Several approaches to the Grothendieck Duality

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Abstract

This is an expository article on Grothendieck duality, and serves as an extended note for the talk in the Condensed Mathematics Seminar, at the University of Michigan, on Nov 7, 2019. We explain the main ideas of the duality, give an application in birational geometry, and introduce four different approaches to the duality, following Hartshorne [Har66], Neeman [Nee96], Deligne [Del], and the condensed math of Clausen-Scholze in [Sch19] separately.

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1 Statements of the Grothendieck duality

Fix a field k. Unless specifically mentioned in the note, all of the schemes are assumed to be finite type over k. In particular they are all quasi-compact and quasi-separated.

1.1 Statements

We first state the duality. Let S be a scheme. Recall that $D_{\mathrm{Qcoh}}(S)$ is the full subcategory of the derived category of sheaves over S, where objects have quasi-coherent cohomology. When S is a finite type scheme, the natural functor from the derived category of quasi-coherent sheaves $D\mathrm{Qcoh}(S)$ to $D_{\mathrm{Qcoh}}(S)$ is an equivalence, and is functorial with respect S under the derived direct image.

Theorem 1.1.1 (Grothendieck Duality). *Let $f : X \to Y$ be a separated map of finite type schemes over k. Then we have the following*

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(i) *The derived direct image $Rf_* : D_{\mathrm{Qcoh}}(X) \to D_{\mathrm{Qcoh}}(Y)$ admits a right adjoint $f^! : D_{\mathrm{Qcoh}}(Y) \to D_{\mathrm{Qcoh}}(X)$. Moreover, for any $K \in D_{\mathrm{Qcoh}}(X)$, $L \in D_{\mathrm{Qcoh}}(Y)$, the counit map $Rf_* f^! \to id$ induces the following natural quasi-isomorphism in $D_{\mathrm{Qcoh}}(Y)$:*

$$Rf_* R\mathcal{H}om_X(K, f^! L) \longrightarrow R\mathcal{H}om_Y(Rf_* K, L).$$

Moreover, both Rf_* and $f^!$ can be restricted to the bounded below derived subcategories, which also forms an adjoint pair. (cf. [Nee96], 6.3)

(ii) *Assume f is proper and of finite Tor-dimension, then for any $L \in D_{\mathrm{Qcoh}}(Y)$, we have a natural quasi-isomorphism*

$$f^! L = f^! \mathcal{O}_Y \otimes^L_{\mathcal{O}_Y} Lf^* L.$$

Moreover, both Rf_* and $f^!$ can be restricted to upper-bounded/lower-bounded/bounded derived category of coherent cohomology. (cf. [Nee18], 2.14, 5.13)

More explicitly, in the Theorem 1.1.1 (i) above, given a map $K \to f^! L$ in $D_{\mathrm{Qcoh}}(X)$, we can apply the functor Rf_* and get

$$Rf_* K \to Rf_* f^! L \to L,$$

where the latter is the counit natural transformation $Rf_* f^! \to id$. So we get a map from $Rf_* K \to L$. This induces a map in the derived category

$$Rf_* R\mathcal{H}om_X(K, f^! L) \longrightarrow R\mathcal{H}om_Y(Rf_* K, L).$$

A upshot is the composition law:

Proposition 1.1.2. *Let $f : X \to Z$ be the composition of maps $g : X \to Y$ and $h : Y \to Z$ of finite type k-schemes. Then we have a natural isomorphism*

$$f^! = g^! h^!.$$

Proof. This follows from the uniqueness of the right adjoint. □

Under the assumption of f being proper and finite Tor-dimension, we call $f^! \mathcal{O}_Y$ the *dualizing complex of* f, and is sometimes denoted by $\omega_f^\bullet$. Moreover, in a slightly better case when $f : X \to Y$ is a Cohen-Macaulay morphism, the dualizing complex is in fact a cohomological twist of a quasi-coherent sheaf.

Theorem 1.1.3 (Dualizing sheaf)**.** *Let $f : X \to Y$ be a morphism of finite type schemes over k. Assume X is connected. Then we have the following:*

(a) *If f is Cohen-Macaulay, then there exists a coherent sheaf ω_f over X such that*

$$f^! \mathcal{O}_Y = \omega_f[n],$$

for some integer $n \in \mathbb{N}$. If the map f is of purely relative dimension d, then n is equal to d.

(b) *If f is furthermore a smooth morphism, then we have*

$$f^! \mathcal{O}_Y = \Omega^n_{X/Y}[n],$$

where $n = \dim(X) - \dim(Y)$.

When $Y = \mathrm{Spec}(k)$ and $f : X \to Y$ is the structure map, we use ω_X to denote the coherent sheaf ω_f in the above situation.

Remark 1.1.4 ([Sta], Tag 0C0Z)**.** As a complement of the Theorem 1.1.3, (a), for a proper morphism $f : X \to Y$, in fact we have the following equivalent conditions:

(i) The morphism f is Cohen-Macaulay at $x \in X$.

(ii) The dualizing complex $f^!\mathcal{O}_Y$ has a unique quasi-coherent cohomology sheaf in a Zariski neighborhood of x.

Remark 1.1.5. Note that as both f^* and the tensor product functor are left adjoint functors, when $f^!$ is equal to $Lf^*(-) \otimes^L f^!\mathcal{O}_Y$, the upper shriek functor commutes with arbitrary colimits. In fact it is proved in [Nee96] 5.4 that for a morphism of schemes $f : X \to Y$ such that Y is quasi-compact and separated, and Rf_* admits the right adjoint $f^!$, the following are equivalent:

(i) There exists a natural isomorphism of functors $D_{\mathrm{Qcoh}}(X) \to D_{\mathrm{Qcoh}}(Y)$

$$f^!L \longrightarrow Lf^*L \otimes^L_{\mathcal{O}_X} f^!\mathcal{O}_Y.$$

(ii) The upper-shriek functor $f^!$ commutes with any coproducts (colimits).

1.2 Serre Duality

Historically speaking, Grothendieck duality is designed to generalize the Serre duality for coherent sheaves over projective varieties to a broader collection of schemes and derived objects. We use the duality theorems above to illustrate how to recover the classical Serre duality of projective spaces as follows (cf. [Har77], III section 7).

Consider the case when $Y = \mathrm{Spec}(k)$, and $X = \mathbb{P}^n$ be the projective space of dimension n over k. Then by the Theorem 1.1.3, (b), we know the dualizing complex $f^!\mathcal{O}_Y = \omega_{\mathbb{P}^n}[n]$, which the sheaf $\omega_{\mathbb{P}^n}$ is equal to the line bundle $\mathcal{O}_{\mathbb{P}^n}(-n-1)$, by taking the top wedge product at the Euler exact sequence

$$0 \longrightarrow \Omega^1_{\mathbb{P}^n/k} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0.$$

We apply the Grothendieck duality 1.1.1 at $f : \mathbb{P}^n \to \mathrm{Spec}(k)$, with $K = \mathcal{F}$ being any coherent sheaf over $\mathbb{P}^n$. Then we get the following functorial quasi-isomorphism

$$R\,\mathrm{Hom}_k(\mathcal{F}, \omega_{\mathbb{P}^n}[n]) \longrightarrow R\,\mathrm{Hom}_k(R\Gamma(\mathbb{P}^n, \mathcal{F}), k).$$

We twist both sides by cohomological degree $[-n]$ and get

$$R\,\mathrm{Hom}_k(\mathcal{F}, \omega_{\mathbb{P}^n}) \longrightarrow R\,\mathrm{Hom}_k(R\Gamma(\mathbb{P}^n, \mathcal{F})[n], k).$$

Note that the functor $\mathrm{Hom}_k(-, k)$ is exact on finite dimensional vector spaces, while $\mathbb{P}^n$ is proper smooth over k. So by taking the i-th cohomology of the above quasi-isomorphism, we get an isomorphism of k-vector spaces:

$$\mathrm{Ext}^i(\mathcal{F}, \omega_{\mathbb{P}^n}) \longrightarrow \mathrm{H}^{n-i}(\mathbb{P}^n, \mathcal{F})'.$$

Here we note that as mentioned after the Theorem 1.1.1, the above isomorphism is induced from the counit transformation $Rf_*f^! \to id$. More explicitly, given a map $\rho : [\backslash]\mathcal{F} \to \{^!\| = \omega_{\mathbb{P}\backslash}[\backslash]$, we can apply $Rf_* = R\Gamma(\mathbb{P}^n, -)$ to its cohomological twist and get the composition

$$R\Gamma(\mathbb{P}^n, \mathcal{F}) \longrightarrow R\Gamma(\mathbb{P}^n, \omega_{\mathbb{P}^n}) \longrightarrow k[-n],$$

whose n-th cohomology is

$$\mathrm{H}^n(\mathbb{P}^n, \mathcal{F}) \longrightarrow \mathrm{H}^n(\mathbb{P}^n, \Omega_{\mathbb{P}^n/k}) \longrightarrow k.$$

It can be showed that the last map above is an isomorphism, and thus we get a perfect pairing:

$$\mathrm{Hom}_{\mathbb{P}^n}(\mathcal{F}, \omega_{\mathbb{P}^n}) \times \mathrm{H}^n(\mathbb{P}^n, \mathcal{F}) \longrightarrow \mathrm{H}^n(\mathbb{P}^n, \omega_{\mathbb{P}^n}) = k,$$

which is exactly the *trace map* in the classical setting.

2 Application in birational geometry: a criterion for the rational singularity

Here we provides an application of the Grothendieck duality to the birational geometry, on the criterion of the rational singularities in characteristic 0. We mostly follow the short paper by Kovács [Kov00]

2.1 Statement

The statement is the following.

Theorem 2.1.1 ([Kov00], Theorem 1)**.** *Let Y $f : X \to Y$ be a morphism of finite type separated schemes over $\mathbb{C}$, and let $\rho : \mathcal{O}_Y \to Rf_*\mathcal{O}_X$ be the canonical map of structure sheaves. Assume Y is normal, X has rational singularities, and there exists a map $\rho' : Rf_*\mathcal{O}_X \to \mathcal{O}_Y$ in the derived category $D_{\mathrm{Coh}}(X)$ such that the composition $\rho' \circ \rho$ is a quasi-isomorphism of $\mathcal{O}_Y$ itself. Then Y also has rational singularities.*

Corollary 2.1.2. *The quotient singularity is rational.*

Remark 2.1.3. The condition for X being rational singular is not hard to achieve: By the resolution of singularities in characteristic 0, given a variety Y over $\mathbb{C}$, we can always find a finite composition of blowups such that the composition $X \to Y$ is birational with X being smooth, hence satisfies the assumption that X has rational singularities.

Remark 2.1.4. Here the existence of $\rho' : Rf_*\mathcal{O}_X \to \mathcal{O}_Y$ should be considered as the existence of the trace operator in the relative situations.

2.2 Proof of the criterion

We first recall the Grauert-Riemenschneider vanishing theorem.

Theorem 2.2.1 (Grauert-Riemenschneider)**.** *Let $f : X' \to X$ be a resolution of singularities of a finite type scheme X over $\mathbb{C}$. Then we have the vanishing of the higher direct image*

$$R^i f_*\omega_{X'} = 0, \ \forall i > 0.$$

Note that the Grauer-Riemenschneider vanishing implies the following observation.

Lemma 2.2.2. *Let X be a normal separated Cohen-Macaulay scheme of finite type over $\mathbb{C}$, and let $\phi : X' \to X$ be a resolution of singularities of X. Assume $\phi_*\omega_{X'} = \omega_X$. Then X has rational singularities.*

Proof. We first notice that under the assumption, the Grauert-Riemenschneider vanishing implies that

$$\omega_X^\bullet = R\phi_*\omega_{X'}^\bullet.$$

By applying twice the duality theorem, we have

$$\begin{aligned} \mathcal{O}_X &\cong R\mathcal{H}om_X(\omega_X^\bullet, \omega_X^\bullet) \\ &= R\mathcal{H}om_X(R\phi_*\omega_{X'}^\bullet, \omega_X^\bullet) \\ &= R\phi_*\mathcal{H}om_{X'}(\omega_{X'}^\bullet, f^!\omega_X^\bullet) \\ &= R\phi_*\mathcal{H}om_{X'}(\omega_{X'}^\bullet, \omega_{X'}^\bullet) \\ &\cong R\phi_*\mathcal{O}_{X'}. \end{aligned}$$

So we are done.

□

Now we can prove the Theorem 2.1.1.

Proof of the Theorem 2.1.1. We first notice that by using the resolution of singularities and the definition of the rational singularity, we may replace X by a resolution of singularities of Y. So we assume the map $f : X \to Y$ is a resolution of singularities.

By the Lemma 2.2.2, it suffices to show that Y is Cohen-Macaulay and $\omega_Y^\bullet = f_*\omega_X$.

We apply the duality functor $R\mathcal{H}om_Y(-, \omega_Y^\bullet)$ to the maps

$$\mathcal{O}_Y \longrightarrow Rf_*\mathcal{O}_X \longrightarrow \mathcal{O}_Y,$$

and get

$$\omega_Y^\bullet \longrightarrow Rf_*\omega_X^\bullet \longrightarrow \omega_Y^\bullet,$$

where the middle term follows from the duality theorem 1.1.1. As the composition $\rho' \circ \rho$ is a quasi-isomorphism, the above induces an injection

$$\mathcal{H}^i(\omega_Y^\bullet) \longrightarrow Rf_*^i\omega_X^\bullet.$$

As X is smooth over $\mathbb{C}$, the dualizing complex $\omega_X^\bullet$ is equal to $\omega_X[d]$ for $d = \dim(X)$. By the Grauert-Riemenschnerder vanishing 2.2.1, $Rf_*^i\omega_X^\bullet = Rf_*^{i+d}\omega_X = 0$ for $i > -d$. So the above injection implies that $\omega_Y^\bullet$ lives in cohomological degree $\leq -d$. However, it can be shown that the dualizing complex $\omega_Y^\bullet$ always lives in degree $[-d, +\infty)$.[1] Hence the dualizing complex is a sheaf living in the cohomological degree $-d$, and thus Y is Cohen-Macaulay.

At last, after a cohomological twist and the Grauet-Riemenschneider vanishing, we are left to check the equality of $\omega_Y \to f_*\omega_X$, or the injection of the section map $f_*\omega_X \to \omega_Y$. We first notice that since f is a resolution of singularities, the injection above becomes the equality on the regular locus. The whole equality follows from the fact that for a CM scheme Y, the dualizing sheaf ω_Y is reflexive, and hence those two are equal.[2]

□

3 Hartshorne's approach: explicit study on dualizing complexes

Here we mentioned Hartshorne's approach on the Grothendieck duality, by studying the dualizing complexes.

Dualizing complexes Let $f : X \to Y$ be a morphism of finite type schemes over k. Assume Rf_* admits a right adjoint $f^!$ such that it commutes with colimits. Then by the Remark 1.1.5, we know $f^!$ can be written as the derived tensor product with the dualizing complexes $f^!\mathcal{O}_Y$. This suggests that in many situations the study of the duality is essentially about the study of the dualizing complex.

We then note that in the case of projective spaces $X = \mathbb{P}^n$ and $Y = \mathrm{Spec}(k)$, the dualizing complex $f^!k = \omega_{\mathbb{P}^n}[n]$ produces an anti-equivalence of the bounded coherent derived category $D^b_{\mathrm{Coh}}(X)$, by

$$K \longmapsto R\mathcal{H}om_{\mathbb{P}^n}(K, f^!k).$$

This suggests a study of broader classes of objects in the derived category as follows:

Definition 3.0.1 ([Har66], Chap V, Section 2)**.** *Let X be a finite type scheme over k, and $C^\bullet \in D^+_{\mathrm{Coh}}(X)$ be a bounded below complex of coherent cohomology. We call $C^\bullet$ a* dualizing complex *if it satisfies the following conditions*

(i) The complex $C^\bullet$ has finite injective dimension.

[1]This can be tested locally, which follows from the Proposition 3.0.5 and the Proposition 3.0.6 in the next section.

[2]More precisely, it can be showed as follows: let $i : Y_{reg} \to Y$ be the open immersion of the regular locus of Y, whose complement is of codimension at least 2 by assumption. Then the fact ω_Y is reflexive on Y implies that $\omega_Y = i_*\omega_{Y_{reg}}$. On the other hand, since $X \to Y$ is a resolution of singularity, the preimage of Y_{reg} along f is isomorphic to Y_{reg}. Furthermore, the open immersion $j : Y_{reg} \to X$ induces an injection $\omega_X \to j_*\omega_{Y_{reg}}$. Thus by taking the direct image along f, we get an injection $f_*\omega_X \to f_*j_*\omega_{Y_{reg}} = i_*\omega_{Y_{reg}} = \omega_Y$.

(ii) The natural map

$$\mathcal{O}_X \longrightarrow R\mathcal{H}om_X(C^\bullet, C^\bullet)$$

is a quasi-isomorphism.

Here we note that as each of the above conditions are local conditions, being a dualizing complex can also be checked locally.

The following observation judges the name of the dualizing complex:

Proposition 3.0.2 ([Har66], Chap V, Proposition 2.1)**.** *Let X be a finite type scheme over k, and $C^\bullet$ a dualizing complex. Then any object $K \in D_{\mathrm{Coh}}(X)$ is reflexive with respect to $C^\bullet$. Namely, the following canonical map is a quasi-isomorphism*

$$K \longrightarrow R\mathcal{H}om_X(R\mathcal{H}om_X(K, C^\bullet), C^\bullet).$$

Here are some examples of dualizing complexes

Example 3.0.3. Let $X = \mathrm{Spec}(k)$, then the image of $k[n]$ for any integer $n \in \mathbb{Z}$ in $D_{\mathrm{Coh}}(X)$ is a dualizing complex.

Example 3.0.4. Let $X = \mathbb{P}^n$ be the projective space over k. Then for any line bundle $\mathcal{L}$ and any integer $n \in \mathbb{Z}$, the object $\mathcal{L}[n]$ is a dualizing complex in $D_{\mathrm{Coh}}(X)$. Here the only non-formal thing to check is the finite injective dimensionality. In fact, over a given regular noetherian scheme X of finite Krull dimension, any coherent sheaf $\mathcal{F}$ has finite injective dimension.

As an upshot, for the structure map $f : \mathbb{P}^n \to \mathrm{Spec}(k)$, the upper shriek $f^!k = \omega_{\mathbb{P}^n}[n]$, which is a cohomological twist of the line bundle $\omega_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-n-1)$, is a dualizing complex in the sense of Definition 3.0.1.

To construct a dualizing complex, we make use of the following two observations.

Proposition 3.0.5 ([Har66], Chap V, 2.4)**.** *Let $f : X \to Y$ be a finite morphism between two finite type schemes over k, and let $C^\bullet$ be a dualizing complex of Y. Then the complex $f^\flat C^\bullet := R\mathcal{H}om_Y(f_*\mathcal{O}_X, C^\bullet) \in D_{\mathrm{Coh}}(X)$ is a dualizing complex of X.*

Proposition 3.0.6 ([Har66], Chap V, 8.3)**.** *Let $f : X \to Y$ be a smooth morphism between two finite type schemes over k, and let $C^\bullet$ be a dualizing complex of Y. Then the complex $f^\# C^\bullet := f^*C^\bullet \otimes^L \omega_{X/Y}[m]$ for $m = \dim(X) - \dim(Y)$ is a dualizing complex.*

The above two results allow us to give a very explicit way of constructing dualizing complexes in many situations.

Example 3.0.7. Let $X = \mathrm{Spec}(A)$ for A a finite types algebra over k. Then by the Noetherian's normalization theorem we can find a subalgebra A_0 of X such that A is finite over A_0, and A_0 is isomorphic to a polynomial ring over k. Take $Y = \mathrm{Spec}(A_0)$, and let $g : X \to Y$ and $h : Y \to \mathrm{Spec}(k)$ be the canonical morphisms. Then the object

$$g^\# h^\flat(k)$$

is a dualizing complex of X.

In fact, the dualizing complex is unique up to a twist. More precisely, we have

Theorem 3.0.8 (Uniqueness; [Har66], Chap V, 3.1)**.** *Let X be a connected finite type scheme over k, $C^\bullet$ be a dualizing complex and $E^\bullet$ be another lower-bounded complex in $D^+_{\mathrm{Coh}}(X)$. Then $E^\bullet$ is a dualizing complex of X if and only if there exists a line bundle $\mathcal{L}$ on X and an integer $n \in \mathbb{Z}$, such that*

$$E^\bullet \cong C^\bullet \otimes \mathcal{L}[n].$$

Digestions The above discussion about the existence and the uniqueness of the dualizing complexes suggest that we can define the upper shriek functor in the following way. Let $f : X \to Y$ be a map of finite type schemes over k, and let $C^\bullet$ be a dualizing complexes. Then we define a functor $f^! : D_{\mathrm{Qcoh}}(Y) \to D_{\mathrm{Qcoh}}(X)$ as follows:

$$L \longmapsto Lf^*L \otimes^L_{\mathcal{O}_X} f^!C^\bullet.$$

This approach is good for several reasons.

- From the construction, we can see that both the dualizing complexes and the upper shriek functor are very explicit and can be computable in many cases.
- Moreover, it is clear that this construction is a local construction, and is compatible when passing to a localization.

However, the main drawback of the approach is that the functor $f^!$ constructed this way cannot be expected to be a right adjoint of any known functors like Rf_* anymore. As mentioned in the Remark 1.1.5, the right adjoint of Rf_* can be written as a tensor product with a dualizing complex only when the right adjoint preserves the colimits. In fact, it is shown in [Nee18] 2.14 that for a proper morphism $f : X \to Y$, the right adjoint of Rf_* preserves the colimits if and only if f is of finite Tor dimension.

4 Neeman's approach: Brown's representable theorem

We then provides an abstract approach given by Neeman [Nee96], using the Brown's representable theorem. We will see soon that the existence of the right adjoint $f^!$ of the derived push-forward Rf_* follows from easily from a very general result.

We first recall the following definition for a triangulated category of being compactly generated.[3]

Definition 4.0.1. *Let $\mathcal{T}$ be a triangulated category. We call it is* compactly generated *if $\mathcal{T}$ admits arbitrary small coproducts, and there exists a small set of compact objects T in $\mathcal{T}$, such that the vanishing of an object $L \in \mathcal{T}$ can be checked by T:*

$$\mathrm{Hom}(K, L) = 0,\ \forall K \in T \Longrightarrow L = 0.$$

Example 4.0.2. Let X be a finite type scheme over the field k. Then the derived category $D_{\mathrm{Qcoh}}(X)$ of quasi-coherent cohomologies admits small coproducts, by taking the direct sum of complexes termwisely.

Moreover, $D_{\mathrm{Qcoh}}(X)$ is compactly generated. When X is affine, the generating set T can be the collection $\{\mathcal{O}_X[n],\ n \in \mathbb{Z}\}$, as

$$\mathrm{Hom}_{D_{\mathrm{Qcoh}}(X)}(\mathcal{O}_X[n], L) = \mathrm{Hom}_{D_{\mathrm{Qcoh}}(X)}(\mathcal{O}_X, L[-n]) = \mathrm{H}^{-n}(X, L).$$

The similar method applies to the case when X is projective. The general situation needs the fact about the compact generatedness of the derived category with supports, which we refer the reader to [Nee96], 2.5 and 2.6.

The Brown's representable theorem as follows.

Theorem 4.0.3 (Brown representability; [Nee96], 3.1)**.** *Let $\mathcal{T}$ be a compactly generated triangulated category, and let $F : \mathcal{T}^{\mathrm{op}} \to \mathrm{Ab}$ be an exact functor (namely it takes distinguished triangles into long exact sequence). Suppose F commutes with small coproduct. Then F is representable.*

Proposition 4.0.4. *Let $f : X \to Y$ be a map of finite type schemes over k, and L be an complex in $D_{\mathrm{Qcoh}}(Y)$. Then the contravariant functor from $D_{\mathrm{Qcoh}}(X)$ to* Ab

$$K \longmapsto \mathrm{Hom}_{D_{\mathrm{Qcoh}}(Y)}(Rf_*K, L)$$

is representable. In particular, the derived direct image Rf_* admits a right adjoint.

[3] The reader who is not familiar with the triangulated category may just assume it is the derived category $D(A)$ of a ring A or $D(X)$ for a scheme X.

Proof. We applies the Brown's representability 4.0.3 to the setting $\mathcal{T} = D_{\mathrm{Qcoh}}(X)$ and the functor

$$D_{\mathrm{Qcoh}}(X) \ni K \longmapsto \mathrm{Hom}_{D_{\mathrm{Qcoh}}(Y)}(Rf_*K, L).$$

The only thing we need to check is the compatibility with the coproducts, which we have:

Claim 4.0.5. Let $f : X \to Y$ be a separated morphism of from a quasi-compact separated scheme X to a scheme Y. Then Rf_* commutes with coproducts.

Proof of the Claim. Let K_i be a collection of objects in $D_{\mathrm{Qcoh}}(X)$ over a small index category. Then we need to check the following natural map is a quasi-isomorphism

$$\coprod_i Rf_*K_i \longrightarrow Rf_*(\coprod_i K_i).$$

As this is a local statement over Y, it suffices to assume Y is affine.

By assumption of the quasi-compactness and separatedness of X, we may write X as the union of finite affine open subsets $X = \cup_{j=1}^n U_j$, with intersection of any two U_j being affine. We induction on the number n of affine open pieces. When $n = 1$, since f is an affine morphism in this case, the functor $Rf_* = f_*$ is just the forgetful functor of chain complexes, which commutes with coproducts. For general n, let $V = \cup_{j=2}^n U_j$ be the union. We then note that since both V and $U_1 \cap V$ are union of $n-1$ open affine pieces (the latter follows from the separatedness), by induction the derived push-forward of f restricted to both of open subsets satisfies the condition. Thus the get the whole $X = U \cup V$, it suffices to use Mayer-Vietories sequence, namely the hypercovering of X by $U \cap V \to U \coprod V \to X$, and the distinguished triangle associated to this. □

□

Remark 4.0.6. The the Brown's representability only shows that for each individual L, we can define an object $f^!L$. This a priori does not imply the map $L \mapsto f^!L$ is functorial. However, pointed by Emanuel Reinecke, the functoriality could follow from that the Yoneda embedding is fully faithful, as a given map $L \to L'$ will induce a natural transformation of functors $\mathrm{Hom}(Rf_*(-), L) \to \mathrm{Hom}(Rf_*(-), L')$.

By the uniqueness of the right adjoint, the functor we get is exactly the $f^!$ in the Duality theorem 1.1.1. So for any $K \in D_{\mathrm{Qcoh}}(X)$ and $L \in D_{\mathrm{Qcoh}}(Y)$, we have the following functorial isomorphism of abelian groups

$$\mathrm{Hom}_{D_{\mathrm{Qcoh}}(X)}(K, f^!L) \longrightarrow \mathrm{Hom}_{D_{\mathrm{Qcoh}}(Y)}(Rf_*K, L),$$

induced from the counit map $Rf_*f^! \to id$. Moreover, by taking the K-injective resolution of K (sorry for the abuse of notation) and change L by $L[n]$ for $n \in \mathbb{Z}$, we get the natural map

$$R\,\mathrm{Hom}_{D_{\mathrm{Qcoh}}(X)}(K, f^!L) \longrightarrow R\,\mathrm{Hom}_{D_{\mathrm{Qcoh}}(Y)}(Rf_*K, L),$$

which is a quasi-isomorphism by checking each Ext group and use the isomorphism above (for a twist of L). Hence we get the (non-sheafifed) version of the Grothendieck duality.

5 Deligne's approach: Ind and pro objects

At last, we give another abstract approach to duality introduced by Deligne (cf. [Del]), and consider the ind objects and pro objects of derived coherent category.

Let $\mathcal{A}$ be an abelian category. Following the definition in [Del], we define the *category of pro-systems* in $\mathcal{A}$, denoted by $\mathrm{pro}\mathcal{A}$, to be the category where objects are pro-systems over a small category. For the derived category $D_{\mathrm{Coh}}(X)$ of coherent cohomologies, we define the full subcategory $\mathrm{pro}D^b_{\mathrm{Coh}}(X)$ of the pro-system derived category $\mathrm{pro}D_{\mathrm{Coh}}(X)$, to be the pro-system of objects $K = $ "$\varprojlim$"K_i in $D_{\mathrm{Coh}}(X)$, such that the cohomological degrees of K_i are uniformly bounded.

Let $f : X \to Y$ be a map of finite type schemes over k. When f is proper, it is shown by Verdier that the Rf_* has a right adjoint $f^!$, which is exactly the one in the Duality theorem 1.1.1. For the general f, since by Nagata's compactification theorem f can be written as a composition of open immersion followed by a proper map, Deligne's idea is to construct the derived version for the *cohomology with proper support* $f_!$, which serves as a generalization of Rf_* in the proper case. When f is an open immersion, the lower shriek functor is the left adjoint to the pullback $f^* : D^+_{\mathrm{Qcoh}}(Y) \to D^+_{\mathrm{Qcoh}}(X)$. However, to define this functor that can serves as the left adjoint, one has to enter the pro-system in order to realize the limit of colimit of hom groups

$$f_! : \mathrm{pro}D^b_{\mathrm{Coh}}(X) \longrightarrow \mathrm{pro}D^b_{\mathrm{Coh}}(Y).$$

Its construction is given as follows:

Construction 5.0.1. Let X and X' be two finite type schemes over k, and a map $g : X \to \overline{X}$ be an open immersion, with $\mathcal{J}$ the defining ideal of the complement $\overline{X}\backslash X$ in $\overline{X}$. Let $\mathcal{F}$ be a coherent sheaf over X. Pick any coherent sheaf $\widetilde{\mathcal{F}}$ over $\overline{X}$ such that $g^*\widetilde{\mathcal{F}} = \mathcal{F}$. Then we can form the pro system "$\varprojlim$"$\mathcal{J}^n\widetilde{\mathcal{F}}$. It can shown that the pro system "$\varprojlim$"$\mathcal{J}^n\widetilde{\mathcal{F}}$ is independent of choices of lifting $\widetilde{\mathcal{F}}$. So we get a functor

$$\mathrm{Coh}(X) \longrightarrow \mathrm{proCoh}(X).$$

This induces a derived functor

$$D^b_{\mathrm{Coh}}(X) \longrightarrow \mathrm{pro}D^b_{\mathrm{Coh}}(X),$$

and by passing to the pro systems of the source, we get

$$g_! : \mathrm{pro}D^b_{\mathrm{Coh}}(X) \longrightarrow \mathrm{pro}D^b_{\mathrm{Coh}}(X).$$

In the case of open immersion, this lower shriek functor $g_!$ can be also regarded as a derived pro version of the extension by zero. The following result observed by Deligne judges the claim.

Proposition 5.0.2 ([Del], Proposition 4)**.** *Let $g : X \to \overline{X}$ be an open immersion, $\mathcal{J}$ the defining ideal for the complement $\overline{X}\backslash X$. Assume $\mathcal{F}$ is a coherent sheaf over X, and $\mathcal{G}$ is a quasi-coherent sheaf over $\overline{X}$. Then we have the following functorial equality*

$$\mathrm{Hom}_X(\mathcal{F}, g^*\mathcal{G}) = \mathrm{Hom}_{\overline{X}}(\text{“}\varprojlim\text{”}\mathcal{J}^n\widetilde{\mathcal{F}}, \mathcal{G}).$$

Here to make sense of the above two Hom groups, we need the following observation.

Proposition 5.0.3 ([Del], Proposition 2)**.** *Let X be a quasi-compact quasi-separated scheme. Then the category* $\mathrm{indCoh}(X)$ *of ind systems of coherent sheaves over X is naturally equivalent to the category* $\mathrm{Qcoh}(X)$ *of quasi-coherent sheaves over X.*

The Proposition makes it clear that in the Hom groups above, the target on both sides can be written as ind systems "$\varinjlim$"G_i and "$\varinjlim$"g^*G_i separately. So the Hom groups are computed by

$$\mathrm{Hom}_X(\mathcal{F}, g^*\mathcal{G}) = \varinjlim_i \mathrm{Hom}_X(\mathcal{F}, g^*\mathcal{G}_i);$$

$$\mathrm{Hom}_{\overline{X}}(\text{“}\varprojlim\text{”}\mathcal{J}^n\widetilde{\mathcal{F}}, \mathcal{G}) = \varinjlim_n \varinjlim_i \mathrm{Hom}_{\overline{X}}(\mathcal{J}^n\widetilde{\mathcal{F}}, \mathcal{G}_i).$$

The above gives the construction of the cohomology of proper support for an open immersion. This allows us to define the cohomology of proper support for general morphism $f = h \circ g : X \to Y$ that is compactifiable, by taking $Rf_! : \mathrm{pro}D^b_{\mathrm{Coh}}(X) \to \mathrm{pro}D^b_{\mathrm{Coh}}(Y)$ as $g_! \circ Rh_*$. We also define the upper shriek functor $Rf^!$ to be $h^! \circ g^*$, where $h^!$ is the right adjoint of Rh_* for the proper map h as in the Theorem 1.1.1. We note that this $Rf^!$ is different from the functor $f^!$ for general f in the Theorem 1.1.1. However, the construction provides us a duality theorem involving the pro systems and ind systems of coherent derived categories, as follows:

Theorem 5.0.4 (Theorem 2, [Del]). *Let $f : X \to Y$ be a compactifiable morphism between two finite type schemes over k. Then the functor $f_! : \mathrm{pro}D^b_{\mathrm{Coh}}(X) \to \mathrm{pro}D^b_{\mathrm{Coh}}(Y)$ is a "left adjoint" of the functor $Rf^! : D^+_{\mathrm{Qcoh}}(Y) \to D^+_{\mathrm{Qcoh}}(X)$, in the sense that for a pro system $K = \text{"}\varprojlim\text{"} K_i \in \mathrm{pro}D^b_{\mathrm{Coh}}(X)$ and an ind system $L = \text{"}\varinjlim\text{"} L_j \in D^+_{\mathrm{Qcoh}}(Y)$ for $L_j \in D^+_{\mathrm{Coh}}(Y)$, we have the natural equality*

$$\varinjlim_i \varinjlim_j \mathrm{Hom}_{D_{\mathrm{Coh}}(X)}(K_i, Rf^! L_j) = \varinjlim_i \varinjlim_j \mathrm{Hom}_{D_{\mathrm{Coh}}(Y)}(f_! K_i, L_j).$$

Remark 5.0.5. In the above Theorem, the construction of the functors $f_!$ and $Rf^!$ are in fact not share the same categorical framework: $f_!$ is a functor between the category of pro-objects, while $Rf^!$ is a functor between ind-objects. This actually suggests to find a bigger category that behaves better in terms of limits and colimits. We will see in the next section that the category of solid modules, introduced by Clausen-Scholze, actually provides a better framework we want. In particular, we will see how Deligne's formalism of $f_!$ and $Rf^!$ can be constructed for solid modules, which leads to a six functor formalism in the coherent setting.

6 Clausen-Scholze's approach: Condensed mathematics

At the last section, we introduce the approach to the Grothendieck duality via the condensed mathematics, introduced by Clausen-Scholze. We will give the statement of the coherent duality in terms of condensed modules, and illustrate how this implies the classical Grothendieck duality, for a proper map of finite type schemes over $\mathbb{Z}$. The only reference currently is the lecture notes [Sch19] by Scholze.

6.1 Condensed math and discrete topology

We first recall the definition of a condensed set/ring/group.[4]

Definition 6.1.1. *[Sch19, Lecture 1] A* condensed set/ring/group $\mathcal{F}$ *is defined as functor*

$$\mathcal{F} : \{profinite\ sets\}^{\mathrm{op}} \longrightarrow \mathbf{Set/Ring/Group},$$

with the condition $\mathcal{F}(\emptyset) = *$, such that it satisfies the following two conditions:

- *For any two profinite sets S_1, S_2, the natural map below is a bijection*

$$\mathcal{F}(S_1 \coprod S_2) \longrightarrow \mathcal{F}(S_1) \times \mathcal{F}(S_2).$$

- *For any surjection $S' \to S$ of profinite sets, the following map is a bijection*

$$\mathcal{F}(S) \longrightarrow \{x \in \mathcal{F}(S') \mid p_1^*(x) = p_2^*(x) \in \mathcal{F}(S' \times_S S')\},$$

  where p_i are two projection maps from the fiber product $S' \times_S S'$.

It can be checked that the above definition is equivalent to the condition of $\mathcal{F}$ being a sheaf of sets/rings/groups over the pro-étale site $*_{\text{proét}}$ of a single point $*$.

For any topological space/ring/group T, there exists a canonical way to associate a condensed set/ring/group $\underline{T}$ to it, by

$$profinite\ set\ S \longmapsto \mathrm{Map}_{\mathrm{cont}}(S, T).$$

This functor is always faithful, and is full when restricted to the subcategory of compactly generated topological spaces. Here we recall that a space X is *compactly generated* if a map $X \to Y$ to a topological space Y is continuous if and only if the composition $S \to X \to Y$ is continuous, for any

[4] Here we follow the convention of [Sch19] and fix a cardinality κ large enough with certain conditions, in order to prevent any possible set-theoretic issue. All of the constructions below is no larger than the given cardinality κ.

profinite set S and any continuous map $S \to X$. Examples of compactly generated spaces include discrete spaces and profinite sets.

The above functor admits a left adjoint

$$\mathcal{F} \longmapsto \mathcal{F}(*)_{\mathrm{Top}},$$

where $\mathcal{F}(*)_{\mathrm{Top}}$ is the topological space defined over the set $\mathcal{F}(*)$ with the quotient topology given from

$$\coprod_{profinite\ S \to \mathcal{F}} S \longrightarrow \mathcal{F}(*).$$

Here each map $S \to \mathcal{F}(*)$ is given by taking the set of sections of the morphism $\underline{S} \to \mathcal{F}$ at the point $*$, and the disjoint union $\coprod_{profinite\ S \to \mathcal{F}} S$ is equipped with the disjoint union topology (the finest one such that each $S \to \coprod S$ is continuous). So the set $\mathcal{F}(*)_{\mathrm{Top}}$ has the topology such that any a subset $U \in \mathcal{F}(*)_{\mathrm{Top}}$ is open if and only if its preimage in S is open, for any map $S \to \mathcal{F}(*)$ given by a section $s \in \mathcal{F}(S)$.

The condensed mathematics is invented to produce a good framework to do algebra that comes with a topology. In fact, the category Cond(Ab) of condensed abelian groups forms a very good abelian category.

Theorem 6.1.2. *[Sch19, Theorem 2.2] The category* Cond(Ab) *of condensed abelian groups is an abelian category that satisfies Grothendieck axioms (AB3), (AB4), (AB5), (AB6), (AB3*), (AB4*).*

In the case when T has the discrete topology, any continuous map (as a section in $\underline{T}(S)$) from a profinite set S to T will factor through a quotient of S onto a finite set, and the preimage of any subset along $s : S \to T$ is open in S. This in particular implies that the space $\underline{T}(*)_{\mathrm{Top}}$ also has the discrete topology on the underlying set T.

When A is a discrete ring or group, the topological group $\underline{A}(*)_{\mathrm{Top}}$ is the same as A with the discrete topology. Moreover, we have the following:

Proposition 6.1.3. *The functor $A \to \underline{A}$ is a fully faithful embedding from the category of discrete abelian groups to the category of condensed abelian groups that preserves filtered colimits. It admits a left adjoint $\mathcal{F} \mapsto \mathcal{F}(*)_{\mathrm{Top}}$, such that the composition $A \mapsto \underline{A} \mapsto \underline{A}(*)_{\mathrm{Top}}$ is the identity.*

Proof. Let $A = \mathrm{colim}_i A_i$ be a filtered colimit of discrete abelian groups. Then for any profinite set S and any continuous map $s : S \to A$, it factors through a finite quotient of S. As A is a filtered colimit, by taking some A_i that contains all finite image we see the map $S \to A$ factors through some $S \to A_i$. Thus we get the bijection

$$\mathrm{Map}_{\mathrm{cont}}(S, \mathrm{colim}_i M_i) = \mathrm{colim}_i \mathrm{Map}_{\mathrm{cont}}(S, M_i).$$

In particular, $A \to \underline{A}$ preserves filtered colimits.

It now suffices to show that there exists a natural isomorphism of abelian groups:

$$\mathrm{Hom}_{\mathrm{Cond(Ab)}}(\underline{A}, \underline{B}) \cong \mathrm{Hom}(A, B),$$

where A and B are two discrete abelian groups. In fact, it is proved in [Sch19, Proposition 4.2] that for two Hausdorff abelian groups A and B such that A is compactly generated, we have a natural isomorphism of condensed abelian groups

$$\underline{\mathrm{Hom}}(\underline{A}, \underline{B}) \cong \underline{\mathrm{Hom}_{\mathrm{cont}}(A, B)},$$

where $\mathrm{Hom}_{\mathrm{cont}}(A, B)$ is the group of continuous homomorphisms with compact-open topology. So the Proposition we want follows by taking the section of the above isomorphism at $*$. □

6.2 Condensed and solid modules

Now we turn to the quasi-coherent theory.

We first introduce the concept of the pre-analytic ring and the analytic ring.

Definition 6.2.1. *A* pre-analytic ring $\mathcal{A}$ *is a condensed ring* $\underline{\mathcal{A}}$ *together with a functor*

$$\{\textit{extremaly disconnected sets}\} \longrightarrow \mathrm{Mod}^{\mathrm{cond}}_{\underline{\mathcal{A}}} : \ S \longmapsto \mathcal{A}[S],$$

where the category $\mathrm{Mod}^{\mathrm{cond}}_{\underline{\mathcal{A}}}$ *is the category of* $\underline{\mathcal{A}}$*-modules in condensed abelian groups, such that the functor take finite disjoint unions into products, and it is equipped with a natural transformation* $S \to \mathcal{A}[S]$.

The pre-analytic ring $\mathcal{A}$ *is called* analytic *if it satisfies the condition in [Sch19, Definition 7.4].*

We will not speak out the explicit definition of the analytic rings; instead, we want to give several examples of analytic rings and introduce the key properties we need.

Example 6.2.2. (i) Let R be a finite type $\mathbb{Z}$-algebra. Then we can define a pre-analytic ring $R_\blacksquare$ by the discrete condensed ring $\underline{R_\blacksquare} := R$ and the functor

$$S \longmapsto R_\blacksquare[S] := \varprojlim_i \underline{R}[S_i],$$

where $S = \varprojlim_i S_i$ is the inverse limit of finite sets S_i. It is showed in [Sch19, Theorem 8.1] that $R_\blacksquare$ is an analytic ring.

(ii) For a map of finite type $\mathbb{Z}$-algebra $R \to A$, we can define a pre-analytic ring $(A, R)_\blacksquare$ by the discrete condensed ring $\underline{(A, R)_\blacksquare} := A$ and the functor

$$S \longmapsto R_\blacksquare[S] \otimes_R A.$$

By [Sch19, Theorem 8.13] we know $(A, R)_\blacksquare$ is an analytic ring. In the special case when $R = A$, by construction we have $(A, A)_\blacksquare = A_\blacksquare$.

One of the important features of an analytic ring $\mathcal{A}$ is that the category $\mathrm{Mod}^{\mathrm{cond}}_{\underline{\mathcal{A}}}$ is condensed $\underline{\mathcal{A}}$-modules admits a well-behaved full subcategory $\mathrm{Mod}^{\mathrm{cond}}_{\mathcal{A}}$ of *solid* $\mathcal{A}$*-modules* where we can build the six functors formalism. The category $\mathrm{Mod}^{\mathrm{cond}}_{\mathcal{A}}$ is defined as the subcategory of $M \in \mathrm{Mod}^{\mathrm{cond}}_{\underline{\mathcal{A}}}$ such that for all extremally disconnected sets S the following map is an isomorphism

$$\mathrm{Hom}_{\underline{\mathcal{A}}}(\mathcal{A}[S], M) \longrightarrow M(S).$$

The relation between the category $\mathrm{Mod}^{\mathrm{cond}}_{\mathcal{A}}$ and $\mathrm{Mod}_{\underline{\mathcal{A}}^{\mathrm{cond}}}$ is given as follows:

Proposition 6.2.3. *[Sch19, Proposition 7.5] Let* $\mathcal{A}$ *be an analytic ring.*

(i) The full subcategory $\mathrm{Mod}^{\mathrm{cond}}_{\mathcal{A}}$ *of solid* $\mathcal{A}$*-modules in* $\mathrm{Mod}^{\mathrm{cond}}_{\underline{\mathcal{A}}}$ *is an abelian category stable under all limits, colimits and extensions. The collection of condensed* $\mathcal{A}$*-modules* $\mathcal{A}[S]$ *for* S *being extremally disconnected forms a family of compact projective generators of* $\mathrm{Mod}^{\mathrm{cond}}_{\mathcal{A}}$.

(ii) The inclusion functor $\mathrm{Mod}^{\mathrm{cond}}_{\mathcal{A}} \subset \mathrm{Mod}^{\mathrm{cond}}_{\underline{\mathcal{A}}}$ *admits a left adjoint functor (called the* solidification*)*

$$M \longmapsto M \otimes_{\underline{\mathcal{A}}} \mathcal{A},$$

which is the unique colimi-preserving extension of the functor $\underline{\mathcal{A}}[S] \mapsto \mathcal{A}[S]$*. There exists a unique symmetric monoidal structure* $\otimes_{\mathcal{A}}$ *on* $\mathrm{Mod}^{\mathrm{cond}}_{\mathcal{A}}$ *such that the solidification functor is symmetric monoidal.*

(iii) The item (i) and (ii) admit a derived version for the fully faithful embedding $\mathcal{D}(\mathrm{Mod}^{\mathrm{cond}}_{\mathcal{A}}) \to \mathcal{D}(\mathrm{Mod}^{\mathrm{cond}}_{\underline{\mathcal{A}}})$ *of derived* ∞*-category.*

Specify to the case when $\mathcal{A} = A_\blacksquare$ for a finite type $\mathbb{Z}$-algebra A, we then have the following result.

Proposition 6.2.4. *Let $R \to A$ be a map of finite type $\mathbb{Z}$-algebras, and let $\mathcal{A}$ be the analytic ring $(A,R)_\blacksquare$. Then the functor*

$$M \longmapsto \underline{M}$$

is a fully faithful embedding from the category of A-modules to the category of solid $\mathcal{A}$-modules that preserves filtered colimits. This functor admits a natural left adjoint, by taking the section at the point $*$

Proof. By the Proposition 6.1.3, the category of A-modules admits a fully faithful embedding into the category of $\underline{\mathcal{A}}$-modules by $M \mapsto \underline{M}$. We take S to be a finite set in the Proposition 6.2.3, then $\underline{A}[S] = R_\blacksquare[S] \otimes_{\underline{R}} \underline{A}$ is a solid $(A,R)_\blacksquare$-module. So we have $\underline{F}$ is a solid $\mathcal{A}$-module for any finite free A-module F (with discrete topology). This in particular implies that $\underline{M}$ is solid for any finite A-module M, for the inclusion $\mathrm{Mod}^{\mathrm{cond}}_{\mathcal{A}} \subset \mathrm{Mod}^{\mathrm{cond}}_{\underline{\mathcal{A}}}$ preserves colimits and any finite A-module is a cokernel of a map of finite free modules.

We then recall that the functor $M \to \underline{M}$ from discrete abelian groups to condensed abelian groups preserves filtered colimits (Proposition 6.1.3). So for any discrete A-module M, we may write it as a filtered colimit $M = \mathrm{colim} M_i$ of finitely generated A-modules. Then $\underline{M} = \underline{\mathrm{colim}_i M_i} = \mathrm{colim}_i \underline{M_i}$ is a colimit of solid $\mathcal{A}$-modules, which by Proposition 6.2.3 again is solid. In this way, the functor

$$M \longmapsto \underline{M}$$

is a fully faithful embedding from the category of A-modules to the category of solid $(A,R)_\blacksquare$-modules that preserves filtered colimits.

As the functor $M \mapsto \underline{M}$ from the category of discrete abelian groups to the category of condensed abelian groups admits a natural left adjoint by taking the section at $*$ (Proposition 6.1.3), the functor

$$\mathrm{Mod}_A \longmapsto \mathrm{Mod}^{\mathrm{cond}}_{\underline{\mathcal{A}}},\ M \longmapsto \underline{M}$$

has the left adjoint by the same formula. So we may compose the inclusion functor $\mathrm{Mod}^{\mathrm{cond}}_{\mathcal{A}} \subset \mathrm{Mod}^{\mathrm{cond}}_{\underline{\mathcal{A}}}$ to get the left adjoint we want. □

In the special case when $R = A$, we get the fully faithful embedding from the category Mod_A to the category $\mathrm{Mod}^{\mathrm{cond}}_{A_\blacksquare}$ of solid $A_\blacksquare$-modules.

We then consider the relative situation.

Proposition 6.2.5. *[Sch19, Proposition 7.7] Let $\mathcal{A} \to \mathcal{B}$ be a map of analytic rings. Then there exists a natural symmetric monoidal functor $- \otimes_{\mathcal{A}} \mathcal{B}$ from the category of solid $\mathcal{A}$-modules to the category of solid $\mathcal{B}$-modules, such that the following diagram commute*

$$\begin{array}{ccc} \mathrm{Mod}^{\mathrm{cond}}_{\mathcal{A}} & \xrightarrow{-\otimes_{\mathcal{A}}\mathcal{B}} & \mathrm{Mod}^{\mathrm{cond}}_{\mathcal{B}} \\ {\scriptstyle -\otimes_{\underline{\mathcal{A}}}\mathcal{A}}\uparrow & & \uparrow{\scriptstyle -\otimes_{\underline{\mathcal{B}}}\mathcal{B}} \\ \mathrm{Mod}^{\mathrm{cond}}_{\underline{\mathcal{A}}} & \xrightarrow{-\otimes_{\underline{\mathcal{A}}}\underline{\mathcal{B}}} & \mathrm{Mod}^{\mathrm{cond}}_{\underline{\mathcal{B}}}, \end{array}$$

where the functor $- \otimes_{\underline{\mathcal{A}}} \mathcal{A} : \mathrm{Mod}^{\mathrm{cond}}_{\underline{\mathcal{A}}} \to \mathrm{Mod}^{\mathrm{cond}}_{\mathcal{A}}$ is the solidification functor (similarly for $\mathcal{B}$).

In the special case when $\mathcal{A}$ and $\mathcal{B}$ are analytic rings coming from finite type $\mathbb{Z}$-algebras, we have the following:

Proposition 6.2.6. *[Sch19, Theorem 8.13] Let $R \to S \to A$ be a map of finite type $\mathbb{Z}$-algebras. Then there exists fully faithful embeddings of categories of solid modules*

$$\mathrm{Mod}^{\mathrm{cond}}_{(A,S)_\blacksquare} \longrightarrow \mathrm{Mod}^{\mathrm{cond}}_{(A,R)_\blacksquare} \longrightarrow \mathrm{Mod}^{\mathrm{cond}}_{R_\blacksquare},$$

where each of them is the forgetful functor. Moreover, each forgetful functor admits a natural left adjoint by the base extension functors

$$\mathrm{Mod}^{\mathrm{cond}}_{(A,S)_\blacksquare} \xleftarrow{-\underset{(A,R)_\blacksquare}{\otimes}(A,S)_\blacksquare} \mathrm{Mod}^{\mathrm{cond}}_{(A,R)_\blacksquare} \xleftarrow{-\underset{R_\blacksquare}{\otimes}(A,R)_\blacksquare} \mathrm{Mod}^{\mathrm{cond}}_{R_\blacksquare}.$$

Proof. The Theorem 8.13 in [Sch19] gives the proof for the pair $\mathrm{Mod}^{\mathrm{cond}}_{(A,S)_\blacksquare} \overset{-\otimes_{(A,R)_\blacksquare}(A,S)_\blacksquare}{\rightleftarrows} \mathrm{Mod}^{\mathrm{cond}}_{(A,R)_\blacksquare}$.

For the pair $\mathrm{Mod}^{\mathrm{cond}}_{(A,R)_\blacksquare} \overset{-\otimes_{R_\blacksquare}(A,R)_\blacksquare}{\rightleftarrows} \mathrm{Mod}^{\mathrm{cond}}_{R_\blacksquare}$, it suffices to check that the forgetful functor is well defined. Namely, given a solid (A,R)-module M, the underlying $\underline{R}$-module structure on M is solid over $R_\blacksquare$.

We check this by the definition of the solidity. Let S be an extramally disconnected set, and M a solid $(A,R)_\blacksquare$-module. Then we have the bijection

$$\mathrm{Hom}_{\underline{(A,R)_\blacksquare}}((A,R)_\blacksquare[S], M) = M(S).$$

By the construction of the analytic ring $(A,R)_\blacksquare$, we know $\underline{(A,R)_\blacksquare} = \underline{A}$ and $(A,R)_\blacksquare[S] = R_\blacksquare[S]\otimes_{\underline{R}}\underline{A}$ in the category of condensed abelian groups. So we have

$$\mathrm{Hom}_{\underline{A}}(R_\blacksquare[S] \otimes_{\underline{R}} \underline{A}, M) = M(S).$$

But notice that the natural map of condensed $\underline{R}$-modules $R_\blacksquare[S] \to R_\blacksquare[S]\otimes_{\underline{R}}\underline{A}$ induces the following bijection

$$\mathrm{Hom}_{\underline{A}}(R_\blacksquare[S]\otimes_{\underline{R}}, M) = \mathrm{Hom}_{\underline{R}}(R_\blacksquare[S], M).$$

In this way, we get the bijection

$$\mathrm{Hom}_{\underline{R}}(R_\blacksquare[S], M) = M(S),$$

and by the definition we see M is a solid $R_\blacksquare$-module.

□

Remark 6.2.7. Here we give a remark about the compatibility of the Proposition 6.2.6 with the discrete modules.

We first note that the forgetful functors in the Proposition 6.2.6 are compatible with the functor in the Proposition 6.2.4

$$M \longmapsto \underline{M},$$

sending a discrete A-module to its associated solid $(A,R)_\blacksquare$-modules. In short, the forgetful functor preserves the discrete objects.

On the other hand, given a discrete R-module N, its associated condensed $\underline{R}$-module $\underline{N}$ is a solid $R_\blacksquare$-module by the Proposition 6.2.4 (for $(R,R)_\blacksquare$). We claim that the base extension $\underline{N}\otimes_{R_\blacksquare}(A,R)_\blacksquare$, as a solid $(A,R)_\blacksquare$-module, is identical to the $\underline{A}$-module $\underline{N \otimes_R A}$ (hence is also a solid $(A,S)_\blacksquare$-module for any map $R \to S \to A$ by the fully faithful embedding in the Proposition 6.2.6).

To compute the base extension, we first notice that by the Proposition 6.2.4 and the Proposition 6.2.3, (ii), for a discrete R-module N the associated solid $R_\blacksquare$-module $\underline{N}$ is equal to its solidification $\underline{N} \otimes_{\underline{R}} R_\blacksquare$. So the diagram in Proposition 6.2.5 implies that the base extension $\underline{N} \otimes_{R_\blacksquare} (A,R)_\blacksquare$ is equal to the $(A,R)_\blacksquare$-solidification of the $\underline{A}$-module $\underline{N} \otimes_{\underline{R}} \underline{A}$. As the tensor product preserves the colimits, by writing N as a filtered colimit of finite R-modules (where each of them is a finite colimit of a pushout diagram of finite free R-modules) we have

$$\underline{N} \otimes_{\underline{R}} \underline{A} = \underline{N \otimes_R A},$$

which is solid over $A_\blacksquare$. So the (A,S)-solidification is equal to $\underline{N \otimes_R A}$ itself and we get the compatibility.

We at last note that we may take the associated derived (∞) functors and get the derived version of these compatibilities.

6.3 Global analytic rings

We then generalize the quasi-coherent theory in the last subsection to the global setting, using the language of adic spaces.

First we recall the definition of adic spaces.

Definition 6.3.1. *A* discrete adic space *is a triple* $(X, \mathcal{O}_X, (|\cdot|_x)_{x\in X})$*, where* X *is a topological space,* $\mathcal{O}_X$ *is a sheaf of rings over* X*, and* $(|\cdot|_x)_{x\in X}$ *is a collection of equivalences of valuations for each* $x \in X$*, such that locally the triple is of the form* $(\mathrm{Spa}(A, A^+), \mathcal{O}_{\mathrm{Spa}(A,A^+)}, (|\cdot|_x)_{x\in \mathrm{Spa}(A,A^+)})$*, for* A *a discrete ring and* A^+ *an integrally closed subring of* A*.*

Here we recall that for a pair of discrete rings (A, A^+), where A^+ is integrally closed inside of A, the space $\mathrm{Spa}(A, A^+)$ is defined as the space of all equivalences of valuations of A with $|f| \leq 1$ for all $f \in A^+$. The topology of $\mathrm{Spa}(A, A^+)$ is the valuation topology defined as in [Sch19, Lecture 9].

For any map of finite type $\mathbb{Z}$-algebra $R \to A$, we can define an *affinoid adic space* associated to the map by

$$\mathrm{Spa}(A, \widetilde{R}),$$

where $\widetilde{R}$ is the integral closure of R in A. This allows us to associate the pair $(A, \widetilde{R})$ (or more directly the pair (A, R)) the abelian category $\mathrm{Mod}^{\mathrm{cond}}_{(A,R)_\blacksquare}$ of solid $(A, R)_\blacksquare$ modules and its derived ∞-category $\mathcal{D}((A, R)_\blacksquare) := \mathcal{D}(\mathrm{Mod}^{\mathrm{cond}}_{(A,R)_\blacksquare})$. In fact, by taking the colimit for all discrete Huber pairs of finite type $\mathbb{Z}$-subalgebras, we can define the (derived) category of solid $(A, A^+)_\blacksquare$-modules for any discrete Huber pair (A, A^+).

It is then natural to ask if the category of solid modules can be glued to a global category of solid modules. Unfortunately, as explained in [Sch19, Lecture 9], the localization behave badly in the abelian level, and only works well after passing to the derived ∞-level (with all of the natural functors becoming their derived versions).

We state this gluing result as follows.

Theorem 6.3.2. *[Sch19, Theorem 9.8] Let* X *be a discrete adic space. Then the association taking any open affinoid subset* $U = \mathrm{Spa}(A, A^+)$ *to the derived* ∞*-category* $\mathcal{D}((A, A^+)_\blacksquare) \subset \mathcal{D}(\mathrm{Mod}^{\mathrm{cond}}_{\underline{A}})$ *defines a sheaf of* ∞*-category on* X*.*

We denote this derived ∞*-category by* $\mathcal{D}((\mathcal{O}_X, \mathcal{O}_X^+)_\blacksquare)$ *or* $\mathcal{D}(X_\blacksquare)$ *in short.*

Here we note that for an inclusion $V = \mathrm{Spa}(B, B^+) \to U = \mathrm{Spa}(A, A^+)$ of affinoid open subsets of X, the transition map

$$\mathcal{D}((A, A^+)_\blacksquare) \longrightarrow \mathcal{D}((B, B^+)_\blacksquare)$$

is the left adjoint to the derived forgetful functor.

In the discrete case, given a map of finite type $\mathbb{Z}$-schemes $X \to Y$, we can also build a discrete adic space $X^{\mathrm{ad}/Y}$, locally of the form $\mathrm{Spa}(A, \widetilde{R})$ for open affine subspaces $U = \mathrm{Spec}(A)$ of X and $V = \mathrm{Spec}(R)$ of Y separately. When $X = Y$, we denote by X^{ad} to be the discrete adic space $X^{\mathrm{ad}/X}$, where locally it is of the form $\mathrm{Spa}(A, A)$ with the associated derived ∞-category of solid $A_\blacksquare$-modules.

Moreover, by the compatibility in the Remark 6.2.7, the functor sending a discrete A-module to its associated solid (A, R)-module in the Proposition 6.2.4 admits a global version. Namely for a map of finite type $\mathbb{Z}$-scheme $f : X \to Y$, denote by Z to be the discrete adic space $X^{\mathrm{ad}/Y}$. We can then construct two derived ∞-functors:

- The functor

$$\mathcal{D}_{\mathrm{Qcoh}}(X) \longrightarrow \mathcal{D}((\mathcal{O}_Z, \mathcal{O}_Z^+)_\blacksquare), \; \mathcal{F} \longmapsto \underline{\mathcal{F}},$$

  which is the derived global version of the functor from discrete A-modules M to the solid $(A, \widetilde{R})_\blacksquare$-modules $\underline{M}$.

- The functor

$$\mathcal{D}((\mathcal{O}_Z, \mathcal{O}_Z^+)_\blacksquare) \longrightarrow \mathcal{D}(X),$$

which is the sheafified version of the functor sending a solid $(A, \widetilde{R})_{\blacksquare}$-module N to its derived section at $*$.

The composition $\mathcal{D}_{\mathrm{Qcoh}}(X) \to \mathcal{D}((\mathcal{O}_Z, \mathcal{O}_Z^+)_{\blacksquare}) \to \mathcal{D}(X)$ is the identity functor on objects in $\mathcal{D}_{\mathrm{Qcoh}}(X)$, as the base extension along an inclusion of affinoid open subsets is compatible with the tensor product of discrete modules (see the Remark 6.2.7). This in particular shows that the functor $\mathcal{D}_{\mathrm{Qcoh}}(X) \to \mathcal{D}(Z^{\mathrm{ad}}_{\blacksquare})$ is a fully faithful embedding of derived ∞-category(cf. Corollary 4.9 in [Sch19]).

Furthermore, the map of finite type $\mathbb{Z}$-schemes $f : X \to Y$ induces a map of discrete adic spaces

$$X^{\mathrm{ad}} \longrightarrow X^{\mathrm{ad}/Y} \longrightarrow Y^{\mathrm{ad}},$$

where the first map is an open immersion when f is separated by [Sch19, Proposition 9.6]. This then produces a natural diagram of adjoint pairs among derived ∞-categories, generalizing the Proposition 6.2.6 to the global situation:

$$\mathcal{D}(X^{\mathrm{ad}}_{\blacksquare}) \rightleftarrows \mathcal{D}(X^{\mathrm{ad}/Y}_{\blacksquare}) \rightleftarrows \mathcal{D}(Y^{\mathrm{ad}}_{\blacksquare})$$

These are the derived direct image funtors $(-)_*$ and the derived pullback functors $(-)^*$ in the solid settings.

6.4 Coherent duality of solid modules, and its application to classical case

We can now state the coherent duality of solid modules.

Theorem 6.4.1. *[Sch19, Theorem 8.13] Let $g : X \to Y$ be a separated map of finite type $\mathbb{Z}$-schemes. Denote by $j : X^{\mathrm{ad}} \to X^{\mathrm{ad}/Y}$ to be the open immersion of discrete adic spaces. Then the derived pullback functor $j* : \mathcal{D}(X^{\mathrm{ad}/Y}_{\blacksquare}) \to \mathcal{D}(X^{\mathrm{ad}}_{b}s)$ admits a left adjoint $j_!$, satisfies*

$$j_! j^* M = M \underset{(\mathcal{O}_{X^{\mathrm{ad}/Y}}, \mathcal{O}^+_{X^{\mathrm{ad}/Y}})_{\blacksquare}}{\otimes^L} j_!(\mathcal{O}_{X^{\mathrm{ad}}}, \mathcal{O}^+_{X^{\mathrm{ad}}})_{\blacksquare}.$$

Theorem 6.4.2. *[Sch19, Theorem 8.14] Let $g : X \to Y$ be a separated map of finite type $\mathbb{Z}$-schemes, with $j : X^{\mathrm{ad}} \to X^{\mathrm{ad}/Y}$, $h : X^{\mathrm{ad}/Y} \to Y$ and $f : X^{\mathrm{ad}} \to Y^{\mathrm{ad}}$ being natural morphisms of discrete adic spaces. Define $f_! : \mathcal{D}(X^{\mathrm{ad}}_{\blacksquare}) \to \mathcal{D}(Y^{\mathrm{ad}}_{\blacksquare})$ to be the composition $\mathcal{D}(X^{\mathrm{ad}}_{\blacksquare}) \xrightarrow{j_!} \mathcal{D}(X^{\mathrm{ad}/Y}_{\blacksquare}) \xrightarrow{h_*} \mathcal{D}(Y^{\mathrm{ad}}_{\blacksquare})$. Then we have*

(i) The functor $f_!$ commutes with all direct sum and satisfies the projection formula

$$f_!(f^* M \underset{(\mathcal{O}_{X^{\mathrm{ad}}}, \mathcal{O}^+_{X^{\mathrm{ad}}})_{\blacksquare}}{\otimes^L} N) \cong M \underset{(\mathcal{O}_{Y^{\mathrm{ad}}}, \mathcal{O}^+_{Y^{\mathrm{ad}}})_{\blacksquare}}{\otimes^L} f_! N.$$

The formation of the lower shriek functor is compatible with compositions.

(ii) The functor $f_!$ admits a right adjoint $f^!$ that preserves discrete objects, and the formation of the upper shriek functor is compatible with compositions. Moreover, the object $f^!(\mathcal{O}_{Y^{\mathrm{ad}}}, \mathcal{O}^+_{Y^{\mathrm{ad}}})_{\blacksquare}$ is a bounded-below complex of discrete coherent sheaves of $\mathcal{O}_X$-modules (via $\mathcal{F} \mapsto \underline{\mathcal{F}}$).

(iii) Assume g is of finite Tor-dimension. Then $f_!$ preserves compact objects, and $f^!$ commutes with all direct sums. In this case the object $f^!(\mathcal{O}_{Y^{\mathrm{ad}}}, \mathcal{O}^+_{Y^{\mathrm{ad}}})_{\blacksquare}$ is a bounded complex of discrete coherent sheaves of $\mathcal{O}_X$-modules, and the functor $f^!$ can be given by

$$f^! M = f^* M \underset{(\mathcal{O}_{X^{\mathrm{ad}}}, \mathcal{O}^+_{X^{\mathrm{ad}}})_{\blacksquare}}{\otimes^L} f^!(\mathcal{O}_{Y^{\mathrm{ad}}}, \mathcal{O}^+_{Y^{\mathrm{ad}}})_{\blacksquare}.$$

Furthermore, when f is a complete intersection, the object $f^!(\mathcal{O}_{Y^{\mathrm{ad}}}, \mathcal{O}^+_{Y^{\mathrm{ad}}})_{\blacksquare}$ is locally quasi-isomorphic to a discrete line bundle concentrated in the same degree.

As an application, we show how the condensed math implies the classical duality for quasi-coherent sheaves over schemes.

Proposition 6.4.3 (Classical Grothendieck duality)**.** *Let $g : X \to Y$ be a proper map of finite type schemes over $\mathbb{Z}$. Then the derived direct image Rg_* admits a right adjoint $g^!$ on the derived category of quasi-coherent sheaves, such that $g^!$ is the restriction of the functor $f^! : \mathcal{D}(Y^{\mathrm{ad}}_{\blacksquare}) \to \mathcal{D}(X^{\mathrm{ad}}_{\blacksquare})$ on the subcategory of discrete objects, for the map of discrete adic spaces $f : X^{\mathrm{ad}} \to Y^{\mathrm{ad}}$.*

Proof. Let $f : X^{\mathrm{ad}} \to Y^{\mathrm{ad}}$ be the natural morphism of discrete adic spaces, and let $K \in \mathcal{D}_{\mathrm{Qcoh}}(X)$ and $L \in \mathcal{D}_{\mathrm{Qcoh}}(Y)$. Consider the following commutative diagram of derived ∞-category

$$\begin{array}{ccc} \mathcal{D}_{\mathrm{Qcoh}}(X) & \underset{g^*}{\overset{g_*}{\rightleftarrows}} & \mathcal{D}_{\mathrm{Qcoh}}(Y) \\ \downarrow & & \downarrow \\ \mathcal{D}(X^{\mathrm{ad}}_{\blacksquare}) & \underset{f^*}{\overset{f_*}{\rightleftarrows}} & \mathcal{D}(Y^{\mathrm{ad}}_{\blacksquare}), \end{array}$$

where every functor is derived and the vertical maps are the fully faithful embedding (constructed by sending a discrete quasi-coherent sheaf to its associated sheaf of solid $X^{\mathrm{ad}}_{\blacksquare}$-modules (resp. $Y^{\mathrm{ad}}_{\blacksquare}$-modules) $\mathcal{F} \mapsto \underline{\mathcal{F}}$). Then by the Theorem 6.4.2, we have a canonical isomorphism

$$R\operatorname{Hom}_{\mathcal{D}(X^{\mathrm{ad}}_{\blacksquare})}(\underline{K}, f^!\underline{L}) \cong R\operatorname{Hom}_{\mathcal{D}(Y^{\mathrm{ad}}_{\blacksquare})}(f_!\underline{K}, \underline{L}).$$

As the map of schemes $g : X \to Y$ is proper, the induced morphism of adic spaces $X^{\mathrm{ad}} \to X^{\mathrm{ad}/Y}$ is an isomorphisms ([Sch19, Proposition 9.6]). In particular, by the construction of $f_!$ we see $f_!$ is equal to the forgetful functor f_*. So we get

$$R\operatorname{Hom}_{\mathcal{D}(X^{\mathrm{ad}}_{\blacksquare})}(\underline{K}, f^!\underline{L}) \cong R\operatorname{Hom}_{\mathcal{D}(Y^{\mathrm{ad}}_{\blacksquare})}(f_*\underline{K}, \underline{L}). \tag{1}$$

Moreover, since $f^!$ preserves discrete object (Theorem 6.4.2 (ii)), the restriction of $f^!$ on the full subcategory $\mathcal{D}_{\mathrm{Qcoh}}(Y)$ defines a derived functor $g^! : \mathcal{D}_{\mathrm{Qcoh}}(Y) \to \mathcal{D}_{\mathrm{Qcoh}}(X)$, and we get

$$R\operatorname{Hom}_{\mathcal{D}(X^{\mathrm{ad}}_{\blacksquare})}(\underline{K}, f^!\underline{L}) \cong R\operatorname{Hom}_{\mathcal{D}(X^{\mathrm{ad}}_{\blacksquare})}(\underline{K}, \underline{g^!L}). \tag{2}$$

In this way, as the diagram above commutes while the vertical underlying functors are fully faithful, by combining (1) and (2) above we can identify $f_*\underline{K}$ as $\underline{g_*K}$ and obtain a natural quasi-isomorphism as below

$$R\operatorname{Hom}_{\mathcal{D}_{\mathrm{Qcoh}}(X)}(K, g^!L) \cong R\operatorname{Hom}_{\mathcal{D}_{\mathrm{Qcoh}}(Y)}(g_*K, L).$$

Thus we are done.

□

Remark 6.4.4. The coherent duality in solid modules above have several improvements compared with the classical theory.

(i) Most important point is that the coherent duality of solid modules provides a six functor formalism for a map of finite type schemes, not necessarily to be proper. In particular, we obtain the existence of the "cohomology with compact support" functor $f_!$ functor.

(ii) Deligne's approach to the Grothendieck duality allows us to build the functor $Rf^!$ and $f_!$ (cf. Section 5). However, as in the Theorem 5.0.4 the functor $f_!$ is between category of pro-objects of coherent sheaves, while $Rf^!$ is between the category of ind-objects of coherent sheaves. Note that the category of solid modules provides a bigger category that contains the colimits of limits of coherent sheaves (via the functor $\mathcal{F} \mapsto \underline{\mathcal{F}}$ for a discrete coherent sheaf $\mathcal{F}$). In particular, we see the objects in Deligne's approach are in fact embedded into the category of solid modules. So the condensed math actually provides a better framework where limits and colimits behave well altogether.

(iii) In the discussion of Hartshorne's approach to dualizing complex, we see the construction of the dualizing complex is not canonical. On the other hand, in the Theorem 6.4.2 we get a natural discrete object $f^!(\mathcal{O}_{Y^{\mathrm{ad}}}, \mathcal{O}^+_{Y^{\mathrm{ad}}})_{\blacksquare}$ that is bounded-below and has coherent cohomology. This provides a canonical construction of the dualizing complex.

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