DISCOURSE

The discourse of a classroom—the ways of representing, thinking, talking, agreeing and disagreeing—is central to what students learn about mathematics as a domain of human inquiry with characteristic ways of knowing. Discourse is both the way ideas are exchanged and what the ideas entail: Who talks? About what? In what ways? What do people write, what do they record and why? What questions are important? How do ideas change? Whose ideas and ways of thinking are valued? Who determines when to end a discussion? The discourse is shaped by the tasks in which students engage and the nature of the learning environment; it also influences them.

Discourse entails fundamental issues about knowledge: What makes something true or reasonable in mathematics? How can we figure out whether or not something makes sense? That something is true because the teacher or the book says so is the basis for much traditional classroom discourse. Another view, the one put forth here, centers on mathematical reasoning and evidence as the basis for the discourse. In order for students to develop the ability to formulate problems, to explore, conjecture, and reason logically, to evaluate whether something makes sense, classroom discourse must be founded on mathematical evidence.

Students must talk, with one another as well as in response to the teacher. When the teacher talks most, the flow of ideas and knowledge is primarily from teacher to student. When students make public conjectures and reason with others about mathematics, ideas and knowledge are developed collaboratively, revealing mathematics as constructed by human beings within an intellectual community. Writing is another important component of the discourse. Students learn to use, in a meaningful context, the tools of mathematical discourse—special terms, diagrams, graphs, sketches, analogies, and physical models, as well as symbols.

The teacher’s role is to initiate and orchestrate this kind of discourse and to use it skillfully to foster student learning. In order to facilitate learning by all students, teachers must also be perceptive and skillful in analyzing the culture of the classroom, looking out for patterns of inequality, dominance, and low expectations that are primary causes of nonparticipation by many students. Engaging every student in the discourse of the class requires considerable skill as well as an appreciation of, and respect for, students’ diversity.
STANDARD 2: THE TEACHER’S ROLE IN DISCOURSE

The teacher of mathematics should orchestrate discourse by—

- posing questions and tasks that elicit, engage, and challenge each student’s thinking;
- listening carefully to students’ ideas;
- asking students to clarify and justify their ideas orally and in writing;
- deciding what to pursue in depth from among the ideas that students bring up during a discussion;
- deciding when and how to attach mathematical notation and language to students’ ideas;
- deciding when to provide information, when to clarify an issue, when to model, when to lead, and when to let a student struggle with a difficulty;
- monitoring students’ participation in discussions and deciding when and how to encourage each student to participate.

Elaboration

Like a piece of music, the classroom discourse has themes that pull together to create a whole that has meaning. The teacher has a central role in orchestrating the oral and written discourse in ways that contribute to students’ understanding of mathematics.

The kind of mathematical discourse described above does not occur spontaneously in most classrooms. It requires an environment in which everyone’s thinking is respected and in which reasoning and arguing about mathematical meanings is the norm. Students, used to the teacher doing most of the talking while they remain passive, need guidance and encouragement in order to participate actively in the discourse of a collaborative community. Some students, particularly those who have been successful in more traditional mathematics classrooms, may be resistant to talking, writing, and reasoning together about mathematics.

One aspect of the teacher’s role is to provoke students’ reasoning about mathematics. Teachers must do this through the tasks they provide and the questions they ask. For example, teachers should regularly follow students’ statements with, “Why?” or by asking them to explain. Doing this consistently, irrespective of the correctness of students’ statements, is an important part of establishing a discourse centered on mathematical reasoning. Cultivating a tone of interest when asking a student to explain or elaborate on an idea helps to establish norms of civility and respect rather than criticism and doubt. Teachers also stimulate discourse by asking students to write explanations for their solutions and provide justifications for their ideas.

Emphasizing tasks that focus on thinking and reasoning serves to provide the teacher with ongoing assessment information. Well-posed questions can simultaneously elicit and extend students’ thinking. The teacher’s skill
at formulating questions to orchestrate the oral and written discourse in
the direction of mathematical reasoning is crucial.

A second feature of the teacher's role is to be active in a different way
from that in traditional classroom discourse. Instead of doing virtually all
the talking, modeling, and explaining themselves, teachers must encour-
age and expect students to do so. Teachers must do more listening,
students more reasoning. For the discourse to promote students' learn-
ing, teachers must orchestrate it carefully. Because many more ideas
will come up than are fruitful to pursue at the moment, teachers must
filter and direct the students' explorations by picking up on some points
and by leaving others behind. Doing this prevents student activity and talk
from becoming too diffuse and unfocused. Knowledge of mathematics, of
the curriculum, and of students should guide the teacher's decisions
about the path of the discourse. Other key decisions concern the
teacher's role in contributing to the discourse. Beyond asking clarifying
or provocative questions, teachers should also, at times, provide infor-
mation and lead students. Decisions about when to let students struggle
to make sense of an idea or a problem without direct teacher input,
when to ask leading questions, and when to tell students something
directly are crucial to orchestrating productive mathematical discourse in
the classroom. Such decisions depend on teachers' understandings of
mathematics and of their students—on judgments about the things that
students can figure out on their own or collectively and those for which
they will need input.

A third aspect of the teacher's role in orchestrating classroom discourse
is to monitor and organize students' participation. Who is volunteering
comments and who is not? How are students responding to one an-
other? What are different students able to record or represent on paper
about their thinking? What are they able to put into words, in what kinds
of contexts? Teachers must be committed to engaging every student in
contributing to the thinking of the class. Teachers must judge when
students should work and talk in small groups and when the whole group
is the most useful context. They must make sensitive decisions about
how turns to speak are shared in the large group—for example, whom to
call on when and whether to call on particular students who do not
volunteer. Substantially, if the discourse is to focus on making sense of
mathematics, on learning to reason mathematically, teachers must
refrain from calling only on students who seem to have right answers or
valid ideas to allow a broader spectrum of thinking to be explored in the
discourse. By modeling respect for students' thinking and conveying the
assumption that students make sense, teachers can encourage students
to participate within a norm that expects group members to justify their
ideas. Teachers must think broadly about a variety of ways for students
to contribute to the class's thinking—using means that are written or
pictorial, concrete or representational, as well as oral.

Vignettes

2.1 Ms. Nakamura has done a lot more number work with her kinder-
garten class this year, and she is pleased with how this is going. Now,
early the end of the year, they have been investigating patterns in the
number of various body parts in the classroom—how many noses or
eyes, for example, there are among the children in the class.

Earlier this week, each child made a nose out of clay. Ms. Nakamura
opens the discussion by revisiting this project. She asks: And how many
noses did we make?
Becky (points to her nostrils): Two of these.
Teacher: But how many actual noses?
Anne: 29.
Teacher: Why? Why were there 29 noses?
Adam: Because every kid in the class made one clay nose and that is the same number as kids in the class.
Teacher (pointing to her nostrils): Now Becky just said—remember what these are called?
Children: Nostrils!
Teacher: So were there 29 nostrils?
Pat: No, there were more.
Gwen: 58! We had 58 nostrils!
Teacher: Why 58?
Gwen: I counted.
Felise: If we had 30 kids, we would be 60. So it is 59 'cause it should be one less.
Teacher: Can you explain that again?
Felise: It's 59 because we don't have 30 kids, we have 29, so it is one less than 60.
Teacher: What does anyone else think?
Adam: I think it is 58. Each kid has 2 nostrils. So if 60 would be for 30 kids, then it has to be two less: 58.
Lawrence: But Felise says 30 kids makes 60....
Felise: No! That makes sense. 58.

The teacher looks around at the children, some of whom are beginning to wriggle. She waits, and then asks: What do the rest of you think?

Two girls chorus: It's 58.
Several others join: 58.
Teacher: So it's 58 because 30 kids would have 60 nostrils and we have to take away 2 for one less kid. Gwen said she counted. Let's count and see.

Ms. Nakamura leads the class through counting nostrils: 1, 2, 3, 4, 5, 6....

She moves on: What else do you think we have on our bodies that would be more than 29?

Graham: More than 29 fingers.
Teacher: More than 29 fingers? Why do you think so?
Graham: Because each kid, we have 10 fingers.
Ricky: More than 29 shoes.
Teacher: More than 29 shoes. And what are those shoes covering?
Ricky: Your feet.
Sarah: Ears.
Beth: More than 29 legs.
Ricky: And that goes with the feet idea.
Teacher: And why do you think that goes with the feet idea?

The teacher consistently asks students to explain and to justify their answers. "Why?" is a standard question, asked about apparently correct as well as about apparently wrong answers.

The teacher probes Felise's answer even though this goes beyond what many of the children are trying to do at this point.

The teacher solicits other students' reactions instead of showing them the right answer. Her tone of voice and her questions show the students that she values their thinking.

She allows time for children to think and remains neutral about the correctness of what is being said. She would like them to monitor whether mathematical ideas make sense by reasoning about them.

Here the teacher summarizes what different children have contributed to investigating Gwen's answer. Because Gwen's reasoning is complicated, the teacher then leads them through another means of verifying the result.

The teacher's question challenges students to think. It is open-ended; more than one right answer exists. Consider the difference between her question here and asking, "Do we have more than 29 fingers?"
Here the teacher lets the pace pick up by allowing these suggestions one right after the other without probing them for their explanations. Still, it might have helped children to hear one another's reasoning. Decisions should take into consideration when to move quickly and when to make sure an idea is thoroughly justified, when to pursue additional issues and when to remain focused on a particular purpose.

Ricky: 'Cause the feet are attached to the shoes.
Teacher: Your shoes. But you said that Beth's idea went with your idea. Why does her leg idea go with your idea?
Ricky: 'Cause you put them on over your legs. Because your feet are attached to your legs.
Teacher: Oh, so your feet are attached to your legs.
Willie: Legs are attached to knees.
Teacher: Legs are attached to knees, so your idea of knees, Beth's idea of legs, and Ricky's idea of feet—they all kind of go together, don't they? They're all attached.
Paul: The feet are attached to the legs, the legs are attached to the knees, the knees are attached to the thighs, and the thighs are attached to the chest.
The teacher chooses to overlook the comment that the thighs are connected to the chest because she is focusing on children's one-to-one reasoning.
Teacher: And how many chests are there altogether?
Children: 29.
Lisa: Because we have one chest.

Ms. Nakamura tells the children that they are to work on a picture now. Choose some body part and draw a picture of how many of those we have in our class and how you know that.

She directs them back to their tables where she has laid out paper and cans of crayons: You did some good thinking today!

2.2 Mrs. Logan is beginning a geometry unit with her students. She opens class by announcing: We're going to be studying about quadrilaterals. What do you know about quadrilaterals?

Several students chorus: Four sides, four-sided figure.

Mrs. Logan draws

and asks: Is this one?

Students: No, it has to connect.

Mrs. Logan: Is this one?

Several students: No, it can't intersect like that.

Mrs. Logan continues drawing and asks: So is this one?

Student: It has to close.

Recording their ideas and providing a representation of their reasoning (justification) help develop students' capacity with the written aspects of discourse.

The teacher chooses to comment on the children's thinking instead of their behavior.

The teacher elicits students' ideas in beginning a discussion. She is able to gather information about what students know and assume that it can guide her subsequent questions.

Reacting directly to what students say, the teacher provokes their thinking by building on the descriptions they have given.

Each time she uses what they have said to construct her next response. This reflects the tight connection between students' thinking and her moves in orchestrating the discourse.
Mrs. Logan: Okay, then is this one?

Students: Yes!

Mrs. Logan pauses and then looks directly at the students: I drew four examples. You said three of these didn’t work. Can you explain what makes the difference?

Several students volunteer pieces of a definition of “quadrilateral.” Mrs. Logan lists their ideas—in their terms—on the board:

**QUADRILATERALS**

4 points
4 segments
no more points intersect
closed curve

Summarizing, Mrs. Logan tells them: You really have all the pieces. The definition in our text is “the union of segments joining four points such that the segments intersect only at the endpoints.”

Now, which of our figures fit with this definition?

After a short discussion of their figures, Mrs. Logan continues: And what are some special kinds of four-sided figures?

Students call out a variety of names: square, rectangle, kite, rhombus, parallelogram.

Mrs. Logan: Another one you should have heard of before is trapezoid.

She draws several figures on the board: Can you classify these and talk about them?

---

The teacher asks a question designed to stimulate students to formulate the pieces of what they have been saying about quadrilaterals.

The teacher connects students’ ideas with the mathematical definition and decides to provide some additional information.

Again, she asks a question designed to elicit the students’ knowledge as a means of developing the class’s consideration of the topic.

The teacher adds to their contributions as needed.

She provides a context for them to discuss quadrilaterals among themselves.

She adds a question to provoke their conversations further.

Instead of responding directly to his question, the teacher decides it is worth everyone’s consideration and redirects it to the group. She also sees it as a way of stimulating this particular student’s participation in the classroom discourse.
The teacher listens closely to find out how students are thinking. She asks questions not to test the students' knowledge but to get them to clarify and refine those ideas. She looks carefully at their written representations of their thinking.

The students resume their work. Some begin making diagrams to represent the interrelationship among the types of quadrilaterals, others are making tables. Mrs. Logan walks around, asking students questions to get them to clarify what they are thinking. She asks one boy to explain his rather complicated chart: And why is the rhombus there, with the parallelogram?

After class, Mrs. Logan contemplates the lesson. In general, she feels it was a good start. Perhaps tomorrow—in order to help the students begin to construct some of the categories—she will engage them in some kind of problem or activity in which they will have to sort quadrilaterals. She muses a bit about how to frame it in a way that will promote discourse.

2.3 Mr. Luu has been working on probability for a few days with his class of sixth graders. Because his textbook is old, there is little about probability in the book. He has been drawing from a variety of sources as well as making up things himself, based on what he hears in the students' comments. He began by asking students to decide whether a coin-tossing game he presented was fair or not. He found out that although most of the students did consider the possible outcomes, they did not analyze the ways those outcomes could be obtained. For example, they thought that when you toss two coins, it is equally likely to get two heads, two tails, or heads-and-tails. He also learned that many of his students were inclined to decide if a game was fair by playing it and seeing if the players tied: If someone won, then the game might be biased in their favor, they thought.
He decides to present them with two dice-tossing games—the sum game and the product game:

**SUM GAME**

Two players:
Choose one player to be "even" and the other to be "odd."
Throw two dice.
Add the numbers on the two faces.
If the sum is even, the even player gets 1 point.
If the sum is odd, the odd player gets 1 point.

**PRODUCT GAME**

Two players:
Choose one to be the "even" player and the other to be "odd."
Throw two dice.
Multiply the numbers on the two faces.
If the product is even, the even player gets 1 point.
If the product is odd, the odd player gets 1 point.

After explaining how each game is played, Mr. Luu challenges the students to figure out if the games are fair or not. He begins by holding a discussion about what it means for something to be "fair." Then he presents the rules for each game, telling the students simply that they are to report back on whether or not either of the games is fair or not and to include an explanation for their judgment.

The students pair off and work on the problem. Some play each of the games first, recording their results, as a means of investigating the question. Others try to analyze the games based on the possible outcomes. Mr. Luu walks around and listens to what the students are saying and poses questions:

"What did you say were all the possible totals you could get? How did you know?"

"Why did you decide you needed to throw the dice exactly 36 times?"

After they have played the game or worked on their analyses for a while, Mr. Luu directs the students to stop, to open their notebooks, and write in their notebooks what they think about the fairness of the two games.

Next, Mr. Luu opens a whole-class discussion about the games. On the basis of what he saw when he was observing, he calls on Kevin and Rania. Rania beams. She explains that they figured out that the sum game is an unfair game "and we didn't even have to play it at all to be sure."

Kevin provides their proof: "There are six even sums possible—2, 4, 6, 8, 10, and 12—but only five odd ones—3, 5, 7, 9, and 11. So the game is unfair to the person who gets points for the odd sums."

The teacher poses questions and problems that both elicit and challenge students' thinking.

The teacher makes a decision about how much to focus a problem, how much to direct the students. Here, he decides that a common understanding of what makes something "fair" is crucial.

The teacher tries to provoke students' thinking. For example, he knows that the students who planned to throw the dice exactly 36 times may be assuming that the experimental results should be the same as their predicted outcomes.

The teacher appreciates the importance of writing about mathematics, and he provides regular occasions for it.

Mr. Luu has called on these two students because on the one hand, they are comfortable with the idea that probabilities can be analyzed, that the game need not actually be played. But on the other hand, these students have made an erroneous conclusion. He thinks that this combination makes their solution a good lead-off for the whole-group discussion.
The teacher expects the students to evaluate Kevin and Rania's argument and to decide together whether or not it makes sense.

The teacher makes careful decisions about when and how to encourage each student to participate. This time he is rewarded; sometimes when he calls on someone in this way, he gets stony silence.

Instead of explaining what Marcus has said, the teacher expects Marcus to provide his own clarification and justification.

Mr. Luu decides to press this issue, for he knows that understanding the concept of "outcome" is central to understanding probability. He thinks they can resolve this themselves, so he nudges the discussion along.

"What do the rest of you think?" asks Mr. Luu, gazing over the group. Several shake their heads. A few others nod.

"Marcus?" he invites. Marcus’s hand was not up, but his face looks up at Mr. Luu. "It don't make sense to me, Mr. Luu. I think that there's more ways to get some of them numbers, like 3—there's two ways to get a 3. But there is only one way to get a 2."

"Huh?" Several children are openly puzzled by this statement.

"Marcus, can you explain what you mean by saying that 3 can be made two ways?" asks Mr. Luu.

"Well, you could get a 1 on one die and a 2 on the other, or you could get a 2 on the first die and a 1 on the other. That's two different ways," he explains quietly.

"But how are those different? One plus two equals the same thing as two plus one!" objects a small girl.

"What do you think, Than?" probes Mr. Luu.

Than remains silent. Mr. Luu waits a long time. Finally Than says, "But they are two different dices, so it is not same."

"Hmmm," remarks Mr. Luu. "Where are other people on this?"

After three or four more comments on both sides of the issue, time is almost up. Mr. Luu assigns the students, for homework, to repeat the coin-tossing game they had investigated last week, to record their results, and to decide if it is fair when three people play it:

<table>
<thead>
<tr>
<th>COIN-TOSSING GAME</th>
</tr>
</thead>
<tbody>
<tr>
<td>Three players:</td>
</tr>
<tr>
<td>One player is &quot;two heads,&quot; one player is &quot;two tails,&quot; and one player is &quot;mixed.&quot;</td>
</tr>
<tr>
<td>Toss two coins.</td>
</tr>
<tr>
<td>If the result is two heads, the &quot;heads&quot; player gets 1 point.</td>
</tr>
<tr>
<td>If the result is two tails, the &quot;tails&quot; player gets 1 point.</td>
</tr>
<tr>
<td>If the result is one head and one tail, the &quot;mixed&quot; player gets 1 point.</td>
</tr>
</tbody>
</table>

Mr. Luu thinks that this game may help them with their thinking about the dice games. He asks them to play the game, to record their results, and to decide if it is fair when three people play it. They are to write about their experiments and explain their conclusion. Mr. Luu suspects that now, if they find out that the "mixed" result person gets about twice as many points as either of the others, they will be able to figure out what is going on and eventually agree with Marcus and Than.

2.4 Toward the end of a unit on quadratic equations, Mr. Santos has decided to assess his algebra students' use of problem-solving processes.
and their ability to make mathematical connections, both among ideas in
the unit and between these ideas and concepts covered earlier. To do
this, he chooses the following problem from the 1988 NCTM Yearbook,
The Ideas of Algebra, K-12 (p. 19):

\[(x^2 - 5x + 5)^2 - 9x + 20 = 1.\]

He decides to ask students to work on the problem in pairs while he
circulates among as many of the pairs as he can, monitoring their
progress. He uses a checklist with students’ names on it as an easy
means of recording observations about students’ thinking, approaches,
and patterns of working.

The first pair of students he visits, Alan and Bettina, groan, “This is really
going to be gross!” “Look, it’s got two different quadratics in the same
equation!” “Yeah, it’s not fair. He never gave us such a complicated one
before!” “Oh, well,” Bettina says, “we might as well get started. Let’s
factor \(x^2 - 9x + 20\) and see what we get.” When they find that \((x - 4)\)
and \((x - 5)\) are the factors, Alan says, “Well, I guess that’s it; \(x = 4\) and \(x = 5\)
are the answers.”

Bettina does not seem certain about Alan’s assertion. “What about this
other quadratic? Don’t we have to check that it works there, too?” she
asks. “Oh, yeah,” agrees Alan, “you check 5 and I’ll check 4.” So, they
substitute 4 and 5 into \(x^2 - 5x + 5\) and find that they get 1 for \(x = 4\) and
they get 5 for \(x = 5\). Alan says, “I get 1 like I’m supposed to,” but Bettina
says, “I don’t get 1. I get 5.” This result puzzles them.

As they look at the problem together, Alan says, “We need to use both
quadratics together,” and Bettina chimes in, “Yeah, that’s this to that
power.” Evaluating the entire expression, they find that for \(x = 4\) they get
1\(^0\) and for \(x = 5\) they get 5\(^0\). They comment that “it’s one either way:
anything to the 0 power is 1.”

Alan leans back, seeming confident that they have solved the problem.
Bettina, still engaged with the problem, says, “Hey, look, if this is 1, then
the exponent could be anything. Can we use that?”

Taking up Bettina’s question, Alan points at the base, \(x^2 - 5x + 5\), and
says, “Okay, you mean we should see if any other values of \(x\) could make
this part equal to 1?”

Out of the corner of his eye, Mr. Santos notices a pair of students
clowning around by the window. He hears them laughing and sees them
pushing one another playfully. He approaches them and asks, “What’s up?”

“No way we can do this problem, Mr. S,” says Diarra.

“And, besides, who CARES?” adds Tommy.

Mr. Santos guesses that part of their frustration is that the problem
looks too complex. He invites them to try the problem by putting in some
numbers.

“How about 1?” suggests Diarra, giggling.

“Yeah,” agrees Tommy.

When they try 1, they are surprised to see that it works out.

He notices that it is headed “Can your Algebra Class Solve This?” and
recognizes that it is likely to be a tough one for them, but he expects
that as a nonroutine problem it will serve as an alternative means of
assessment. He hopes to gain insight into the students’ learning and
development of mathematical disposition.

The teacher’s skill in orchestrating discourse is enhanced by close
knowledge of students.

The teacher does not take these complaints too seriously, noting that
these students exhibit an improving mathematical disposition by getting
down to work on the problem.

The teacher suspects that these students may be getting a right
answer for a wrong reason. But since
they seem about to verify their
solution, he decides to observe and
listen to them a little longer to see
what happens.

The teacher notices with pleasure
that Bettina seems persistent,
reflective, and on the lookout for
additional solutions. He notes that
Alain expects to reach the answer
quickly and is satisfied with a single
answer. He thinks that it would be
worthwhile for Alain to engage in
more tasks of this sort and makes a
mental note to think further about
how to develop students’ persistence
and reflectiveness. Alain and Bettina
have a new idea to work on, and Mr.
Santos moves on to observe another
group.

The teacher monitors students’
engagement in the mathematics. He
communicates that he expects them
to be involved.

The teacher decides to give them a
hint to head them toward solving the
problem. He expects that this will
help to get them involved.
"Hey, this is easy, men!" exclaims Tommy. At this, other students crowd around.

"Are there other solutions?" asks Mr. Santos, relieved that the students are becoming engaged.

"I'll try 2," volunteers one. Others are trying other numbers. As he walks away, Mr. Santos hears another burst of excitement as a pair of students discovers that 2 works also. He also hears a groan from a student who has tried 0.

Looking around the classroom, Mr. Santos notices a pair of students, Geraldo and Linnea, using a graphing calculator. When he goes over to them, they tell him that they have graphed the functions $y = 1$ and

$$y = (x^2 - 5x + 5)^2 - 5x + 20$$

and are now zooming in to look for the points of intersection. When he asks them to explain what they have been doing, they say that they decided from the beginning to use a graphing calculator. They describe moving from graphing the two quadratics separately to using the exponentiation key and graphing the whole function at once. Linnea says, "We finally realized that what we had here was a polynomial to a polynomial power."

Mr. Santos asks them about the section of discontinuity on the graph and if their "picture" represents a complete graph. He suddenly realizes that this pair of students has provided him with additional insight into this problem and makes a mental note to change his lesson plans for later in the week. He will bring in the demonstration computer so that the whole class can participate in further discussion on using technology to solve this problem.

Mr. Santos looks around the class for another group to visit and notices another pair, Peter and Ona, lounging with nothing to do. "How are you two doing?" he inquires pleasantly.

"Great!" Peter replies. "We got the answer; it's 4 and 5." They show Mr. Santos how they did it. They have used an approach similar to the one used by Alan and Bettina.

Mr. Santos asks, "Didn't you just say that when $x = 4$ you got this polynomial [pointing to the base, $(x^2 - 5x + 5)]$ to be equal to 1?" He pauses, hoping that they will notice the importance of the base having the value 1.

After some consideration, Peter says, "Yes, but we were worried more about the exponent being 0; but if the base is 1, the exponent wouldn't have to be 0." Ona says, "Okay, let's see if we can solve $x^2 - 5x + 5 = 1$. So they set out to factor the $x^2 - 5x + 5$, ignoring the fact that it is not set equal to 0.

Mr. Santos glances at his watch and sees that the period is almost over. He decides to end the class by reminding the students to write their journal entries for the day. They are to record the problems and successes they encountered during the period, any new insights, and anything that stood out to them about other students' arguments or solutions in class. Mr. Santos also tells the students that there are more than two solutions to the problem and that they will have another period to work on the problem on their own before class discussion of the problem takes place.
STANDARD 3:
STUDENTS' ROLE IN DISCOURSE

The teacher of mathematics should promote classroom discourse in which students—

- listen to, respond to, and question the teacher and one another;
- use a variety of tools to reason, make connections, solve problems, and communicate;
- initiate problems and questions;
- make conjectures and present solutions;
- explore examples and counterexamples to investigate a conjecture;
- try to convince themselves and one another of the validity of particular representations, solutions, conjectures, and answers;
- rely on mathematical evidence and argument to determine validity.

Elaboration
The nature of classroom discourse is a major influence on what students learn about mathematics. Students should engage in making conjectures, proposing approaches and solutions to problems, and arguing about the validity of particular claims. They should learn to verify, revise, and discard claims on the basis of mathematical evidence and use a variety of mathematical tools. Whether working in small or large groups, they should be the audience for one another's comments—that is, they should speak to one another, aiming to convince or to question their peers. Above all, the discourse should be focused on making sense of mathematical ideas, on using mathematical ideas sensibly in setting up and solving problems.

Vignettes

3.1 It is late September in a sixth-grade class. Mrs. Fondant wants to engage her students in a problem that will yield multiple solutions to help break down their image of mathematics as a domain of single right answers. One aim she has right now is to establish the norms and routines of discourse in the class. She knows that much of what she does is different from what the students have grown accustomed to in previous grades. Therefore, Mrs. Fondant takes this aspect of her task at the beginning of the year seriously. She thinks she is finally making some progress with this group, after a month of concentrating on this dimension of her work with them.

She writes the following problem on the board:

The Wolverines scored 30 points in the first half of last night's basketball game. The unusual thing is that they did it without scoring a single foul shot. How did they score the 30 points?
Immediately, one student yells, "That's easy! They scored ten 3-point shots!"

Another quickly interjects, "There are a lot of possibilities. It could be fifteen 2-point shots." Mrs. Fondant has one of the students explain scoring possibilities.

Several students offer possible answers. The teacher directs them to work alone or with a partner to figure out as many answers as they can. Walking around, she notices that some are constructing tables, others are using formulas, and still others are randomly writing down combinations as they occur to them.

After a few minutes, Mrs. Fondant asks if they are ready to talk about what they have found. They seem to be, so she asks one girl for one of the combinations that she worked out.

The girl says, "Two 3-pointers and twelve 2-pointers—$2 \times 3$ is 6 and $12 \times 2$ is 24 and $6 + 24$ is 30."

Several others add possible combinations.

Another girl asks, "How can we keep track of all these? Let's make a table."

Mrs. Fondant looks expectantly at the group. "Does someone want to make a table on the board?"

One student comes up and makes a two-column chart:

<table>
<thead>
<tr>
<th>3-point shots</th>
<th>2-point shots</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>3</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>15</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>

He stands at the board recording the combinations others suggest.

One student questions the solution, six 3-point shots and six 2-point shots. "That makes 33 points."
A girl explains, "6 x 3 is 18 and 6 x 2 is 12. 18 + 12 is 30."

"Oh, yeah," he says.

Another student suggests redoing the table so that the combinations are in order. "Maybe we'll see a combination we missed." The teacher asks him to come up and do that.

Looking at the revised table, one student raises her hand. "Why are there no odd numbers in the 3-point box?"

Several seconds pass. Everyone seems to be pondering this. A couple of students whisper to one another.

The teacher asks what they think about this. One girl says that maybe they missed some. A few students begin searching for other combinations. After a few moments, a student says she thinks it has something to do with the fact that an odd number times an odd number equals an odd number, but she's not sure what that tells her.

In the third row, a few students are leaning together, talking quietly. One says, "I think that's important."

Mrs. Fondant asks why it would matter.

"Because for 2-point shots, the total number of points will always be even," begins one student and then pauses.

"Oh! Is it because 30 is even? And you need two even numbers to equal an even?" bursts in one of the boys who has been talking in the little group.

"What do you think?" asks Mrs. Fondant.

Before class ends, Mrs. Fondant asks the students, "What if they had scored 31 points—would that have changed our table?"

3.2 Ms. Chavez has rolled the math department computer into her class for the morning and has connected it to her LCD viewer. Her 28 first-year algebra students, seated at round tables in groups of threes and fours, are working on a warm-up problem. The day before they had a test on functions. For the warm-up to today's class, Ms. Chavez has asked students to set up a table of values and graph the function \( y = |x| \).

She has chosen this problem as a way to introduce some ideas for a new unit on linear, absolute value, and quadratic functions. During the warm-up, students can be heard talking quietly to one another about the problem: "Does your graph look like a V-shape?" "Did you get two intersecting lines?" Walking around the room, Ms. Chavez listens to these conversations while she takes attendance. After about five minutes, she signals that it is time to begin the whole-group discussion.

A girl volunteers and carefully draws her graph on a large wipe-off grid board at the front of the room. As she does this, most students are watching closely, glancing down at their own graphs, checking for correspondence. A few students are seen helping others who had some difficulties producing the graph.
Students use a variety of tools to reason together about mathematics. They do not rely on the teacher to initiate all ideas or to certify results.

By directing them to think through comparisons, the teacher creates a context that is likely to promote students' reasoning and communication about these functions and their graphs.

The students communicate with one another about mathematics without the teacher asking them questions or directing their comments. They also use mathematical language that they have developed through the discourse.

The teacher listens to students carefully.

Students initiate conjectures publicly. They make connections between this graphing activity and transformational geometry.

Another student suggests that they enter the function into the computer and watch it produce the graph. Several other students chime in, "Yes!" The first girl does this, and the class watches as the graph appears on the overhead screen. It matches the graph she sketched, and the class cheers, "Way to go, Elena!

Ms. Chavez then asks the class to sketch the graphs of \( y = |x| + 1 \), \( y = |x| + 2 \), and \( y = |x| - 3 \) on the same set of axes and write a paragraph that compares and contrasts the results with the graph of \( y = |x| \). "Feel free to work alone or with the others in your group," she tells them.

After a few minutes, two students exclaim, "All the graphs have the same shape!"

A few other students look up. Another student observes, "They're like angles with different vertex points." "Then they're really congruent angles," adds his partner.

Ms. Chavez circulates through the class, listening to the students' discussions, asking questions, and offering suggestions. She notices one group has produced only one branch of the graphs. "Why don't you choose a few negative values for \( x \) and see what happens?" Another group asks, "What would happen if we tried \( |x - 3| \)?" "Try it!" urges Ms. Chavez.

The students continue working, and the conversation is lowered to murmurs once again. Then the members of one group call out, "Hey, we've got something! All these graphs are just translations of \( y = |x| \), just like we learned in the unit on geometry."

National Council of Teachers of Mathematics
"That's an interesting conjecture you have," remarks Ms. Chavez. She looks expectantly at the other students. "Do the rest of you agree?" They are still, many looking hard at their graphs. One student says, slowly, "I'm not sure I get it."

A boy in the group that made the conjecture about translations explains, "Like \( y = |x| + 2 \) is like \( y = |x| \) moved up two spaces and \( y = |x| - 3 \) moved down three spaces. It's like what Louella said about them being like angles with different vertex points."

Ms. Chavez decides to provoke the class to pursue this. She asks if anyone thinks they can graph \( y = |x| + 4 \) without first setting up a table of values. Hands shoot up. "Ooooh!" Scanning the class, Ms. Chavez notices Lionel, who does not volunteer often, has his hand up. He looks pleased when she invites him to give it a try.

Lionel sketches his graph on the dry-erase board. Elena again enters the equation of the graph into the computer and the class watches as the graph is produced. The computer-generated graph verifies Lionel's attempt. Again there are cheers. Lionel gives a sweeping bow and sits down.

Ms. Chavez asks the students to write in their journals, focusing on what they think they understand and what they feel unsure about from today's lesson. They lean over their notebooks, writing. A few stare into space before beginning. She gives them about ten minutes before she begins to return the tests. She will read the journals before tomorrow's class.

At the end of the of the period, she distributes the homework that she has prepared. The worksheet includes additional practice on the concept of \( y = |x| + c \) as well as something new, to provoke the next day's discussion: \( y = b|x| + c \).

Over the next couple of weeks, students explore linear, quadratic, and absolute value functions. Nearing the end of this unit, Mrs. Chavez decides to engage students in reflecting on and assessing how far they have come.

As she assigns homework for that evening, she announces, "I'd like each of you to write two questions that you think are fair and would demonstrate that you understand the major concepts of this unit. I'll use several of your ideas to create the test. And here's a challenge for the last part of your assignment: you just drew the graph of \( f(x) = x^2 - 2x \) as a part of the review. Think about everything we've done so far this semester, and see if you can remember any ideas that will help you draw the graph of \( f(|x|) \)."

3.3 Ms. Pizzo has been working on fractions for a little over a week with her thirty-six students, the biggest class she has ever had. She feels that she is not connecting very well with them—the group is simply too large. Many students have a conception of fractions that they picked up last year, which is that a fraction is a certain size piece of something. For example, "one-fourth" is this:
"Three-quarters" looks, as one student said, like "a baby carriage."

Ms. Pizzo is worried, though, for her students do not seem to understand fractions as numbers, nor do they see fractions as relational to some referent whole: for example, the idea of three-fourths of eighteen makes no sense to them. Three-fourths is the baby carriage shape.

Trying to think of something that will engage them and get them talking about fractions in some other ways, Ms. Pizzo decides to give them the following problem:

\[
\frac{1}{2} \text{ of the crayons in James's box of 12 crayons are broken.}
\]

\[
\frac{1}{3} \text{ of the crayons in Fred's box of 12 crayons are broken.}
\]

Who should be sadder and why?

The students seem interested in this. They set to work, some drawing pictures, some getting out real crayons. Ms. Pizzo overhears Hilda tell Robbie, "Look, one-half is this much—

and one-third is this much—

so you can just tell that James should be sadder."

Robbie, staring at Hilda's drawing, exclaims suddenly, "Look! One-third is smaller than one-half, even though three is more than two!"

"Hey!" answers Hilda. "Does that work with others?"

Robbie quickly draws one-fourth. The two children look at each other, excited.

Steven leans over. "It doesn't work. Look it!" He draws the three-fourths baby carriage shape. "Three-fourths. It is bigger than one-half, and four is more than two."

Ms. Pizzo is enjoying overhearing this interchange. "What was your idea, your conjecture?" she asks Hilda and Robbie. She hopes that she can get them to articulate what they were noticing.

By now, several other children have wandered over to Hilda's desk, having overheard this excitement.

"I guess we were saying that, with fractions, if there is a bigger number on the bottom, the fraction is smaller. It's like opposite," answers Robbie.

"But now with what Steven showed us, we see we were wrong," adds Hilda.
"Let's talk about this in the whole-group discussion," suggests Ms. Pizzo, pleased both that the students are finding this interesting, that they are beginning—despite the size of the group—to produce little mathematical sparks that kindle the rest of the students' interest, and that they are on to a key concept in fractions. She makes a note to herself to develop some kind of task that will help students investigate their ideas about the relationships between the size of denominators and the size of fractions.

The teacher does not press to elaborate or qualify the students' partial understanding of the meaning of the denominator in relation to the size of the fraction—for instance, showing them that their conjecture would work with unit fractions. Instead, she assumes that the class as a whole can work with and clarify what Hilda, Robbie, and Steven have been working on.
STANDARD 4:
TOOLS FOR ENHANCING DISCOURSE

The teacher of mathematics, in order to enhance discourse, should encourage and accept the use of—

- computers, calculators, and other technology;
- concrete materials used as models;
- pictures, diagrams, tables, and graphs;
- invented and conventional terms and symbols;
- metaphors, analogies, and stories;
- written hypotheses, explanations, and arguments;
- oral presentations and dramatizations.

Elaboration

In order to establish a discourse that is focused on exploring mathematical ideas, not just on reporting correct answers, the means of mathematical communication and approaches to mathematical reasoning must be broad and varied. Teachers must value and encourage the use of a variety of tools rather than placing excessive emphasis on conventional mathematical symbols. Various means for communicating about mathematics should be accepted, including drawings, diagrams, invented symbols, and analogies. The teacher should introduce conventional notation at points when doing so can further the work or the discourse at hand. Teachers should also help students learn to use calculators, computers, and other technological devices as tools for mathematical discourse. Given the range of mathematical tools available, teachers should often allow and encourage students to select the means they find most useful for working on or discussing a particular mathematical problem. At other times, in order to develop students’ repertoire of mathematical tools, teachers may specify the means students are to use.

Vignettes

4.1 Mr. Johnson has presented his first-grade class with several pairs of numbers and asked them to decide which number is greater and to justify their responses. He has also been encouraging them to find ways to write these comparisons.

Ben: I think 5 is greater than 3 because [he walks to the board and sticks five magnets up and then carefully sticks three magnets in another row].

Mr. Johnson asks whether that makes sense to other people. The children nod. He asks if anyone wants to show how they would write this down.

Kevin, up at the board, writes:

5 \rightarrow 3
Next, Betsy writes:

Mr. Johnson: Can you explain what you were thinking? Kevin?

Kevin explains that his arrow shows that 5 is more than 3 because the bigger number "can point at" the smaller one.

Mr. Johnson asks Betsy to explain hers, and she says that she thinks you should just circle the smaller one.

Mr. Johnson: What if the two numbers you were comparing were 6 and 5? What would you do? How would you write that?

Several seconds pass. Ruth shoots her hand in the air. Several others also have their hands up.

Ruth: You could draw an arrow to both of them.

Annie: You could circle both of them because they are the same.

Jimmy: You shouldn't mark either one, either way. They are not greater or less. They are the same.

Mr. Johnson nods at their suggestions. He writes an equals sign (=) on the board and explains that this is a symbol that people have invented for the ideas the children have been talking about.

Rashida: That's like what Ruth said.

Ruth beams and Annie calls out: It's like mine too.

4.2 Mrs. Martinez and Mr. Golden, who have teamed up to teach eighth grade this year, have divided their students into groups of four. The teachers have challenged them to show why the text says that division by zero is "undefined." The teachers want their students to know why "you can't divide by zero." Usually that is all that students have learned. Once the students figure out why division by zero is undefined, they are to prepare something that they could use to justify their explanation to the rest of the class.

Mrs. Martinez suggests that the calculator may be a useful tool for this problem. "Making up some kind of story problem for a situation that involves division might be helpful for others," adds Mr. Golden. The two teachers have arranged their large classroom so that calculators, graph paper, Unifix cubes and base ten blocks, felt-tip markers and blank overhead transparencies, rulers, and other materials are out where students can freely use them. This facilitates the use of alternative tools. Students are encouraged and expected to make decisions about which tool to use. Several students are, in fact, preparing overheads to display their conclusions about division by zero. Others are excitedly punching calculator buttons.

"The answer keeps getting larger and larger!" exclaim a pair of girls as they watch the results obtained by successively dividing 4 by smaller and smaller divisors with the calculator. "Why is that important?" asks Mrs. Martinez as she watches over one girl's shoulder. "Well, because each...

The teacher accepts more than one way of representing the idea with symbols; both are nonstandard but sensible.

The teacher poses a challenge that requires students to invent a means of recording an idea.

The teacher gives students time to think before responding. He doesn't repeat the question or call on children; he is silent.

The teacher connects the students' approaches and reasoning to the conventional notation. Because the students have thought about what it means for two numbers to be equal, they are ready to learn how that is conventionally represented. In this case, the notation follows the development of the concept in a meaningful context.

The teachers have posed a task that requires students to communicate about mathematics—in pairs as well as in the whole group—and that lends itself to a variety of tools.

The teachers suggest particular tools in order to stimulate students to make choices about what might help them work on the problem.

The teacher suggests another tool—a graph—that might help the students make mathematical connections, examine the pattern they are seeing, and present their work to the rest of the group.
of the numbers we are dividing by is getting closer and closer to zero but isn't zero." "Maybe you could make a graph to show what you are finding," suggests Mrs. Martinez.

Mr. Golden finds two students slouching sullenly in their chairs behind the room divider. "We don't understand what to do," grumbles one. Sitting down next to them, Mr. Golden begins, "Let's see if I can help. You are trying to figure out what the special problem is in trying to divide by zero. Maybe you can use some things you already know about division. How do you know that 8 + 2 is 4? How could you prove that if someone challenged your answer?" The students look at him disbelievingly. He waits. Then one says, "Well, I'd just say that 4 times 2 is 8 so 8 divided by 2 has to be 4." "Can that help you at all with this problem?" asks Mr. Golden. He stands up. The two students look at one another and then, sitting up a bit, begin talking. "Well, that doesn't work if you take 8 ÷ 0," Mr. Golden hears one say as he walks away.

**Summary: Discourse**

Because the discourse of the mathematics class reflects messages about what it means to know mathematics, what makes something true or reasonable, and what doing mathematics entails, it is central to both what students learn about mathematics as well as how they learn it. Therefore, the discourse of the mathematics class should be founded on mathematical ways of knowing and ways of communicating. The nature of the activity and talk in the classroom shapes each student's opportunities to learn about particular topics as well as to develop their abilities to reason and communicate about those topics. Students' dispositions toward mathematics are also fundamentally influenced by the experiences they have with mathematical activity. Although teachers may seem quieter at times, the teacher is nevertheless central in fostering worthwhile mathematical discourse within the classroom community. Teachers' skills in developing and integrating the tasks and discourse in ways that promote students' learning depend on the construction and maintenance of a learning environment that supports and grows out of these kinds of thinking and activity. It is to this issue that we now turn.
ANALYSIS

A central question for which teachers must be responsible is, "How well are the tasks, discourse, and environment working to foster the development of students' mathematical literacy and power?"

Trying to understand as much as possible about the effects of the mathematics classroom on each student is essential to good teaching. Teachers must monitor classroom life using a variety of strategies and focusing on a broad array of dimensions of mathematical competence, as outlined in the Curriculum and Evaluation Standards for School Mathematics. What do students seem to understand well, what only partially? What connections do they seem to be making? What mathematical dispositions do they seem to be developing? How does the group work together as a learning community making sense of mathematics? What teachers learn from this should be a primary source of information for planning and improving instruction in both the short and the long term.
STANDARD 6:  
ANALYSIS OF TEACHING AND LEARNING

The teacher of mathematics should engage in ongoing analysis of teaching and learning by—

- observing, listening to, and gathering other information about students to assess what they are learning;
- examining effects of the tasks, discourse, and learning environment on students' mathematical knowledge, skills, and dispositions;

in order to—

- ensure that every student is learning sound and significant mathematics and is developing a positive disposition toward mathematics;
- challenge and extend students' ideas;
- adapt or change activities while teaching;
- make plans, both short- and long-range;
- describe and comment on each student's learning to parents and administrators, as well as to the students themselves.

Elaboration

Assessment of students and analysis of instruction are fundamentally interconnected. Mathematics teachers should monitor students' learning on an ongoing basis in order to assess and adjust their teaching. Observing and listening to students during class can help teachers, on the spot, tailor their questions or tasks to provoke and extend students' thinking and understanding. Teachers must also use information about what students are understanding to revise and adapt their short- and long-range plans for the tasks they select and for the approaches they choose to orchestrate the classroom discourse. Similarly, students' understandings and dispositions should guide teachers in shaping and reshaping the learning environment of the classroom. Additionally, teachers have the responsibility of describing and commenting on students' learning to administrators, to parents, and to the students themselves.

Students' mathematical power depends on a varied set of understandings, skills, and dispositions. Teachers must attend to the broad array of dimensions that contribute to students' mathematical competence as outlined in the Curriculum and Evaluation Standards for School Mathematics. They should assess students' understandings of concepts and procedures, including the connections they make among various concepts and procedures. Teachers must also assess the development of students' ability to reason mathematically—to make conjectures, to justify and revise claims on the basis of mathematical evidence, and to analyze and solve problems. Students' dispositions toward mathematics—their confidence, interest, enjoyment, and perseverance—are yet another key dimension that teachers should monitor.

Paper-and-pencil tests, although one useful medium for judging some aspects of students' mathematical knowledge, cannot suffice to provide
teachers with the insights they need about their students' understandings in order to make instruction as effectively responsive as possible. Teachers need information gathered in a variety of ways and using a range of sources. Observing students participating in a small-group discussion may contribute valuable insights related to their abilities to communicate mathematically. Interviews with individual students will complement that information and also provide information about students' conceptual and procedural understanding. Students' journals are yet another source that can help teachers appraise their students' development. Teachers can also learn a great deal from closely watching and listening to students during whole-group discussions.

As they monitor students' understandings of, and dispositions toward, mathematics, teachers should ask themselves questions about the nature of the learning environment they have created, of the tasks they have been using, and of the kind of discourse they have been fostering. They should seek to understand the links between these and what is happening with their students. If, for example, students are having trouble understanding inverse functions, is it because of the kinds of tasks in which they have been engaged? Is it related to the ways in which the group has explored and discussed ideas about functions and their inverses? Although it may be that the students lack prerequisite understandings, it could also be that this is a difficult piece of mathematics or that the teacher needs to consider alternative ways to help students "unpack" the ideas. Or, if students quickly give up when a direct route for solving a problem is not apparent, teachers must consider how the experiences that students have been having and the environment in which they have been working may not have helped them to develop the perseverance and confidence they need. Teachers need to analyze continually what they are seeing and hearing and explore alternative interpretations of that information. They need to consider what such insights suggest about how the environment, tasks, and discourse could be enhanced, revised, or adapted in order to help students learn.

Vignettes

6.1 Some teachers begin to change by allocating one day a week to "different" mathematics activities. Although the Curriculum and Evaluation Standards makes clear that the goal is for problem solving, reasoning, and communication to be interwoven throughout the curriculum, teachers must experiment with alternative approaches to changing their practice. This "one day a week" strategy is one such approach—not the goal, but for some teachers, a viable first step.

Ms. Levesque has been having students working in groups of four on Fridays, solving nonroutine problems. Last week, she had them work on the handshake problem. (If ten people are at a party and everyone shakes everyone else's hand exactly once, how many handshakes take place?) Things seem to be going quite well. The students appear to enjoy these Friday sessions and she looks forward to them herself.

She notices, however, that when she listens to students talking among themselves before class, they still groan about the word problems on their daily homework. Many students leave this part of their work unfinished because, they say, the problems are too hard.

As Ms. Levesque compares what she is learning about her students with her goals for them, she is troubled, because she wants her students to feel confident about solving mathematical problems and to stick to them even when the problems are hard.

The teacher gets information about her students from informal as well as formal sources. Students' conversations often give her clues about how they are feeling about mathematics or mathematics class.
She thinks about those Friday sessions. Why aren't they fostering these dispositions toward mathematical problem solving? Ms. Levesque wonders whether perhaps these special sessions seem to the students to be separate from "real math." It is, after all, just one day a week—and what they do on the other four days is quite different in spirit and in content.

Ms. Levesque decides to try working on word problems together for part of the period every day for a while to see if that makes a difference. She will try having them discuss the problems, examining different approaches and solutions, instead of just going over the answers together. In addition, the students will keep a journal or notebook in which to record strategies and reflections. When she talks to her department head about this over lunch, her department head says that he has had a similar concern with his classes and that she, too, will try Ms. Levesque's plan and they can compare notes after a few weeks.

5.2 The second graders have just finished working on addition and subtraction with regrouping. On a written test, many of them "forget" to regroup when they need to in subtraction. Instead, they do this:

\[
\begin{align*}
50 -38 &= 28 \\
\end{align*}
\]

The teacher, Mr. Lewis, thinks they are being careless. He feels a little annoyed because this is something on which he has spent a lot of time. He decides, though, that he should sit down with the children one by one for a few minutes and have them talk through a couple of the problems and how they solved them. He thinks he may be able to tell what they are doing wrong this way.

He chooses a couple of problems from the test and asks the children to justify their answers using bundles of Popsicle sticks. He discovers that most of them are not connecting the work they did in class with manipulatives to these written problems. When they have the Popsicle
The teacher analyzes what he finds from talking individually with the students. He reflects on how he worked with the class on this topic and conjectures that his approach had some flaws. He begins a search for how he can revise what he was doing.

The teacher knows that he must find some ways of documenting and assessing what students are learning, especially in view of her new goals for them. She finds it helpful to work with a colleague.

The teacher wants the parent to understand both what her child is doing and what is being held as important in her mathematics class.

Because it enables her to give the parent specific examples, her system of cards as an index to the children's journals helps her to do both.

The teacher got this idea from the NCTM Curriculum and Evaluation Standards for School Mathematics, (pp. 235, 236). She and her colleague found several ideas there for assessing and keeping track of students' learning.

The teacher has a systematic way of collecting and analyzing information about her own teaching.

... sticks, they find that their answers don't make sense, and they revise them to match what they do with the sticks.

Mr. Lewis had assumed that if they "saw" the concepts by actually touching the objects, they would understand. He now thinks that maybe he didn't do enough to help them build the links between the concrete model and the algorithm. He starts wondering what he could do to help them make that connection better. He hypothesizes that maybe they know how to regroup but may not understand why or when regrouping is necessary. He decides to make up a worksheet with examples where regrouping is necessary and some where it is not and have the children discuss whether or not they would have to regroup in each case and how they know that.

6.3 Ms. Lundgren has been trying to change her approach to teaching mathematics so that students are learning to reason and communicate about mathematics, to make sense of mathematical ideas, and to make connections. She believes she has been successful in moving the discourse of her classroom away from a focus on right answers and the teacher as authority.

Although she finds it difficult, she has also been devising better mathematical tasks, she thinks. With the help of the other fifth-grade teacher, Ms. Lundgren has also come up with some ways of keeping track of what students are learning. Today she is meeting with parents to go over their children's report cards, and she has decided to draw on her new records for these conferences.

When she is meeting with Mrs. Byers, Stacy's mother, Ms. Lundgren wants to show her how Stacy is making connections in division. Looking at her card on Stacy, Ms. Lundgren tells the mother that Stacy was able to explain how, for 28 ÷ 8, 3 r 4 was the same answer as 3.5 (a quotient obtained on the calculator) but also how the two answers differed. Ms. Lundgren, having made a note of it, opened Stacy's mathematics journal to the page where Stacy had worked this out. Then, referring to the index card again, Ms. Lundgren shows Stacy's mother all the ways that Stacy found to represent $\frac{1}{2}$ in her journal. Because she also wants to talk with Mrs. Byers about Stacy's disposition toward mathematics, Ms. Lundgren refers to a chart she is keeping on her students' mathematical attitudes. With this chart, she has periodically made notes to herself. She has also had her colleague next door come in and observe once a month and make notes on the chart for her. Mrs. Byers finds all these specific examples very useful and comments that she thinks that what Ms. Lundgren is trying to do in math is great and she wishes she had had a mathematics class like this when she was in school.

6.4 Ms. Weissmann has been audiotaping her mathematics classes each day this year. She listens to as much of each tape as possible while she plans for the next day's class. In listening to herself and to the students, she begins to notice a pattern.

On the one hand, the girls are very quiet and speak softly and say "I don't know" at least as often as they say anything. The boys, on the other hand, are loud, and she hears herself calling them by name a lot. They participate actively in the mathematics discussions as well as in their
own little games and fooling around. She begins tallying the frequency with which she calls on boys and on girls. She also begins a chart for what the boys and girls each contribute to class discussions, not just how often.

At the same time, Ms. Weissmann gets a couple of books from the library, both centered on discourse and on women’s patterns of interaction in different settings. She decides to make this a project for herself: to improve the balance of kinds and frequency of participation among boys and girls in the class discussion. She also plans to be alert if there are other such patterns underlying the boy-girl split.

Summary: Analysis

Analysis of instruction recognizes the intimate relationship between teaching and assessment. To improve their mathematics instruction, teachers must constantly analyze what they and their students are doing and how that is affecting what the students are learning. Using a variety of strategies, teachers must continuously monitor students’ capacity and inclination to analyze situations, frame and solve problems, and make sense of mathematical concepts and procedures. Teachers should use such information about students to assess not just how students are doing, but also to appraise how well the tasks, discourse, and environment are working together to foster students’ mathematical power and to adapt their instruction in response.

This pattern is not uncommon, but it is troubling to this teacher, who has always been interested in, and relatively successful with, mathematics. She also is convinced that things do not have to be like this.

The teacher selects some simple ways of maintaining a record of what is going on in her class.

The teacher’s “project” helps her to focus on an issue that is of great importance to her.