Degree of scale invariance of inflationary perturbations

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Most inflationary models predict that the power-law index of the spectrum of density perturbations is close to 1, though not precisely equal to 1, $|n-1| \sim \mathcal{O}(0.1)$, implying that the spectrum of density perturbations is nearly, but not exactly, scale invariant. However, there are models where n is significantly less than 1 ($n \sim 0.7$); a spectral index significantly greater than 1 is more difficult to achieve. Without recourse to specific models, we show very generally and very explicitly that $n \approx 1$ is a consequence of the slow-roll conditions for inflation and ''naturalness,'' and thus, that near scale invariance is a generic prediction of inflation and a test of the inflationary framework. We derive the conditions needed to deviate significantly from scale invariance, and then show, by explicit construction, the existence of smooth potentials that satisfy all the conditions for successful inflation and give n as large as 2.

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I. INTRODUCTION

Inflation generates adiabatic density perturbations that can seed the formation of structure in the Universe. They arise from quantum fluctuations in the field that drives inflation and are stretched to astrophysical size by the enormous growth of the scale factor during inflation [1]. The magnitude of these perturbations was recognized early on to be important in constraining inflationary models. The nearly scale-invariant value for the scalar spectral index, $n \approx 1$, is considered to be one of the three principal predictions of inflation, and the deviation of n from unity is an important probe of the underlying dynamics of inflation [2].

The advantage of scale-invariant primordial density perturbations was first spelled out nearly three decades ago [3,4]: any other spectrum, in the absence of a long-wavelength or short-wavelength cutoff, will have excessively large perturbations on small scales or large scales.[1] Even though inflation provided the first realization of such a spectrum, long before inflation many cosmologists considered the scale-invariant spectrum to be the only sensible one. For this reason, the inflationary prediction of a deviation from scale invariance—even if small—becomes all the more important, as it provides a test of the inflationary framework.

One of the pioneering papers on inflationary fluctuations [5] emphasized that the fluctuations were not precisely scale invariant; the first quantitative discussion followed a year later [6]. The Cosmic Background Explorer Differential Microwave Radiometer (COBE DMR) detection of cosmic microwave background (CMB) anisotropy awakened the inflationary community to the testability of the inflationary density-perturbation prediction. The connection between $(n-1)$ and the underlying inflationary potential was pointed out soon thereafter [7,8], and the possibility of reconstructing the inflationary potential from measurements of CMB anisotropy began being discussed [9]. It is now quite clear that the degree of deviation from scalar invariance is an important test and probe of inflation.

Particular inflationary potentials and the values of n they predict have been widely discussed in literature (see, e.g., Refs. [10,11]). Lyth and Riotto [11], for example, remark that many inflationary potentials can be written in the form $V(\phi) = V_0(1 \pm \mu \phi^p)$ (in the interval relevant for inflation), and conclude that virtually all potentials of this form give $0.84 < n < 0.98$ or $1.04 < n < 1.16$ (also see Ref. [6]). Experimental limits on n, derived from CMB anisotropy measurements, are not yet very stringent, $0.7 < n < 1.2$ [12,13]. Even the stronger bound claimed by Bond and Jaffe [14], $n = 0.95 \pm 0.06$, falls far short of the potential of future CMB experiments [e.g., the Microwave Anisotropy Probe (MAP) and Planck satellites], $\sigma_n \sim 0.01$ [15].

It is our purpose here to address the issue of the deviation from scale invariance in full generality and without regard to specific models, and, in so doing, to show explicitly why scale invariance is a generic feature of inflation. We take a very agnostic approach to models, both for purposes of generality and because our knowledge about the physics of the scalar sector and of the inflationary-energy scale is very limited. We note there are excellent reviews of specific particle-physics models of inflation and their motivations within fundamental physics (see, e.g., Ref. [11]). In addition, we limit our analysis to standard, one-field inflation models. While this encompasses the vast majority of the models discussed in the literature, there are other, nonstandard approaches to inflation (see, e.g., Refs. [16]).

In the next section we show how the slow-roll conditions for inflation and naturalness limit the deviation from scale invariance. In Sec. III, to illustrate what must be done to

[1]Inflation provides a natural cutoff on comoving scales smaller than ~ 1 km, the horizon size at the end of inflation; perturbations on scales larger than the present horizon will not be important until long into the future. Thus, for inflation exact scale invariance is not necessary to avoid problems with excessively large perturbations.

achieve significant deviation from scale invariance, we construct inflationary models based upon smooth potentials where n is much smaller than and much larger than unity. We end with some concluding remarks in Sec. IV.

II. WHY INFLATIONARY PERTURBATIONS ARE NEARLY SCALE INVARIANT

The equations governing inflation are well known [17]:

$$\ddot{\phi}+3H\dot{\phi}+V'(\phi)=0 \tag{1}$$

$$H^2\equiv\left(\frac{\dot{a}}{a}\right)^2=\frac{8\pi}{3m_{PL}^2}\left[V(\phi)+\frac{1}{2}\dot{\phi}^2\right] \tag{2}$$

$$N\equiv\ln(a_f/a_i)=\int_{\phi_i}^{\phi_f}H dt \tag{3}$$

$$\delta_H^2(k)\simeq V^3/V'^2\propto k^{n-1} \tag{4}$$

where $a(t)$ is the cosmic scale factor, derivatives with respect to the field ϕ are denoted by a prime, and derivatives with respect to time by an overdot. The quantity δ_H is the post-inflation horizon-crossing amplitude of the density perturbation, which, if the perturbations are not precisely scale invariant is a function of comoving wave number k (i.e., $n=1$). (δ_H also corresponds to the dimensionless amplitude of the fluctuations in the gravitational potential.)

In computing the density perturbations, the value of the potential and its first derivative are evaluated when the scale k crossed outside the horizon during inflation. Because both V and V' can vary, $\delta_H^2\propto k^{n-1}$ in general depends upon scale. For most models, δ_H^2 is not a true power law, but rather n varies slowly with scale, typically $|dn/d\ln k|\leqslant 10^{-3}$ [18]. In fact, both n and $dn/d\ln k$ are measurable cosmological parameters and can provide important information about the potential.

In the slow-roll approximation the $\ddot{\phi}$ term is neglected in the equation of motion for ϕ and the kinetic term is neglected in the Friedmann equation [6,17]:

$$\dot{\phi}\simeq-\frac{V'}{3H} \tag{5}$$

$$N\simeq-\frac{8\pi}{m_{PL}}\int_{\phi_i}^{\phi_f}\frac{d\phi}{x(\phi)}. \tag{6}$$

The power-law index n is given by [8,10]

$$(n-1)=-\frac{x_{60}^2}{8\pi}+\frac{m_{PL}x_{60}'}{4\pi} \tag{7}$$

where

$$x(\phi)\equiv m_{PL}V'(\phi)/V(\phi)$$

measures the steepness of the potential and $x'\equiv dx/d\phi$ measures the change in steepness. (Higher-order corrections are discussed and the next correction is given in Ref. [19].) The subscript "60" indicates that these parameters are evaluated roughly 60 e-folds before the end of inflation, when the scales relevant for structure formation crossed outside the horizon.

Deviation from scale invariance is a generic prediction since the inflationary potential cannot be absolutely flat. It is controlled by the steepness of the potential and the change in steepness. Significant deviation from scale invariance requires a steep potential or one whose steepness changes rapidly. Further, Eq. (7) immediately hints that it is easier to make models with a "red spectrum" ($n<1$), than with a "blue spectrum" ($n>1$), because the first term in Eq. (7) is manifestly negative, while the second term can be of either sign. In addition, $x_{60}^2/8\pi$ is usually larger in absolute value than $m_{PL}x_{60}'/4\pi$.

The two conditions on the potential needed to ensure the validity of the slow-roll approximation are (see, e.g., Refs. [6,17])

$$m_{PL}V'/V=x\lesssim\sqrt{48\pi} \tag{8}$$

$$m_{PL}^2V''/V\lesssim 24\pi. \tag{9}$$

Note that the first slow-roll condition constrains the first term in the expression for $(n-1)$, $x^2/8\pi\lesssim 6$, and the second slow-roll condition constrains the second term since $m_{PL}x'/4\pi=m_{PL}^2V''/(4\pi V)-x^2/(4\pi)$.

A model that can give n significantly less than 1 is power-law inflation [20,21] (there are other models too [6,22]). It illustrates the tension between sufficient inflation and large deviation from scale invariance. The potential for power-law inflation is exponential,

$$V=V_0\exp(-\beta\phi/m_{PL}), \tag{10}$$

and the scale factor of the Universe evolves according to a power law

$$a(t)\propto t^{16\pi/\beta^2}\equiv t^p \quad \text{with} \quad p\equiv 16\pi/\beta^2 \tag{11}$$

and

$$\dot{\phi}=\sqrt{\frac{p}{4\pi}}\frac{m_{PL}}{t}. \tag{12}$$

Further, n can be calculated exactly in the case of power-law inflation [23]:

$$(n-1)=\frac{2}{1-p}\rightarrow-\frac{2}{p}\ \ (\text{slow-roll limit}). \tag{13}$$

For this potential $x=-\beta$, $x'=0$ (constant steepness), and the slow-roll constraint implies $|\beta|\lesssim 7$, or $p\gtrsim 1$. This is not very constraining as $p>1$ is required for the superluminal expansion necessary for inflation [24]. The quantitative requirement of sufficient inflation to solve the horizon problem and a safe return to a radiation-dominated universe before big-bang nucleosynthesis (reheat temperature $T_{\rm RH}\gg 1$ MeV and reheat age $t_{\rm RH}\ll 1$ sec) and baryogenesis ($T_{\rm RH}>1$ TeV and $t_{\rm RH}<10^{-12}$ sec) restrict p more seriously.

In particular, the amount of inflation depends upon when inflation ends:

$$N=-\frac{8\pi}{m_{PL}}\int_{\phi_i}^{\phi_f}\frac{d\phi}{x(\phi)}=p\ln(H_i/H_f) \tag{14}$$

where $H_i=p/t_i$ and $H_f=p/t_f$. The number of e-folds N required to solve the horizon problem (i.e., expand a Hubble-sized patch at the beginning of inflation to comoving size larger than the present Hubble volume) is approximately 60, but depends upon H_i and H_f if p is not $\gg 1$ (see, e.g., Ref. [17]):

$$N>74+\ln(H_i/H_f)+\frac{1}{2}\ln(H_f/m_{PL}). \tag{15}$$

Bringing everything together, the constraint to p is

$$p>1+\frac{74}{\ln(H_i/H_f)}+\frac{1}{2}\frac{\ln(H_f/m_{PL})}{\ln(H_i/H_f)}. \tag{16}$$

Based upon the gravity-wave contribution to CMB anisotropy H_i must be less than about $10^{-5}m_{PL}$ and the baryogenesis constraint implies $H_f\gtrsim(1\ \mathrm{TeV})^2/m_{PL}\sim10^{-32}m_{PL}$. Since reheating is not expected to be very efficient and baryogenesis may require a temperature much greater than 1 TeV (if it involves GUT, rather than electroweak, physics), we can safely say that $H_f\gg10^{-32}m_{PL}$. Thus, sufficient inflation and safe return to a radiation-dominated universe before baryogenesis requires

$$p\gg 2 \tag{17}$$

$$(1-n)\ll 2. \tag{18}$$

Even insisting that $H_f\gtrsim(10^{13}\ \mathrm{GeV})^2/m_{PL}$, a typical inflation scale, only leads to $p\gtrsim5$ and $n\gtrsim0.5$, which is still a large deviation from scale invariance.

While the exponential potential allows a very large deviation from $n=1$, it illustrates the tension between achieving sufficient inflation and large deviation from scale invariance: because $(1-n)=2/(p-1)$, large deviation from scale invariance implies a slow, prolonged inflation, $\ln(t_f/t_i)\simeq N(1-n)/2$, with the change in the inflaton field being many times the Planck mass, $\Delta\phi\simeq N\sqrt{(1-n)/8\pi}m_{PL}\gg m_{PL}$. Other models also exhibit this tension: For example, for the potential $V(\phi)=V_0-m^2\phi^2/2+\lambda\phi^4/4$, the lower limit to n is set by the condition of sufficient inflation [6].

Achieving n significantly greater 1 provides a different challenge. Since the first term in the equation for $(n-1)$ is negative, all the work must be done by the change-in-steepness term, $m_{PL}x'/4\pi$. To see the difficulty of doing so, let us assume that we can expand the slow-roll parameter $x(\phi)$ around a point ϕ_* in the slow-roll region:

$$x(\phi)\approx x_*+x'_*(\phi-\phi_*). \tag{19}$$

This expression holds for potentials whose steepness does not change much in the slow-roll region. N can now be evaluated explicitly:

$$N=-\frac{8\pi}{m_{PL}}\int_{\phi_i}^{\phi_f}\frac{d\phi}{x(\phi)}=\frac{8\pi}{x'_{60}m_{PL}}\ln\left(\frac{x_i}{x_f}\right), \tag{20}$$

where x_i and x_f are understood to have been evaluated according to expression (19). Combining expressions (7) and (20), we get

$$n-1=\frac{2}{N}\ln\left(\frac{x_i}{x_f}\right)-\frac{x_{60}^2}{8\pi}. \tag{21}$$

The difficulty of obtaining large $n-1$ is now more transparent. For example, to get $n\approx1.5$ with $N\geq60$ we need $\ln(x_i/x_f)>15$—more, if $x_{60}^2/8\pi$ is not negligible. Not only does such a large change seem unnatural, but it probably invalidates the expansion in Eq. (19).

Note that Eq. (21) (and others below) make it appear that $(n-1)$ depends directly upon the amount of inflation. This is not really the case, because N is the number of e-folds that occur during the time x evolves from x_i to x_f. In relating $(n-1)$ to properties of the potential it is probably most useful to set $N=60$, the e-folds relevant to creating our present Hubble volume.

Now we further specialize to the case $x_{60}^2/8\pi\ll|m_{PL}x'_{60}|/4\pi$ and $|x_{60}|\gg|x'_{60}\Delta\phi|$, where $\Delta\phi=\phi_f-\phi_i$. Here we have explicitly assumed that the change in the steepness of the potential is small. It now follows that

$$N\simeq\frac{8\pi}{m_{PL}}\left|\frac{\Delta\phi}{x_{60}}\right| \tag{22}$$

$$(n-1)\simeq\frac{2}{N}\left|\frac{\Delta\phi}{x_{60}}\right|x'_{60}<\frac{2}{N} \tag{23}$$

(note that $\Delta\phi$ and x_{60} are of opposite sign). Thus, we get a very strong constraint on n in this case, $(n-1)<0.04$, and learn that to achieve n significantly greater than unity, the scalar field must change by much more than m_{PL}.

One well-known class of inflationary models that gives $n\geq1$ is hybrid inflation [25]; in the slow-roll region, $V(\phi)\simeq V_0(1+\mu\phi^2)$. In these models,

$$N\simeq\frac{4\pi}{\mu m_{PL}^2}\ln(\phi_i/\phi_f) \tag{24}$$

$$(n-1)\simeq\frac{m_{PL}x'}{4\pi}=\frac{\mu m_{PL}^2}{2\pi}\simeq\frac{2}{N}\ln(\phi_i/\phi_f). \tag{25}$$

Thus, n significantly larger than 1 can be achieved, albeit at the expense of an exponentially long roll, $\phi_i/\phi_f=\exp[N(n-1)/2]$. However, ϕ_f may not be arbitrarily small here—in fact, the smallest value it can take in the semi-classical approximation is equal to the magnitude of quantum fluctuations of the field, $H/2\pi$ (this is further discussed in the next section). This constraint, in combination with the other constraints, limits the maximum value of n in hybrid inflation scenarios to $n\leq1.2$ [11].

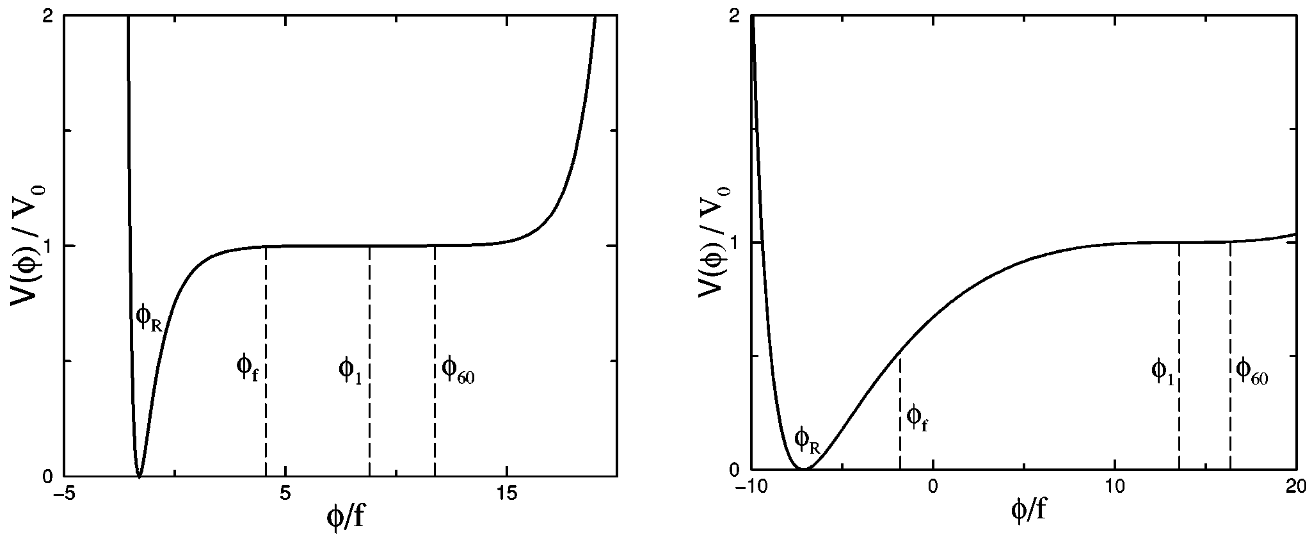

FIG. 1. Two potentials with $n\approx 2$. In each case inflation starts at ϕ_{60} and ends at ϕ_f; the potential has a point of approximate inflection at $\phi=\phi_1$ and its minimum at $\phi=\phi_R$. Top: $V=V_0+M^4[\sinh[(\phi-\phi_1)/f]+\exp(-\phi/g)]$. Bottom: $V=V_0+M^4[(\phi-\phi_1)/f+[(\phi-\phi_1)/f]^3+\exp(-\phi/f)]$. Potential parameters are given in the text.

To summarize this discussion, let us rewrite Eq. (21) by expressing $x_{60}^2/8\pi$ in terms of N and $\Delta\phi$ by assuming that $x(\phi)$ does not change too much:

$$(n-1)\simeq\frac{2}{N}\ln(x_i/x_f)-\frac{8\pi}{N^2}\left(\frac{\Delta\phi}{m_{PL}}\right)^2. \tag{26}$$

As this equation illustrates, unless $\Delta\phi/m_{PL}$ is large or the steepness changes significantly, $|n-1|\lesssim 2/N\approx 0.04$. This is certainly borne out by inflationary model building: with a few notable exceptions all models predict $|n-1|\lesssim\mathcal{O}(0.1)$ [11].

III. MODELS WITH VERY BLUE SPECTRA

A. Constraints

The conditions for successful inflation were spelled out a decade ago [6,17]. The *règles de jeu* are the following:

(i) Slow-roll conditions.

(ii) Sufficient number of e-folds to solve the horizon problem ($N\gtrsim 60$).

(iii) Density perturbations of the correct amplitude

$$\delta_H\sim V_{60}^{3/2}/V_{60}'\sim 10^{-5}. \tag{27}$$

(iv) The distance that ϕ rolls in a Hubble time must exceed the size of quantum fluctuations, otherwise the semiclassical approximation breaks down:

$$\dot\phi H^{-1}\gg H/2\pi\Rightarrow V'\gg V^{3/2}/m_{PL}^3. \tag{28}$$

This is automatically satisfied if the density perturbations are small. Additionally, no aspect of inflation should hinge upon ϕ_i or ϕ_f being smaller than $H/2\pi$, the size of the quantum fluctuations.

(v) ''Graceful exit'' from inflation. The potential should have a stable minimum with zero energy around which the field oscillates at the stage of reheating. The reheat temperature must be sufficiently high to safely return the Universe to a radiation-dominated phase in time for baryogenesis and big bang nucleosynthesis (BBN).

(vi) No overproduction of undesired relics such as magnetic monopoles, gravitinos, or other nonrelativistic particles.

There are additional constraints that the potential should obey in order to give $n\gg 1$:

(a) $m_{PL}x_{60}'/4\pi$ has to be large and positive, while $x_{60}^2/8\pi$ should be negligible.[2] Therefore $|x_{60}|\lesssim O(1)$ and $m_{PL}x_{60}'\simeq 4\pi(n-1)$. In other words, at 60 e-folds before the end of inflation the potential should be nearly flat and starting to slope upwards.

(b) To obtain 60 e-folds of inflation, the potential should be nearly flat in some region during inflation. However, the potential must not become too flat, since then density perturbations diverge ($\delta_H\propto 1/V'$). Therefore, the potential should have a point of approximate inflection where $V'(\phi)$ is small but not zero.

B. Example 1

A potential with the characteristics just mentioned is

$$V=V_0+M^4\left[\sinh\left(\frac{\phi-\phi_1}{f}\right)+e^{-\phi/g}\right], \tag{29}$$

where M, f, g, and ϕ_1 are constants with dimension of mass. The plot of the potential, with the parameters calculated below, is shown in the top panel of Fig. 1. The hyperbolic sine was invoked to satisfy requirements (a) and (b), while the exponential was used to produce a stable minimum.

[2]Of course, $x_{60}^2/8\pi$ is not required to be negligible, but it is even more difficult to get large $(n-1)$ without this assumption.

We make the following assumptions to make the analysis simpler (later justified by our choice of parameters below):

(1) V_0 dominates the potential in the slow-roll region,

$$V_0 \gg M^4 \sinh\left(\frac{\phi-\phi_1}{f}\right) \quad \text{for} \quad \phi_i > \phi > \phi_f. \tag{30}$$

(2) $f \gg g$ so that the factor $\exp(-\phi/g)$ can be completely ignored in the slow-roll region.

(3) $(\phi-\phi_1)/f$ is at least of the order of a few for $\phi_i > \phi > \phi_f$, so that $\sinh[(\phi-\phi_1)/f] \gg 1$.

(4) For simplicity we take $\phi_i = \phi_{60}$.

In terms of the dimensionless parameter $K \equiv M^4 m_{PL}/V_0 f$:

$$x \simeq K \cosh\left(\frac{\phi-\phi_1}{f}\right), \tag{31}$$

$$x' \simeq \frac{K}{f}\sinh\left(\frac{\phi-\phi_1}{f}\right) - \frac{K^2}{m_{PL}}\cosh^2\left(\frac{\phi-\phi_1}{f}\right). \tag{32}$$

The condition that $x_{60} \lesssim \mathcal{O}(1)$ becomes

$$K \cosh\left(\frac{\phi_{60}-\phi_1}{f}\right) \lesssim \mathcal{O}(1), \tag{33}$$

and the end of inflation occurs one of the slow-roll conditions breaks down; in this case $m_{PL}^2 V''/V \simeq 24\pi$ or

$$\frac{m_{PL}K}{f}\sinh\left(\frac{\phi_f-\phi_1}{f}\right) \simeq 24\pi. \tag{34}$$

We can now write

$$(n-1) \simeq \frac{m_{PL}K}{4\pi f}\sinh\left(\frac{\phi_{60}-\phi_1}{f}\right). \tag{35}$$

That inflation produces density perturbations of the correct magnitude implies

$$\sqrt{V_0} \approx 4.3\times 10^{-6} x_{60} m_{PL}^2. \tag{36}$$

The expression for the number of e-folds can be calculated analytically. Introducing $\alpha = (\phi-\phi_1)/f$, we have

$$N = -\frac{8\pi}{m_{PL}}\int_{\phi_i}^{\phi_f}\frac{d\phi}{x} = -\frac{8\pi f}{K m_{PL}}\tan^{-1}[\sinh(\alpha)]\Big|_{\alpha_i}^{\alpha_f} \approx \frac{8\pi^2 f}{K m_{PL}}. \tag{37}$$

In the last equality we used the fact that both α_i and $|\alpha_f|$ are at least of the order of a few, so that $\tan^{-1}[\sinh(\alpha_i)] \approx -\tan^{-1}[\sinh(\alpha_f)] \approx \pi/2$. This assumption will also be fully justified with our choice of parameters below.

Finally, the potential should have a stable minimum (with $V=0$) at some $\phi=\phi_R$. This implies that $V(\phi_R)=0$ and $V'(\phi_R)=0$.

Before proceeding, we must specify n. We choose, somewhat arbitrarily, $n=2$. Of course, for such a large n we should include terms beyond the lowest order, complicating the analysis. But we are not looking for accuracy—if $n=2$ is obtainable to first order, then one can certainly say that $n \gg 1$ is obtainable. (In fact, for the two potentials chosen, the second-order correction decreases $n-1$ only slightly.)

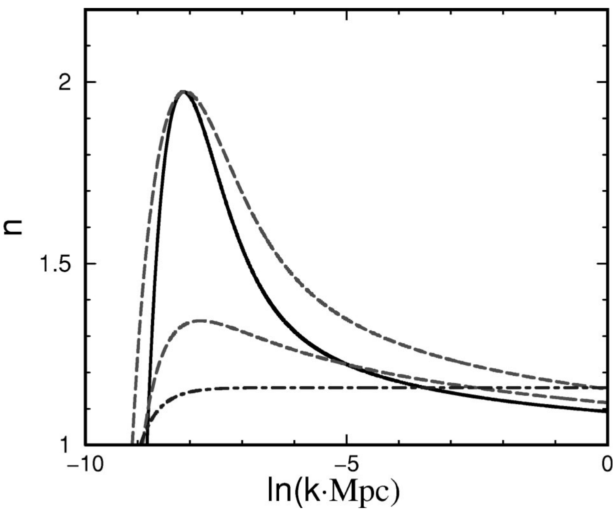

FIG. 2. The power-law index n for the two inflationary potentials constructed to give $n\sim 2$ as a function of $\ln k$. The solid curve corresponds to the hyperbolic sine potential and the upper dashed curve to the cubic "$\phi+\phi^3$" potential. While both potentials achieve $n\sim 2$, neither has a very good power-law spectrum. Also shown is a cubic potential model with $n\simeq 1.4$ (lower dashed curve), where the variation of n is less severe. For comparison, the hybrid inflation model (dash-dotted curve) with $n\simeq 1.2$ is also shown; here n is fairly constant over the astrophysically interesting range.

We now have to choose parameters V_0, M, f, g, ϕ_1, ϕ_{60}, ϕ_f and ϕ_R to satisfy Conditions (33)–(37), as well as $V(\phi_R)=0$ and $V'(\phi_R)=0$. The choice of these parameters is by no means unique, however. Here is such a set:

$$V_0 = 1.7\times 10^{-13} m_{PL}^4 \qquad \frac{\phi_1}{f} = 8.80$$

$$M^4 = 1.3\times 10^{-17} m_{PL}^4 \qquad \frac{\phi_f}{f} = 4.10$$

$$f = 7.6\times 10^{-3} m_{PL} \qquad \frac{\phi_{60}}{f} = 11.75$$

$$g = f/5. \tag{38}$$

To verify our analytic results we integrated the equation of motion for ϕ numerically and computed the spectrum of density perturbations. We did so neglecting the $\ddot{\phi}$ in the equation of motion for ϕ and the kinetic energy of the field (slow-roll approximation) and taking both these quantities into account (exact calculation). The result is that $N_{\rm slowroll} = 57.3$ and $N_{\rm exact} = 57.9$. Thus, the field really rolls as predicted by analytic methods ($N\approx 60$), and the slow-roll approximation holds well for this potential.

The numerical results for the spectrum of density perturbations did contain a surprise, shown in Fig. 2. While this potential achieved large n, slightly smaller than 2, over a few e-folds n falls to a smaller value. Indeed, even restricting the spectrum to astrophysically interesting scales, $1-10^4$ Mpc, the spectrum is not a good power law, $|dn/d\ln k|\sim 0.3$, and is reminiscent of the ''designer spectra'' with special features constructed in Ref. [26]. The reason is simple: in achieving $x'\sim 1$ an even larger value of x'' was attained.

C. Example 2

Is there anything special about the hyperbolic sine? Not really—for example, a potential of the form ''$\phi+\phi^3$'' also works. Consider the potential

$$V=V_0+M^4\left[\left(\frac{\phi-\phi_1}{f}\right)+\left(\frac{\phi-\phi_1}{f}\right)^3+e^{-\phi/g}\right]. \quad (39)$$

Again, we assume that V_0 dominates during inflation, that $\phi_i=\phi_{60}$ and that $\exp(-\phi/g)$ can be ignored in the inflationary region. To evaluate N, we further assume that $|(\phi_{60}-\phi_1)/f|\gtrsim 1$ and $|(\phi_f-\phi_1)/f|\gtrsim 1$. All of these assumptions are justified by the choice of parameters below.

The analysis of the inflationary constraints is similar. We conclude that large n (here $n=2$) is possible, with the following parameters:

$$V_0=1.09\times 10^{-12}m_{PL}^4 \qquad \frac{\phi_1}{f}=13.54$$

$$M^4=1.46\times 10^{-16}m_{PL}^4 \qquad \frac{\phi_f}{f}=-1.82$$

$$f=g=1.33\times 10^{-2}m_{PL} \qquad \frac{\phi_{60}}{f}=16.34. \quad (40)$$

This potential is shown in the bottom panel of Fig. 1. Numerical integration of the equation of motion shows that our ''60 e-folds'' is actually $N_{\rm slowroll}=55.0$ and $N_{\rm exact}=56.0$. Further, just as with the hyperbolic sine potential, $n\sim 2$ is achieved, but the spectrum of perturbations is not a good power law. Both potentials achieve a large change in steepness by having inflation occur near an approximate inflection point; however, the derivative of the change in steepness is also large, and n varies significantly. The change in n can be mitigated at the expense of a smaller value of n; see Fig. 2.

IV. CONCLUSIONS

The deviation of inflationary density perturbations from exact scale invariance is controlled by the steepness of the potential and the change in steepness; cf. Eq. (7). The steepness of the potential also controls the relationship between the amount of inflation and change in the field driving inflation, $N\sim 8\pi(\Delta\phi/m_{PL})/x$. A very ''red spectrum'' can be achieved at the expense of a steep potential and prolonged inflation ($t_f/t_i\gg 1$ and $\Delta\phi\gg m_{PL}$); the simplest example is power-law inflation. A very ''blue spectrum'' can be achieved at the expense of a large change in steepness near an inflection point in the potential and a poor power law. In both cases there appears to be a degree of unnaturalness.

The robustness of the inflationary prediction of approximately scale-invariance density perturbations is expressed by Eq. (26):

$$(n-1)\simeq\frac{2}{N}\ln(x_i/x_f)-\frac{8\pi}{N^2}\left(\frac{\Delta\phi}{m_{PL}}\right)^2.$$

Unless the change in steepness of the potential is large, $|\ln(x_f/x_i)|\gg 1$, or the duration of inflation is very long, $\Delta\phi\gg m_{PL}$, the deviation from scale invariance must be small, $|n-1|\lesssim\mathcal{O}(2/N)\sim 0.1$. Even for an extreme range in n, say from $n=0.5$ to $n\sim 1.5$, the variation of δ_H over astrophysically interesting scales, ~ 1 Mpc to $\sim 10^4$ Mpc, is not especially large—a factor of 10 or so—but is easily measurable.

Inflation also predicts a nearly scale-invariant spectrum of gravitational waves (tensor perturbations). The deviation from scale invariance is controlled solely by the first term in $(n-1)$ [8,10], $n_T=-x_{60}^2/8\pi$. Thus, only a red spectrum is possible, with the same remarks applying as for density (scalar) perturbations with $n\ll 1$. In addition, the relative amplitude of the scalar and tensor perturbations is related to the deviation of the tensor perturbations from scale invariance, $T/S\simeq -7n_T$ (S and T are respectively the scalar and tensor contributions to the variance of the quadrupole anisotropy of the CMB). Detection of the gravity-wave perturbations is an important, but very challenging, test of inflation; if, in addition, the spectral index of the tensor perturbations can be measured, it provides a consistency test of inflation [27].

The deviation of n from unity is a key test of inflation and provides valuable information about the underlying potential [9]. Measurements of the anisotropy of the CMB and of the power spectrum of inhomogeneity today which will be made over the next decade will probe the nature of the primeval density perturbations and determine n very precisely, $\sigma_n\sim 0.01$ [15]. By so doing they will provide a key test of inflation and provide insight into the underlying dynamics. On the basis of our work here, as well as previous studies (see, e.g., Ref. [11]), one would expect $|(n-1)|\sim\mathcal{O}(0.1)$, but not precisely zero. A determination that $|(n-1)|\sim\mathcal{O}(0.1)$ would be a confirmation of the basic inflationary framework. On the other hand, a determination that $|n-1|\gtrsim\mathcal{O}(0.2)$ would point to a handful of less generic potentials. Finally, in the context of inflation, it would be very surprising to find that $n=1$.

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