Relaxed linearized algorithms for faster X-ray CT image reconstruction

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Statistical image reconstruction (SIR)

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Statistical image reconstruction for X-ray CT

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\hat{x} = \arg \min_{x} \left\{ \Psi_{PWLS}(x) \equiv \frac{1}{2} \| y - Ax \|_2^2 W + R(x) + \iota \Omega(x) \right\}
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Statistical image reconstruction for X-ray CT

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First-order algorithms with ordered-subsets

Figure: Chest: Existing first-order algorithms with ordered-subsets (OS).

[Graph showing RMS difference in HU vs. number of iterations for OS-SQS, OS-FGM2, OS-LALM, and OS-OGM2]
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Equality-constrained composite minimization

Consider an equality-constrained minimization problem:

$$(\hat{x}, \hat{u}) \in \arg \min_{x,u} \left\{ \frac{1}{2} \|u - y\|_2^2 + h(x) \right\} \text{ s.t. } u = Ax,$$

where $h$ are closed and proper convex functions.
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In particular, the quadratic loss function penalizes the difference between the linear model \(Ax\) and noisy measurement \(y\), and \(h\) is a regularization term that introduces the prior knowledge of \(x\) to the reconstruction.
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where \( h \) are closed and proper convex functions.

In particular, the quadratic loss function penalizes the difference between the linear model \( Ax \) and noisy measurement \( y \), and \( h \) is a regularization term that introduces the prior knowledge of \( x \) to the reconstruction. For example,

\[
(\hat{x}, \hat{u}) \in \arg \min_{x,u} \left\{ \frac{1}{2} \|u - W^{1/2}y\|_2^2 + (R + \iota \Omega)(x) \right\} \text{ s.t. } u = W^{1/2}Ax
\]

represents an X-ray CT image reconstruction problem.
Standard AL method

The standard AL method finds a saddle-point of the augmented Lagrangian (AL) of (1):

$$\mathcal{L}_A(x, u, d; \rho) \triangleq \frac{1}{2} \|u - y\|^2_2 + h(x) + \frac{\rho}{2} \|Ax - u - d\|^2_2$$

in an alternating direction manner:

$$\begin{cases} 
  x^{(k+1)} \in \arg \min_x \left\{ h(x) + \frac{\rho}{2} \|Ax - u^{(k)} - d^{(k)}\|^2_2 \right\} \\
  u^{(k+1)} \in \arg \min_u \left\{ \frac{1}{2} \|u - y\|^2_2 + \frac{\rho}{2} \|Ax^{(k+1)} - u - d^{(k)}\|^2_2 \right\} \\
  d^{(k+1)} = d^{(k)} - Ax^{(k+1)} + u^{(k+1)},
\end{cases}$$

where $d$ is the scaled Lagrange multiplier of $u$, and $\rho > 0$ is the AL penalty parameter.

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d^{(k+1)} &= d^{(k)} - Ax^{(k+1)} + u^{(k+1)} ,
\end{align*}$$

where $d$ is the scaled Lagrange multiplier of $u$, and $\rho > 0$ is the AL penalty parameter.

Linearized AL method

The linearized AL method adds an additional $G$-proximity term to the $x$-subproblem in the standard AL method:

$$
\begin{align*}
\mathbf{x}^{(k+1)} & \in \arg \min_{\mathbf{x}} \left\{ h(\mathbf{x}) + \frac{\rho}{2} \| \mathbf{Ax} - \mathbf{u}^{(k)} - \mathbf{d}^{(k)} \|_2^2 + \frac{\rho}{2} \| \mathbf{x} - \mathbf{x}^{(k)} \|_G^2 \right\} \\
\mathbf{u}^{(k+1)} & \in \arg \min_{\mathbf{u}} \left\{ \frac{1}{2} \| \mathbf{u} - \mathbf{y} \|_2^2 + \frac{\rho}{2} \| \mathbf{Ax}^{(k+1)} - \mathbf{u} - \mathbf{d}^{(k)} \|_2^2 \right\} \\
\mathbf{d}^{(k+1)} & = \mathbf{d}^{(k)} - \mathbf{Ax}^{(k+1)} + \mathbf{u}^{(k+1)},
\end{align*}
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where $G \triangleq \mathbf{D}_L - \mathbf{A}'\mathbf{A}$, and $\mathbf{D}_L$ is a diagonal majorizing matrix of $\mathbf{A}'\mathbf{A}$.

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\mathbf{u}^{(k+1)} & \in \arg \min_{\mathbf{u}} \left\{ \frac{1}{2} \| \mathbf{u} - \mathbf{y} \|^2_2 + \frac{\rho}{2} \| \mathbf{A} \mathbf{x}^{(k+1)} - \mathbf{u} - \mathbf{d}^{(k)} \|^2_2 \right\} \\
\mathbf{d}^{(k+1)} & = \mathbf{d}^{(k)} - \mathbf{A} \mathbf{x}^{(k+1)} + \mathbf{u}^{(k+1)},
\end{align*}
\]

where $G \triangleq \mathbf{D}_L - \mathbf{A}' \mathbf{A}$, and $\mathbf{D}_L$ is a diagonal majorizing matrix of $\mathbf{A}' \mathbf{A}$. Here, we linearize the algorithm in a sense that the Hessian matrix of the “augmented” quadratic AL term is diagonal.

Linearized AL algorithm for X-ray CT

**Algorithm:** OS-LALM for CT reconstruction.

**Input:** $K \geq 1$, $M \geq 1$, and an initial (FBP) image $x$.

set $\rho = 1$, $\zeta = g = M\nabla L_M(x)$

**Algorithm**:

for $k = 1, 2, \ldots, K$ do

for $m = 1, 2, \ldots, M$ do

$s = \rho \zeta + (1 - \rho) g$

$x^+ = x - (\rho D_L + D_R)^{-1} (s + \nabla R(x)) \Omega$

$\zeta^+ = M\nabla L_m(x^+)$

$g^+ = \frac{\rho}{\rho + 1} \zeta^+ + \frac{1}{\rho + 1} g$

decrease $\rho$ gradually

end

end

**Output:** The final image $x$. 

Relaxed AL method

The relaxed AL method accelerates the standard AL method with under- or over-relaxation:

\[
\begin{align*}
    x^{(k+1)} & \in \arg \min_x \left\{ h(x) + \frac{\rho}{2} \| Ax - u^{(k)} - d^{(k)} \|_2^2 \right\} \\
    u^{(k+1)} & \in \arg \min_u \left\{ \frac{1}{2} \| u - y \|_2^2 + \frac{\rho}{2} \| r_{u,\alpha}^{(k+1)} - u - d^{(k)} \|_2^2 \right\} \\
    d^{(k+1)} &= d^{(k)} - r_{u,\alpha}^{(k+1)} + u^{(k+1)}
\end{align*}
\]

where

\[
r_{u,\alpha}^{(k+1)} \triangleq \alpha Ax^{(k+1)} + (1 - \alpha) u^{(k)}
\]

is the relaxation variable of \( u \), and \( 0 < \alpha < 2 \) is the relaxation parameter.

[Eckstein and Bertsekas, Math. Prog., 1992]
The relaxed AL method accelerates the standard AL method with under- or over-relaxation:

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    \begin{cases}
        x^{(k+1)} & \in \arg\min_x \left\{ h(x) + \frac{\rho}{2} \| Ax - u^{(k)} - d^{(k)} \|^2 \right\} \\
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    r_{u,\alpha}^{(k+1)} \triangleq \alpha Ax^{(k+1)} + (1 - \alpha) u^{(k)}
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is the relaxation variable of \( u \), and \( 0 < \alpha < 2 \) is the relaxation parameter. When \( \alpha = 1 \), it reverts to the standard AL method.

[Eckstein and Bertsekas, Math. Prog., 1992]
Implicit linearization via redundant variable-splitting

Consider an equivalent equality-constrained minimization problem with a redundant equality constraint:

\[(\hat{x}, \hat{u}, \hat{v}) \in \arg \min_{x,u,v} \left\{ \frac{1}{2} \| u - y \|_2^2 + h(x) \right\} \quad \text{s.t.} \quad \begin{cases} u = Ax \\ v = G^{1/2}x \end{cases} \]

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\[(\hat{x}, \hat{u}, \hat{v}) \in \arg\min_{x,u,v} \left\{ \frac{1}{2} \|u - y\|^2_2 + h(x) \right\} \quad \text{s.t.} \quad \begin{cases} u = Ax \\ v = G^{1/2} x. \end{cases} \]

Suppose we use the same AL penalty parameter $\rho$ for both equality constraints in AL methods. The quadratic AL term

\[\frac{\rho}{2} \left\| Ax - u^{(k)} - d^{(k)} \right\|^2_2 + \frac{\rho}{2} \left\| G^{1/2} x - v^{(k)} - e^{(k)} \right\|^2_2 \]

has a diagonal Hessian matrix $H_{\rho} \triangleq \rho A'A + \rho G = \rho D_L$, leading to a separable quadratic AL term.

Proposed relaxed linearized AL method

The proposed relaxed linearized AL method solves the equivalent minimization problem with a redundant equality constraint using the relaxed AL method:

\[
\begin{align*}
\mathbf{x}^{(k+1)} &\in \operatorname{arg\,min}_{\mathbf{x}} \left\{ h(\mathbf{x}) + \frac{\rho}{2} \| \mathbf{A}\mathbf{x} - \mathbf{u}^{(k)} - \mathbf{d}^{(k)} \|_2^2 \right. \\
&\quad \left. + \frac{\rho}{2} \| \mathbf{G}^{1/2}\mathbf{x} - \mathbf{v}^{(k)} - \mathbf{e}^{(k)} \|_2^2 \right\} \\
\mathbf{u}^{(k+1)} &\in \operatorname{arg\,min}_{\mathbf{u}} \left\{ \frac{1}{2} \| \mathbf{u} - \mathbf{y} \|_2^2 + \frac{\rho}{2} \| r^{(k+1)}_{\mathbf{u},\alpha} - \mathbf{u} - \mathbf{d}^{(k)} \|_2^2 \right\} \\
\mathbf{d}^{(k+1)} &= \mathbf{d}^{(k)} - r^{(k+1)}_{\mathbf{u},\alpha} + \mathbf{u}^{(k+1)} \\
\mathbf{v}^{(k+1)} &= r^{(k+1)}_{\mathbf{v},\alpha} - \mathbf{e}^{(k)} \\
\mathbf{e}^{(k+1)} &= \mathbf{e}^{(k)} - r^{(k+1)}_{\mathbf{v},\alpha} + \mathbf{v}^{(k+1)}.
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When \( \alpha = 1 \), the proposed method reverts to the linearized AL method.
Proposed relaxed linearized AL method

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\begin{align*}
    x^{(k+1)} &\in \arg\min_x \left\{ h(x) + \frac{\rho}{2} \|Ax - u^{(k)} - d^{(k)}\|^2 + \frac{\rho}{2} \|G^{1/2}x - v^{(k)} - e^{(k)}\|^2 \right\} \\
    u^{(k+1)} &\in \arg\min_u \left\{ \frac{1}{2} \|u - y\|^2 + \frac{\rho}{2} \|r_{u,\alpha}^{(k+1)} - u - d^{(k)}\|^2 \right\} \\
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When \( \alpha = 1 \), the proposed method reverts to the linearized AL method.
Proposed relaxed linearized AL method (cont’d)

The proposed relaxed linearized AL method further simplifies as follows:

\[
\begin{align*}
\gamma^{(k+1)} &= (\rho - 1) g^{(k)} + \rho h^{(k)} \\
x^{(k+1)} &\in \arg\min_x \left\{ h(x) + \frac{1}{2} \| x - (\rho D_L)^{-1} \gamma^{(k+1)} \|_2^2 \right\} \\
\zeta^{(k+1)} &= \nabla L(x^{(k+1)}) \triangleq A' (Ax^{(k+1)} - y) \\
g^{(k+1)} &= \frac{\rho}{\rho+1} \left( \alpha \zeta^{(k+1)} + (1 - \alpha) g^{(k)} \right) + \frac{1}{\rho+1} g^{(k)} \\
h^{(k+1)} &= \alpha (D_L x^{(k+1)} - \zeta^{(k+1)}) + (1 - \alpha) h^{(k)},
\end{align*}
\]

where \( L(x) \triangleq (1/2) \| Ax - y \|_2^2 \) denotes the quadratic data-fidelity term, and \( g^{(k)} \triangleq A'(u^{(k)} - y) \) denotes the split gradient of \( L \).
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\zeta^{(k+1)} &= \nabla L(x^{(k+1)}) \triangleq A' (Ax^{(k+1)} - y) \\
g^{(k+1)} &= \frac{\rho}{\rho + 1} \left( \alpha \zeta^{(k+1)} + (1 - \alpha) g^{(k)} \right) + \frac{1}{\rho + 1} g^{(k)} \\
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Relaxed OS-LALM for faster CT reconstruction

To solve X-ray CT image reconstruction problem:

\[ \hat{x} \in \arg\min_x \left\{ \frac{1}{2} \| y - Ax \|_W^2 + R(x) + \iota_\Omega(x) \right\} \]

using the proposed relaxed linearized AL method, we apply the following substitution:

\[
\begin{cases}
A \leftarrow W^{1/2} A \\
y \leftarrow W^{1/2} y
\end{cases}
\]

and set \( h \triangleq R + \iota_\Omega \).
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The image update now is a diagonally weighted denoising problem. We solve it using a projected gradient descent step from \( x^{(k)} \).
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    A &\leftarrow W^{1/2} A \\
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\end{cases}$$

and set \( h \triangleq R + \iota_\Omega \).

The image update now is a \textbf{diagonally weighted denoising problem}. We solve it using a \textbf{projected gradient descent step from} \( x^{(k)} \). For speed-up, \textbf{ordered subsets (OS)} or incremental gradients are used.
We also use a **continuation technique** to speed up convergence; that is, we decrease $\rho$ gradually with iteration.
Speed-up with decreasing continuation sequence

We also use a continuation technique to speed up convergence; that is, we decrease $\rho$ gradually with iteration.

Based on a second-order recursive system analysis, we use

$$\rho_i(\alpha) = \begin{cases} 
1, & \text{if } i = 0 \\
\frac{\pi}{\alpha(i+1)} \sqrt{1 - \left(\frac{\pi}{2\alpha(i+1)}\right)^2}, & \text{otherwise.} 
\end{cases}$$

Therefore, we use a faster-decreasing continuation sequence in a more over-relaxed linearized AL method.

Proposed relaxed linearized algorithm

**Algorithm:** Relaxed OS-LALM for CT reconstruction.

**Input:** $K \geq 1$, $M \geq 1$, $0 < \alpha < 2$, and an initial (FBP) image $x$.

set $\rho = 1$, $\zeta = g = M \nabla L_M(x)$, $h = D_L x - \zeta$

for $k = 1, 2, \ldots, K$ do

for $m = 1, 2, \ldots, M$ do

$s = \rho (D_L x - h) + (1 - \rho) g$

$x^+ = \left[ x - (\rho D_L + D_R)^{-1} (s + \nabla R(x)) \right] \Omega$

$\zeta = M \nabla L_m(x^+)$

$g^+ = \frac{\rho}{\rho + 1} (\alpha \zeta + (1 - \alpha) g) + \frac{1}{\rho + 1} g$

$h^+ = \alpha (D_L x^+ - \zeta) + (1 - \alpha) h$

decrease $\rho$ using (2)

end

end

**Output:** The final image $x$. 
Chest region helical scan

We reconstruct a $600 \times 600 \times 222$ image from an $888 \times 64 \times 3611$ helical (pitch 1.0) CT scan.

![Figure: Chest: Cropped images of the initial FBP image $x^{(0)}$ (left), the reference reconstruction $x^*$ (center), and the reconstructed image $x^{(20)}$ using relaxed OS-LALM with 10 subsets after 20 iterations (right).]
Figure: Chest: Convergence rate curves of different OS algorithms with 10 (left) and 20 (right) subsets.
Figure: Chest: Difference images of the initial FBP image $x^{(0)} - x^*$ and the reconstructed image $x^{(10)} - x^*$ using OS algorithms with 10 subsets after 10 iterations.
Figure: Chest: Difference images of the initial FBP image $\mathbf{x}^{(0)} - \mathbf{x}^*$ and the reconstructed image $\mathbf{x}^{(5)} - \mathbf{x}^*$ using OS algorithms with 20 subsets after 5 iterations.
Chest region helical scan (cont’d)

Figure: Chest: Difference images of the initial FBP image $\mathbf{x}^{(0)} - \mathbf{x}^*$ and the reconstructed image $\mathbf{x}^{(10)} - \mathbf{x}^*$ using OS algorithms with 20 subsets after 10 iterations.
Chest region helical scan (cont’d)

**Figure:** Chest: Difference images of the initial FBP image $\mathbf{x}^{(0)} - \mathbf{x}^*$ and the reconstructed image $\mathbf{x}^{(20)} - \mathbf{x}^*$ using OS algorithms with 20 subsets after 20 iterations.  

More results
Conclusions and future work

In summary,

▶ We proposed a relaxed variant of linearized AL methods for faster X-ray CT image reconstruction
▶ Experimental results showed that the proposed algorithm converges $\alpha$-fold faster than its unrelaxed counterpart
▶ The speed-up means that one needs fewer subsets to reach an RMS difference criteria in a given number of iterations
▶ Empirically, the proposed algorithm is reasonably stable when we use moderate numbers of subsets

For future work,

▶ We want to work on the convergence rate analysis of the proposed algorithm
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Acknowledgments

- Supported in part by NIH grant U01 EB-018753
- Equipment support from Intel Corporation
- Research support from GE Healthcare
Shoulder region helical scan

We reconstruct a $512 \times 512 \times 109$ image from an $888 \times 32 \times 7146$ helical (pitch 0.5) CT scan.

Figure: Shoulder: Cropped images of the initial FBP image $x^{(0)}$ (left), the reference reconstruction $x^*$ (center), and the reconstructed image $x^{(20)}$ using relaxed OS-LALM with 20 subsets after 20 iterations (right).
Shoulder region helical scan (cont’d)

**Figure:** Shoulder: Convergence rate curves of different OS algorithms with 20 (left) and 40 (right) subsets.
Figure: Shoulder: Difference images of the initial FBP image $x^{(0)} - x^*$ and the reconstructed image $x^{(20)} - x^*$ using OS algorithms with 20 subsets after 20 iterations.
Abdomen region helical scan

We reconstruct a $600 \times 600 \times 239$ image from an $888 \times 64 \times 3516$ helical (pitch 1.0) CT scan.

Figure: Abdomen: Cropped images of the initial FBP image $x^{(0)}$ (left), the reference reconstruction $x^*$ (center), and the reconstructed image $x^{(20)}$ using relaxed OS-LALM with 10 subsets after 20 iterations (right).
Abdomen region helical scan (cont’d)

Figure: Abdomen: Convergence rate curves of different OS algorithms with 10 (left) and 20 (right) subsets.
Abdomen region helical scan (cont’d)

Figure: Abdomen: Difference images of the initial FBP image $x^{(0)} - x^*$ and the reconstructed image $x^{(20)} - x^*$ using OS algorithms with 10 subsets after 20 iterations.
Simple vs. proposed relaxed OS-LALM

Figure: Chest: Convergence rate curves of different relaxed algorithms with a fixed AL parameter $\rho = 0.05$ (left) and the decreasing $\rho$ (right).
Wide-cone axial scan

We reconstruct a $718 \times 718 \times 440$ image from an $888 \times 256 \times 984$ axial CT scan.

Figure: Wide-cone: Cropped images of the initial FBP image $\mathbf{x}^{(0)}$ (left), the reference reconstruction $\mathbf{x}^*$ (center), and the reconstructed image $\mathbf{x}^{(20)}$ using relaxed OS-LALM with 24 subsets after 20 iterations (right).
Wide-cone axial scan (cont’d)

Figure: Wide-cone: Convergence rate curves of different OS algorithms with 12 (left) and 24 (right) subsets.