathrm{H}^1(D_i,\mathcal{O}) \arrow[r, "d_1"] \arrow[dr, dotted, "d_2"] & \oplus\mathrm{H}^1(D_{ij},\mathcal{O}) \arrow[r] & 0 & 0 \\
\oplus\mathrm{H}^0(D_i,\mathcal{O}) \arrow[r] & \oplus\mathrm{H}^0(D_{ij},\mathcal{O}) \arrow[r] & \oplus\mathrm{H}^0(D_{ijk},\mathcal{O}) \arrow[r] & 0.
\end{tikzcd}$$

Here we note that since D_{ij} is of dimension one, the term $\mathrm{H}^2(D_{ij},\mathcal{O}) = 0$. Moreover, since $\mathrm{H}^2(D,\mathcal{O}) = 0$ by the rational singularity, we have $\mathrm{H}^2(D_i,\mathcal{O}) = 0$ for any i.

Now we make the following two claims.

Claim 2.0.3. The natural map $d_1 + \overline{d}_1 : \oplus\mathrm{H}^1(D_i,\mathbb{C}) \to \oplus\mathrm{H}^1(D_{ij},\mathbb{C})$ is surjective.

Claim 2.0.4. The natural map $\mathrm{H}^2(D,\mathbb{C}) \longrightarrow \oplus\mathrm{H}^2(D_i,\mathbb{C})$ is injective.

The two claims need a little bit of Hodge structure and the computation of spectral sequence: in the first claim we need the decomposition

$$\mathrm{H}^1(D_i, \mathbb{C}) = \mathrm{H}^1(D_i, \mathcal{O}) \oplus \overline{\mathrm{H}^1(D_i, \mathcal{O})};$$

in the second claim we make use of this decomposition and deduce the surjectivity of the map $E_2^{0,1}(\mathbb{C}) \to E_2^{0,1}(\mathcal{O})$. We leave the detail as exercises to the reader.

The universal coefficient theorem and the second claim above imply that $\oplus \mathrm{H}_2(D_i, \mathcal{Q}) \to \mathrm{H}^2(D, \mathcal{Q})$ is surjective. On the other hand, as D_{ijk} are points and D_{ij} are smooth projective curves, we have the torsion-freeness of $\mathrm{H}_0(D_{ijk}, \mathbb{Z}), \mathrm{H}_1(D_{ij}, \mathbb{Z})$. We base change them to $\mathbb{Q}$, then the surjectivity above implies that both $\mathrm{H}_0(D_{ijk}, \mathbb{Z})$ and $\mathrm{H}_1(D_{ij}, \mathbb{Z})$ are zero. In particular, we get the surjection

$$\oplus \mathrm{H}_2(D_i, \mathbb{Z}) \longrightarrow \mathrm{H}_2(D, \mathbb{Z}).$$

In this way, by the Lefschetz (1,1) theorem for D_i, we get the algebraicity of elements in $\mathrm{H}^2(D, \mathbb{Z})$. $\square$

Sketch of the proof for Theorem 2.0.2. Let $\mathcal{L}$ be an ample line bundle on Y, and let S be a smooth surface as an object in the linear system $|\mathcal{L}|$. By an argument of Goresky and MacPherson, the fact that Y has lci singularities implies that we have the following surjection of homology groups

$$\mathrm{H}_2(S, \mathbb{Z}) \longrightarrow \mathrm{H}_2(Y, \mathbb{Z}).$$

Now consider the Hilbert scheme $\mathcal{H}$ of smooth surfaces in Y whose homology class in $\mathrm{H}^4(Y, \mathbb{Z})$ is the same as that of S (here recall that the cycle map take $[S] \in \mathrm{CH}_2(X)$ into the degree 4 homology group). By taking a high power of $\mathcal{L}$ if necessary, we may assume $\mathcal{H}$ to be smooth. [2] Then for each $t \in \mathcal{H}$, we get a surjection $\mathrm{H}_2(S_t, \mathbb{Z}) \longrightarrow \mathrm{H}_2(Y, \mathbb{Z})$ as a variation of Hodge structure over $\mathcal{H}$. Moreover, any path in $\mathcal{H}$ connecting two points t_1 and t_2 in $\mathcal{H}$ will induces a map of translation between the Hodge structure $\mathrm{H}_2(S_{t_1}, \mathbb{Z})$ to the Hodge structure $\mathrm{H}_2(S_{t_2}, \mathbb{Z})$.

We denote by S_{t_0} to be the surface S chosen at the beginning and fix it. Let $\mathrm{H}_2(S, \mathbb{Z})_{van}$ be the kernel of the surjection $\mathrm{H}_2(S, \mathbb{Z}) \to \mathrm{H}_2(Y, \mathbb{Z})$. We use the Poincaré duality to identify $\mathrm{H}_2(S, \mathbb{Z})$ with $\mathrm{H}^2(S, \mathbb{Z})$, and denote by $\mathrm{H}^2(S, \mathbb{Z})_{van}$ to be the subgroup of $\mathrm{H}^2(S, \mathbb{Z})$ corresponding to $\mathrm{H}_2(S, \mathbb{Z})_{van}$ under the identification. Let $\mathrm{H}^2(S, \mathbb{R})_{van}$ be the base extension of $\mathrm{H}^2(S, \mathbb{Z})_{van}$ along $\mathbb{Z} \to \mathbb{R}$. We then have the following:

Proposition 2.0.5 ([Tot19], 4.2)**.** *Let C be a nonempty contractible cone in $\mathrm{H}^2(S, \mathbb{R})_{van}$. Suppose Y satisfies the following condition*

> *There exists an open contractible neighborhood U of t_0 in $\mathcal{H}$, such that any element η in $\mathrm{H}^2(S, \mathbb{Z})_{van} \cap C$ can be translate to a Hodge class for some $t \in U$.*

Then every Hodge class of Y of dimension one is algebraic.

The relation of those cohomology groups can be seen as follows:

$$\begin{tikzcd}
\eta \in \square \arrow[r, hook] \arrow[dd] & \mathrm{H}_2(S, \mathbb{Z})_{van} \arrow[d, "\sim"', "PD"] \\
 & \mathrm{H}^2(S, \mathbb{Z})_{van} \arrow[d] \\
C \arrow[r, hook] & \mathrm{H}^2(S, \mathbb{R})_{van}.
\end{tikzcd}$$

Proposition 2.0.6 ([Tot19], 5.3)**.** *Assume the terminal projective 3-fold Y has trivial canonical bundle. Then the condition in the Proposition 2.0.5 is satisfied.*

As I do not have any insight about those two Propositions, I am not going to say anything here. $\square$

[2] I don't quite understand this statement.

3 Application to integral Tate conjecture in characteristic 0

At last, we use the integral Hodge conjecture to prove the integral Tate conjecture of 3-folds X in characteristic 0, assuming either X is rational connected or is of Kodaira dimension zero with $h^0(X, K_X) > 0$.

Before we prove the main result, we first relate the Tate conjecture and the integral Tate conjecture of codimension one together. Here is a quick observation.

Lemma 3.0.1. *Let X be a smooth projective varieties over a finitely generated field k. Then the Tate conejcture of codimension one of X implies its integral Tate conjecture of codimension one.*

Proof. Let ℓ be a prime number invertible in k. Consider the short exact sequence of étale sheaves

$$0 \longrightarrow \mu_{\ell^n} \longrightarrow \mathbb{G}_m \xrightarrow{\ell^n} \mathbb{G}_m \longrightarrow 0.$$

Its long exact sequence induces a short exact sequence below

$$0 \longrightarrow \mathrm{NS}(X)/\ell^n \longrightarrow \mathrm{H}^2(X_{k^s}, \mu_{\ell^n}) \longrightarrow \mathrm{Hom}(\mathbb{Z}/\ell^n, \mathrm{Br}_X) \longrightarrow 0.$$

Here we use the fact that the kernel of the surjection $\mathrm{Pic}(X) \to \mathrm{NS}(X)$ is ℓ-divisible. As X is smooth projective, its Néron-Severi group is finitely generated, so the inverse limit with respect to n above gives a short exact sequence

$$0 \longrightarrow \mathrm{NS}(X) \otimes \mathbb{Z}_\ell \longrightarrow \mathrm{H}^2(X_{k^s}, \mu_{\ell(1)}) \longrightarrow \mathrm{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, \mathrm{Br}_X) \longrightarrow 0.$$

Here we note that the transition maps above are $\cdot\ell, \cdot\ell$, and the identity separately. We also notice that the ℓ-adic Tate group $T_\ell(X) = \mathrm{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, \mathrm{Br}_X)$ is ℓ-torsion free.

Now notice that the cycle class map $\mathrm{CH}^1(X) \to \mathrm{H}^2(X_{k^s}, \mathbb{Z}_\ell(1))$ factors through $\mathrm{CH}^1(X) \to \mathrm{NS}(X)$. So the rational Tate conjecture of codimension zero implies the vanishing of $\mathrm{Hom}(\mathbb{Z}_\ell, \mathrm{Br}_X) \otimes \mathbb{Q}_\ell$, which implies the vanishing of $\mathrm{Hom}(\mathbb{Z}_\ell, \mathrm{Br}_X)$. So we get the integral Tate conjecture of X. $\square$

We then provides a collection of 3-folds that satisfy the Tate conjecture of codimension one.

Proposition 3.0.2. *Let X be a smooth projective 3-fold over a finitely generated field k of characteristic 0. Assume X is either rational connected, or has Kodaira dimension zero with $h^0(X, K_X) > 0$. Then X satisfies the Tate conjecture of codimension one.*

At last, we obtain the integral Tate conjecture, and in particular prove the Theorem 1.1.4. We fix a finitely generated field k of characteristic 0, together with an embedding of k and k^s into $\mathbb{C}$.

Proposition 3.0.3. *Let X be smooth projective 3-fold over k, such that it satisfies the Tate conjecture of codimension one and integral Hodge conjecture. Then the integral Tate conjecture holds for X.*

Proof. By assumption, it suffices to prove the integral Tate conjecture in codimension 2. Let H be a very ample line bundle on X that is defined over k. Consider the following diagram

$$\begin{tikzcd}
\mathrm{CH}^1(X) \arrow[r, "\cup H"] \arrow[d] & \mathrm{CH}^2(X) \arrow[d] \arrow[dr] & \\
\mathrm{H}^2(X_{k^s}, \mathbb{Z}_\ell(1))^{\mathrm{Gal}(k^s/k')} \arrow[r] \arrow[d] & \mathrm{H}^4(X_{k^s}, \mathbb{Z}_\ell(2))^{\mathrm{Gal}(k^s/k')} \arrow[d] & \mathrm{H}^4(X_{\mathbb{C}}, \mathbb{Z}) \cap \mathrm{H}^{2,2}(X) \arrow[l] \\
\mathrm{H}^2(X_{k^s}, \mathbb{Q}_\ell(1))^{\mathrm{Gal}(k^s/k')} \arrow[r, "\sim"'] & \mathrm{H}^4(X_{k^s}, \mathbb{Q}_\ell(2))^{\mathrm{Gal}(k^s/k')} &
\end{tikzcd},$$

where the map of Chow groups is given by the intersection with H, which is compatible with the cup product by its image in ℓ-adic cohomology and is Galois equivariant. Here the Hard Lefschetz theorem for rational ℓ-adic cohomology (proved by Deligne in his second paper on Weil's conjecture) implies that the bottom horizontal map is an isomorphism. Now let u be an element in $\mathrm{H}^4(X_{k^s}, \mathbb{Z}_\ell(2))^{\mathrm{Gal}(k^s/k')}$. By the Hard Lefschetz theorem and the integral Tate conjecture for codimension one, there exists an algebraic 1-cycle v and an positive integer n such that $\ell^n \cdot u$ is equal to the cup product $v \cup H$, which is an algebraic class. On the other hand, we can identify u as an element in $\mathrm{H}^4(X_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{Z}_\ell$ by the abstract isomorphic of $\mathbb{Z}_\ell$-modules

$$\mathrm{H}^4(X_{k^s}, \mathbb{Z}_\ell(2)) \cong \mathrm{H}^4(X_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{Z}_\ell.$$

Notice that the singular cohomology $\mathrm{H}^4(X_{\mathbb{C}}, \mathbb{Z})$ has a direct summand decomposition by the Hodge structure where the subgroup of the Hodge class is a direct summand by the integral Hodge conjecture. In this way, the element $\ell^n u$ is a ($\mathbb{Z}_\ell$-linear combination of) Hodge class implies that the element u itself is a Hodge class, which is algebraic.[3] So we get the result. □

References

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[3] I think I don't understand this statement: by the direct summand decomposition, and the identification of $\mathbb{Z}_\ell$ modules: $\mathrm{H}^4(X_{k^s}, \mathbb{Z}_\ell(2)) \cong \mathrm{H}^4(X_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{Z}_\ell$, we can write u as $a + b$, where a is a $\mathbb{Z}_\ell$-combination of Hodge classes and b is the complement. The condition $\ell^n \cdot u$ is Hodge merely implies that b is killed by ℓ^n, but seems not suggest b itself vanishes. Do I miss something here?