Model-based X-ray CT image reconstruction using variable splitting methods with ordered subsets

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University of Michigan, Ann Arbor

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Acknowledgements

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- Sinogram data are provided by GE Healthcare

Outline of the thesis

1. Introduction
2. Background of X-ray CT and its reconstruction
3. Fast X-ray CT image reconstruction using VS methods with OS
   [HN & J A Fessler, Fully 3D, 2013]
   [HN & J A Fessler, SPIE MI, 2014]
   [HN & J A Fessler, CT Meeting, 2014]
   [HN & J A Fessler, arXiv:1402.4381, 2014]
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6. Conclusion and future work

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An imaging technique that uses X-rays to view the internal structure of a non-uniformly composed and opaque object such as the human body.

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Computed tomography

An imaging technique that combines a series of X-ray projections taken from many different angles and computer processing (i.e., reconstruction methods) to create cross-sectional images of the bones and soft tissues inside to the human body.

What is computed tomography?

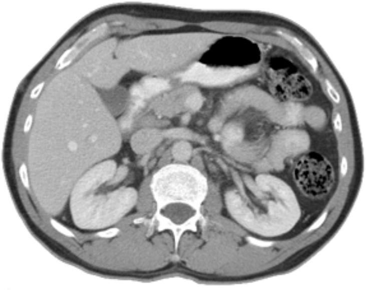

Figure: Chest X-ray image (left) and cross-sectional image of abdomen (right).

Basics of X-ray computed tomography

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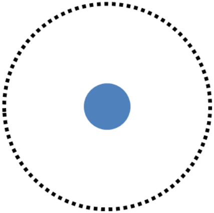

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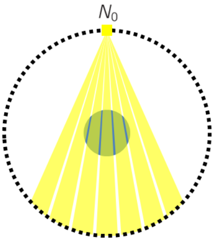

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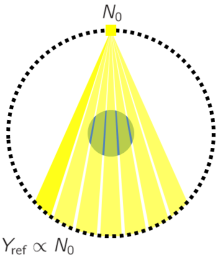

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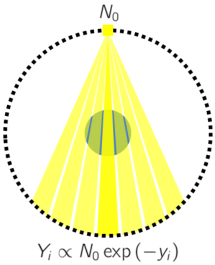

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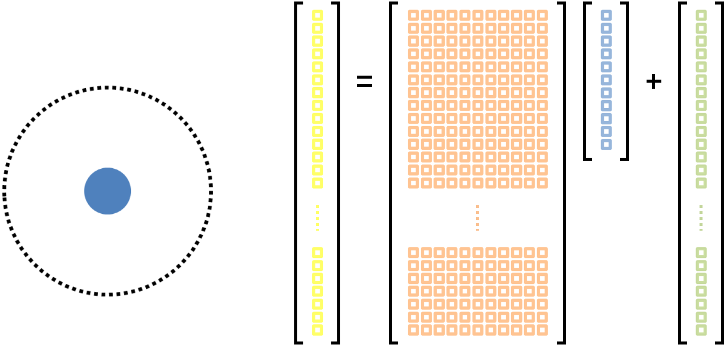

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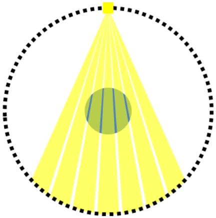

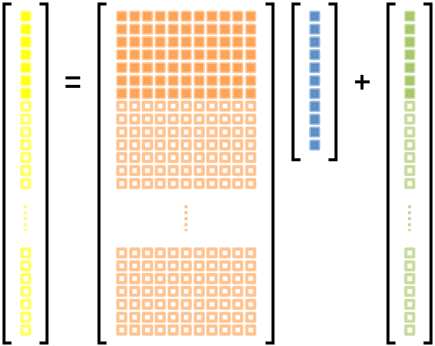

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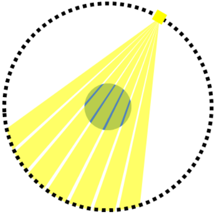

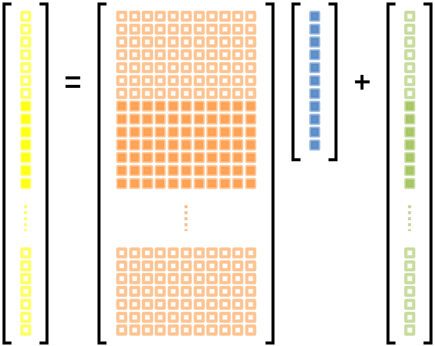

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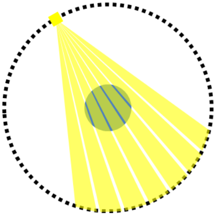

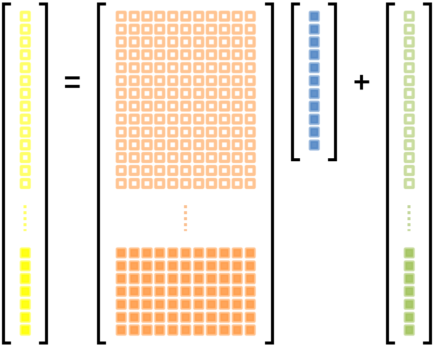

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Image reconstruction methods

Non-iterative methods

Iterative methods

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- Direct Fourier reconstruction

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- Penalized weighted least-squares (PWLS) formulation [TS+07]

$$\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}\in\Omega} \left\{ \Psi(\mathbf{x}) \triangleq \frac{1}{2} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_{\mathbf{W}}^2 + \mathrm{R}(\mathbf{x}) \right\}$$

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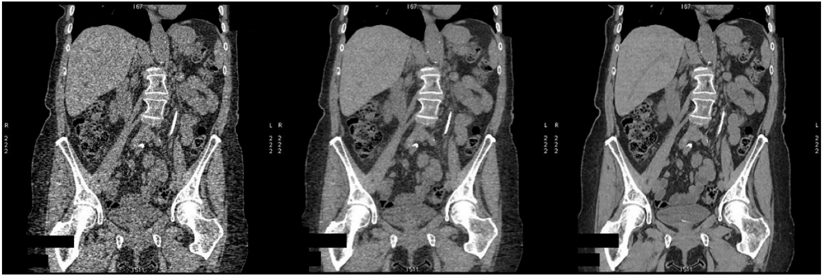

Figure: Dose reduction: FBP (left), ASiR (middle), and MBIR (right).

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Image reconstruction methods

Large problem size

- $\mathbf{x}$: $512 \times 512 \times 100 \approx 3 \cdot 10^7$ unknown image volume
- $\mathbf{y}$: $888 \times 32 \times 7000 \approx 2 \cdot 10^8$ measured noisy sinogram
- $\mathbf{A}$: $\left(3 \cdot 10^7\right) \times \left(2 \cdot 10^8\right)$ system matrix
- $\mathbf{A}$ is sparse but still too large to store
- Projection $\mathbf{Ax}$ and back-projection $\mathbf{A}'\mathbf{r}$ operations computed on the fly
- Computing gradient $\nabla\Psi(\mathbf{x}) = \mathbf{A}'\mathbf{W}\left(\mathbf{Ax} - \mathbf{y}\right) + \nabla \mathrm{R}(\mathbf{x})$ requires projection and back-projection operations that dominate computation

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Enormous dynamic range of transmission data

- The dynamic range of weighting $\mathbf{W}$ is huge
- $\mathbf{A}'\mathbf{WA}$ is highly shift-variant, and the problem is very ill-conditioned

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- Very slow (hours) but noise robust (low dose)

Image reconstruction methods

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Fast iterative methods

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CT reconstruction using split Bregman method

Equivalent formulation and split Bregman method

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- PWLS CT reconstruction: $\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}} \left\{ \frac{1}{2} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_{\mathbf{W}}^2 + \Phi(\mathbf{C}\mathbf{x}) \right\}$

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- Equivalent formulation [GO09]:

$$(\hat{\mathbf{x}}, \hat{\mathbf{v}}) \in \arg\min_{\mathbf{x},\mathbf{v}} \left\{ \frac{1}{2} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_{\mathbf{W}}^2 + \Phi(\mathbf{v}) \right\} \text{ s.t. } \mathbf{v} = \mathbf{C}\mathbf{x}$$

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- Corresponding (scaled) augmented Lagrangian:

$$\mathcal{L}_{\mathrm{A}}(\mathbf{x}, \mathbf{v}, \mathbf{e}; \eta) \triangleq \frac{1}{2} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_{\mathbf{W}}^2 + \Phi(\mathbf{v}) + \frac{\eta}{2} \|\mathbf{C}\mathbf{x} - \mathbf{v} - \mathbf{e}\|_2^2$$

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Equivalent formulation and split Bregman method (cont'd)

- Split Bregman iterates [GO09]:

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- Convergent with inexact updates [NF14a]

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- Convergent with inexact updates [NF14a]
- Slow $\mathbf{x}$-update due to the highly shift-variant Hessian $\mathbf{A}'\mathbf{W}\mathbf{A} + \eta\mathbf{C}'\mathbf{C}$

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Alternative formulation and ADMM

- Alternative formulation [RF12]:

$$(\hat{\mathbf{x}}, \hat{\mathbf{u}}, \hat{\mathbf{v}}) \in \arg\min_{\mathbf{x},\mathbf{u},\mathbf{v}} \left\{ \frac{1}{2} \|\mathbf{y} - \mathbf{u}\|_{\mathbf{W}}^2 + \Phi(\mathbf{v}) \right\} \text{ s.t. } \mathbf{u} = \mathbf{A}\mathbf{x}, \mathbf{v} = \mathbf{C}\mathbf{x}$$

Better conditioning with additional VS

Alternative formulation and ADMM (cont'd)

- ADMM iterates [RF12]:

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Better conditioning with additional VS

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- $\rho \mathbf{A}'\mathbf{A} + \eta \mathbf{C}'\mathbf{C}$ can be well preconditioned by an appropriate circulant preconditioner in 2D CT

Better conditioning with additional VS

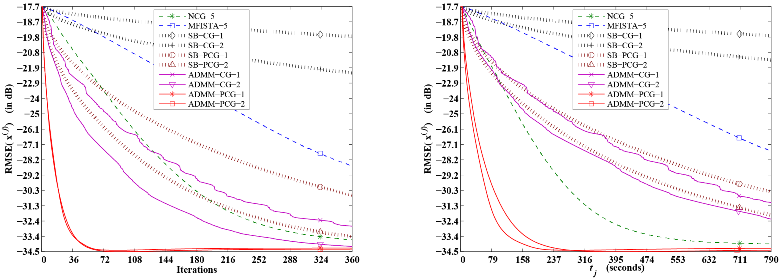

Figure: 2D NCAT: RMS errors as a function of iteration (left) and time (right).

Better conditioning with additional VS

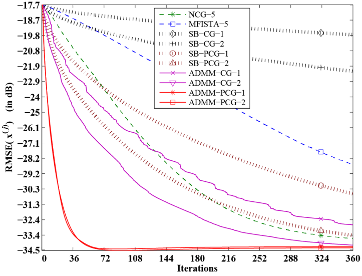

Figure: 2D NCAT: RMS errors as a function of iteration (left) and time (right).

The fact is ...

$\mathbf{A}'\mathbf{A}$ is still highly shift-variant in 3D CT due to the different geometries and scan trajectories, so this method is still slow in 3D CT

Outline

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1. Background
   - Model-based CT reconstruction
   - Fast (2D) CT reconstruction using ADMM
2. OS-LALM: a splitting-based OS algorithm for PWLS problems
   - Linearized AL method with OS acceleration
   - Deterministic downward continuation approach
   - Low-memory OS-LALM with additional variable splits
3. Experimental results
   - Low-dose CT with edge-preserving regularizers
   - Sparse-view CT with TV-like regularizers
4. Conclusion and future work

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What's wrong with ADMM in CT recon?

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- Proposed formulation [NF13]:

$$(\hat{\mathbf{x}}, \hat{\mathbf{u}}) \in \arg\min_{\mathbf{x}\in\Omega,\mathbf{u}} \left\{ \frac{1}{2} \left\| \mathbf{W}^{1/2}\mathbf{y} - \mathbf{u} \right\|_2^2 + \mathrm{R}(\mathbf{x}) \right\} \text{ s.t. } \mathbf{u} = \mathbf{W}^{1/2}\mathbf{A}\mathbf{x}$$

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Inexact linearized AL method

The idea is ...

Linearized AL method and proposed variants

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- Majorizing $\theta_k(\mathbf{x}) \triangleq \frac{\rho}{2}\left\|\tilde{\mathbf{A}}\mathbf{x} - \mathbf{u}^{(k)} - \mathbf{d}^{(k)}\right\|_2^2$ simplifies image updates

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- Inexact updates? Convergence rate? Many subsets? [NF14d]

Outline

Faster convergence using continuation

An inconvenient truth ... of ADMM

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- We can adjust the step size by varying the penalty parameter (How?)

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- Inspired by a second-order recursive system analysis Derivation
- An adaptive restart condition takes care of the dependence on $\tilde{\mathbf{A}}$

Outline

OS-LALM with an additional VS

Is additional VS really beneficial?

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CT reconstruction with "high-memory" VS

- PWLS formulation:

$$\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}\in\Omega} \left\{ \frac{1}{2} \left\| \tilde{\mathbf{y}} - \tilde{\mathbf{A}}\mathbf{x} \right\|_2^2 + \Phi(\mathbf{C}\mathbf{x}) \right\}$$

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- Equivalent formulation [NF14c]:

$$(\hat{\mathbf{x}}, \hat{\mathbf{u}}, \hat{\mathbf{v}}) \in \arg\min_{\mathbf{x}\in\Omega,\mathbf{u},\mathbf{v}} \left\{ \frac{1}{2} \left\| \tilde{\mathbf{y}} - \mathbf{u} \right\|_2^2 + \Phi(\mathbf{v}) \right\} \text{ s.t. } \mathbf{u} = \tilde{\mathbf{A}}\mathbf{x}, \mathbf{v} = \mathbf{C}\mathbf{x}$$

OS-LALM with an additional VS

"High-memory" OS-LALM

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\begin{cases}
\mathbf{s}^{(k+1)} = \rho \nabla \ell\left(\mathbf{x}^{(k)}\right) + (1-\rho)\,\mathbf{g}^{(k)} \\
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- Remarkable memory and computational overhead

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“Low-memory” OS-LALM

- At the kth iteration, replace Φ by

$$\breve{\Phi}\left(\mathbf{v}; \mathbf{C}\mathbf{x}^{(k+1)}\right) \propto \mathbf{v}'\nabla\Phi\left(\mathbf{C}\mathbf{x}^{(k+1)}\right) + \tfrac{L_{\Phi}}{2}\left\|\mathbf{v} - \mathbf{C}\mathbf{x}^{(k+1)}\right\|_2^2$$

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- The $\mathbf{v}$-update has a linear approximate solution:

$$\mathbf{v}^{(k+1)} \approx \mathbf{C}\mathbf{x}^{(k+1)} - \left(\tfrac{\bar{\eta}}{\bar{\eta}+L_\Phi}\mathbf{e}^{(k)} + \tfrac{L_\Phi}{\bar{\eta}+L_\Phi} L_\Phi^{-1}\nabla\Phi\left(\mathbf{C}x^{(k+1)}\right)\right)$$

Low-memory OS-LALM with "compressed" VS

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Outline

1. Background
   - Model-based CT reconstruction
   - Fast (2D) CT reconstruction using ADMM
2. OS-LALM: a splitting-based OS algorithm for PWLS problems
   - Linearized AL method with OS acceleration
   - Deterministic downward continuation approach
   - Low-memory OS-LALM with additional variable splits
3. Experimental results
   - Low-dose CT with edge-preserving regularizers
   - Sparse-view CT with TV-like regularizers
4. Conclusion and future work

Setup and notation

Setup and notation

OS configuration

- OS with the bit-reversal order
- Separable quadratic surrogate with Hessian $\mathbf{D}_{\mathsf{L}} \triangleq \mathrm{diag}\{\mathbf{A}'\mathbf{W}\mathbf{A}\mathbf{1}\}$

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Naming conventions

- OS-SQS-M: the standard OS algorithm [EF99]
- OS-Nes05-M: the state-of-the-art OS+momentum algorithm [KR+13]
- OS-LALM-M-ρ-n: the proposed one-split algorithm [NF14b; NF14d]
- OS-LALM-M-c-η (low-mem): the proposed two-split algorithm

Outline

Shoulder region helical scan

Specification

- Image size: $512 \times 512 \times 109$
- Sinogram size: $888 \times 32 \times 7146$ (about 7 turns, pitch $= 0.5$)

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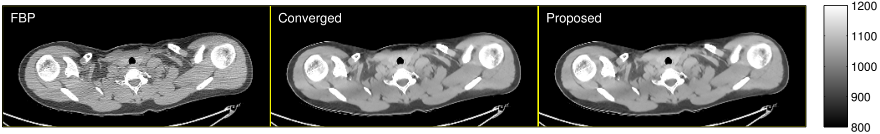


Figure: Shoulder: the initial FBP image (left), the reference reconstruction (middle), and the reconstructed image using OS-LALM after 30 iterations (right).

Shoulder region helical scan

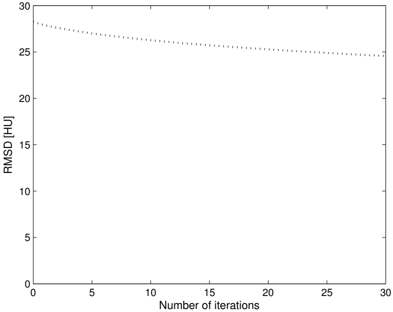
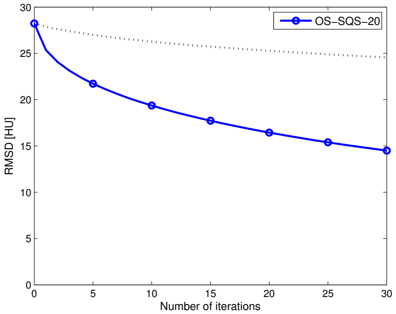


Figure: Shoulder: RMS differences as a function of iteration using different OS-based algorithms with 20 subsets (left) and 40 subsets (right).

Shoulder region helical scan



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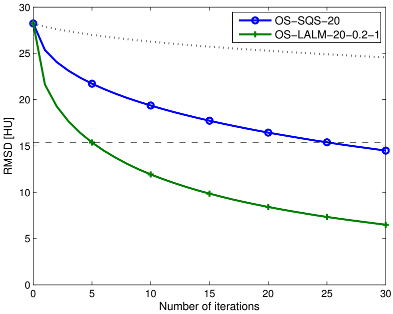

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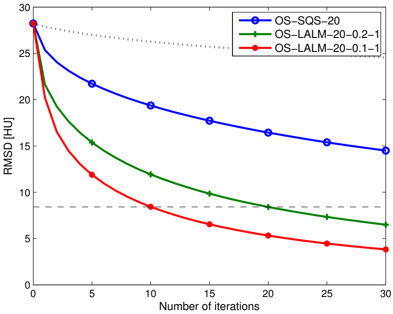

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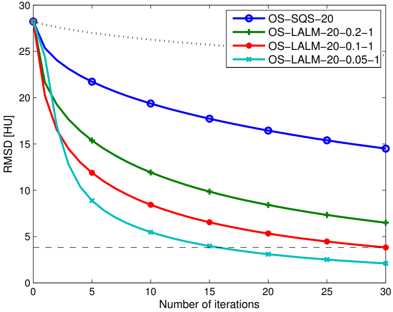

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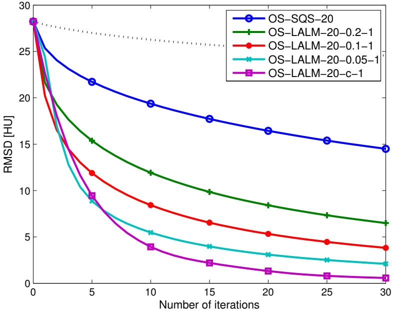

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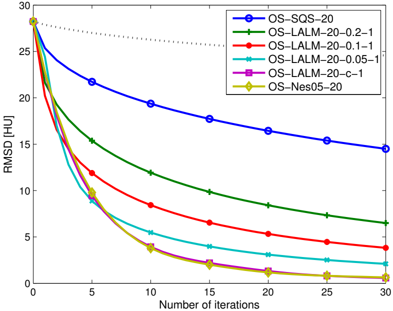

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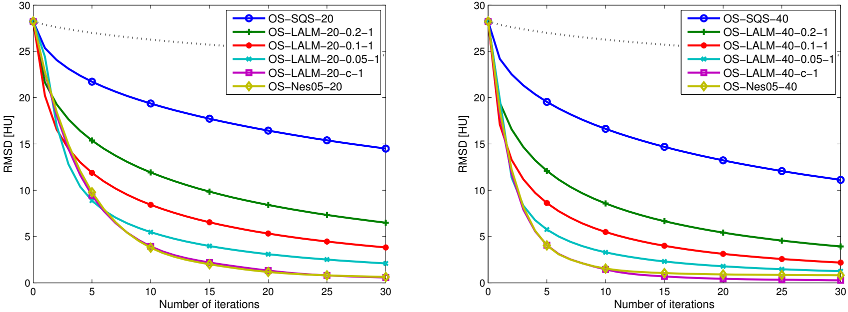
Shoulder region helical scan



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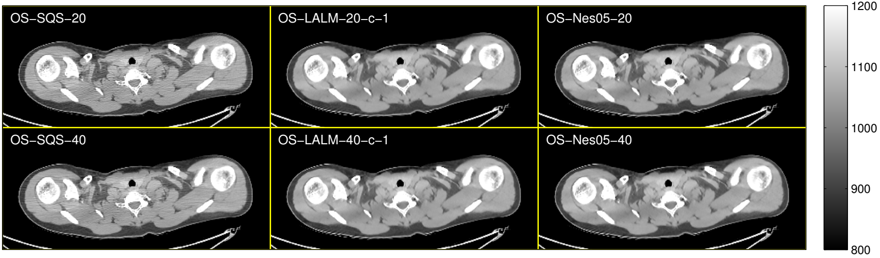

Figure: Shoulder: reconstructed images using different OS-based algorithms after 30 iterations.

Shoulder region helical scan

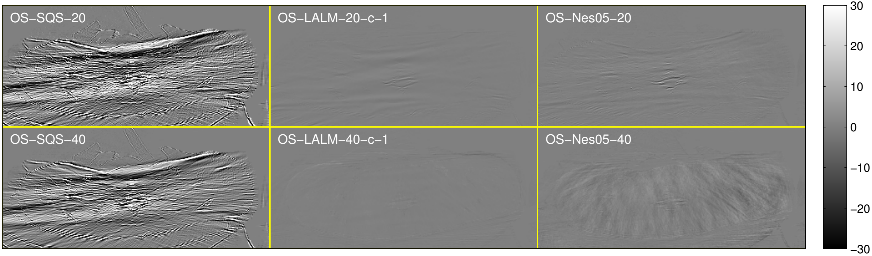

Figure: Shoulder: difference images of the reconstructed images using different OS-based algorithms after 30 iterations.

GE performance phantom axial scan

Specification

- Image size: $1024 \times 1024 \times 90$
- Sinogram size: $888 \times 64 \times 984$ (less view redundancy, cf. helical scan)

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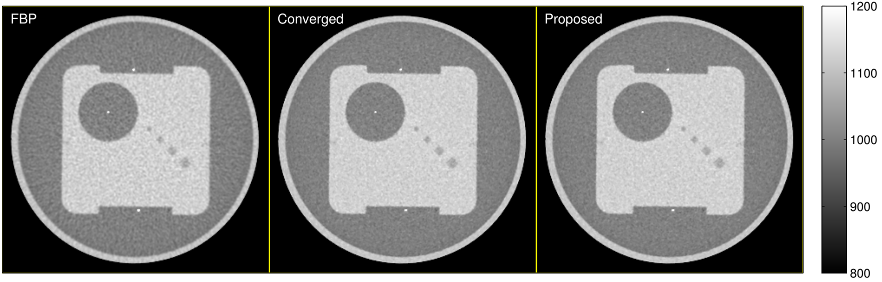


Figure: GEPP: the initial FBP image (left), the reference reconstruction (middle), and the reconstructed image using OS-LALM after 30 iterations (right).

GE performance phantom axial scan

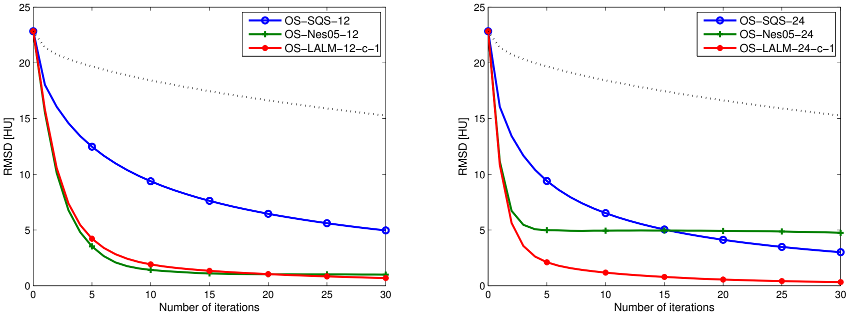
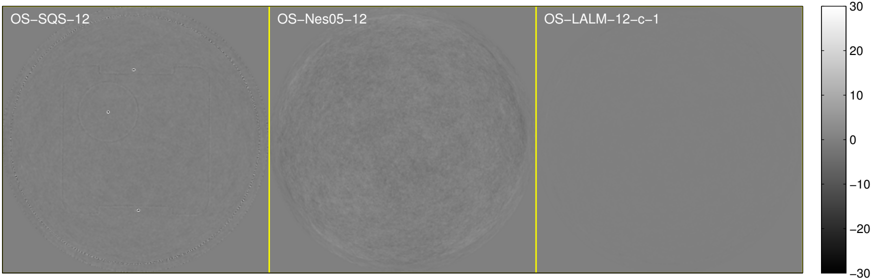


Figure: GEPP: RMS differences as a function of iteration using different OS-based algorithms with 12 subsets (left) and 24 subsets (right).

GE performance phantom axial scan



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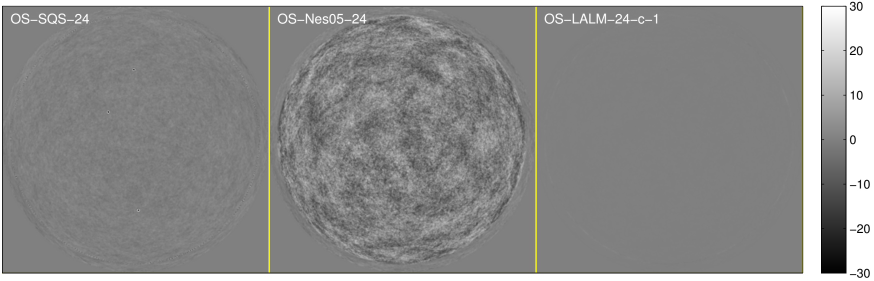

Figure: GEPP: difference images of the reconstructed images using different OS-based algorithms with 24 subsets after 30 iterations.

Outline

Chest region half scan

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- Image size: $718 \times 718 \times 122$
- Sinogram size: $888 \times 64 \times 81$ (about 12.6% views are used)

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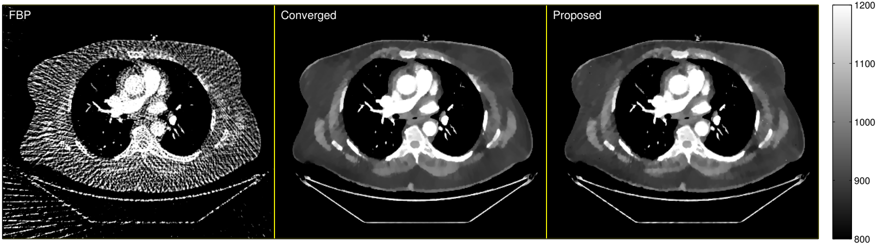

Figure: Chest: the initial FBP image (left), the reference reconstruction (middle), and the reconstructed image using OS-LALM after 100 iterations (right).

Chest region half scan

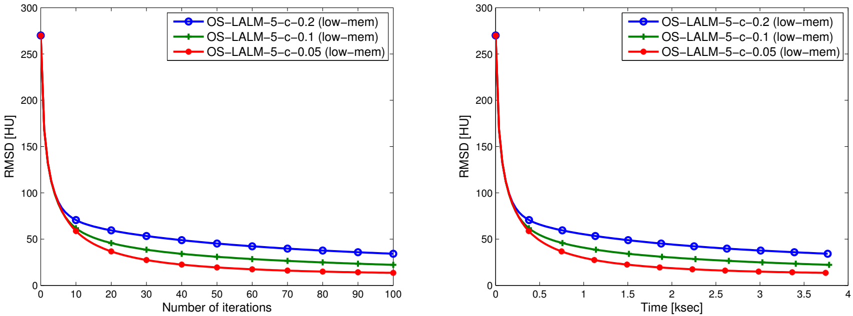

Figure: Chest: RMS differences as a function of iteration (left) and time (right) using OS-LALM with $M = 5$ and different values of η.

Chest region half scan

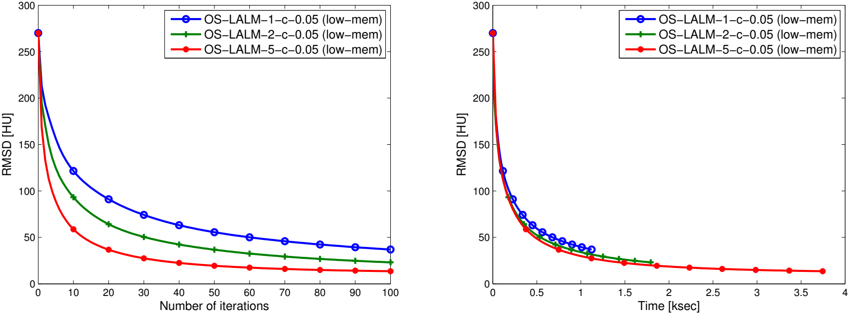

Figure: Chest: RMS differences as a function of iteration (left) and time (right) using OS-LALM with different values of M and $\eta = 0.05$.

Chest region half scan

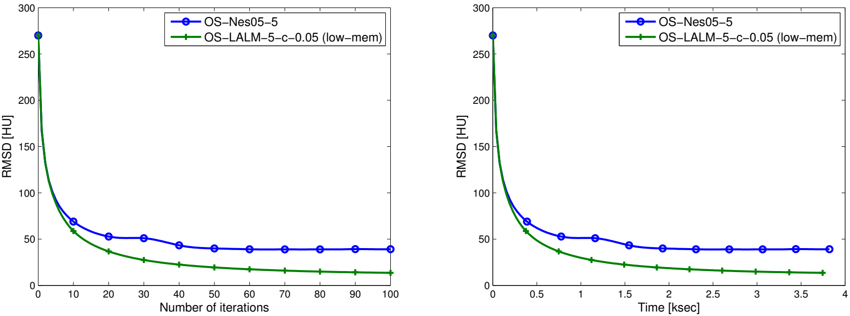

Figure: Chest: RMS differences as a function of iteration (left) and time (right) using different OS-based algorithms with $M = 5$.

Chest region half scan

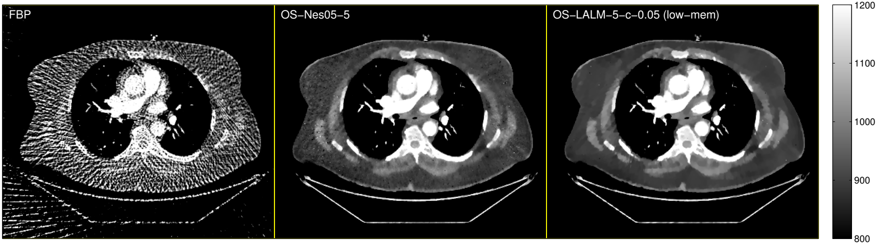

Figure: Chest: reconstructed images using different OS-based algorithms with $M = 5$ after 100 iterations.

Outline

Conclusion

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We proposed ...

- A splitting-based OS algorithm, OS-LALM, for solving PWLS X-ray CT image reconstruction problems
- A deterministic downward continuation approach for accelerating the proposed algorithm
- A low-memory variant of the proposed algorithm when considering additional variable splits

Conclusion

We proposed ...

- A splitting-based OS algorithm, OS-LALM, for solving PWLS X-ray CT image reconstruction problems
- A deterministic downward continuation approach for accelerating the proposed algorithm
- A low-memory variant of the proposed algorithm when considering additional variable splits

Experimental results showed that ...

- The proposed algorithm significantly accelerates the convergence of X-ray CT image reconstruction with negligible overhead
- The proposed algorithm is stable when using many subsets for OS acceleration

Future work

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Theory

- Convergence analysis of OS-LALM when $M > 1$
- Convergence analysis of OS-LALM with downward continuation
- Optimal downward continuation and restart condition
- Convergence analysis of low-mem OS-LALM
- Parameter selection for low-mem OS-LALM

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Extension

- Non-quadratic data-fitting term
- Low-mem OS-LALM with non-smooth potential functions

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THANK YOU!

any question?

Second-order recursive system analysis

Minimize a quadratic function

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- LALM iterates (with a fixed ρ):

$$\begin{cases} \mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - (1/L)\left(\mathbf{V}\boldsymbol{\Lambda}\mathbf{V}'\mathbf{x}^{(k)} + \left(\rho^{-1} - 1\right)\mathbf{g}^{(k)}\right) \\ \mathbf{g}^{(k+1)} = \frac{\rho}{\rho+1}\mathbf{V}\boldsymbol{\Lambda}\mathbf{V}'\mathbf{x}^{(k+1)} + \frac{1}{\rho+1}\mathbf{g}^{(k)} \end{cases}$$

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- The diagonalized system $\bar{\mathbf{x}} \triangleq \mathbf{V}'\mathbf{x}$ and $\bar{\mathbf{g}} \triangleq \mathbf{V}'\mathbf{g}$ satisfies a 2nd-order recursive system determined by the characteristic polynomial:

$$(1+\rho)\, r^2 - 2\left(1 - \lambda_i/L + \rho/2\right) r + \left(1 - \lambda_i/L\right)$$

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- For $\lambda_i < L/2$, the critically damped system converges fastest

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System behaviors based on the value of ρ (cont'd)

To attain the fastest asymp. convergence rate, we would like to choose

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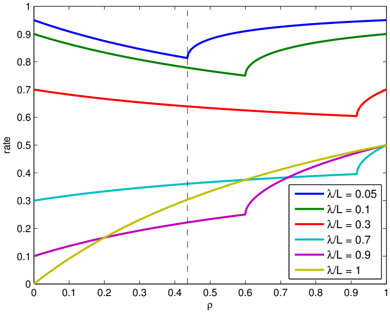

Figure: Asymptotic convergence rate of a system with 6 distinct eigenvalues.

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- If the algorithm is restarted when $\xi(k) > 0$, we shall observe the next restart signal after a further $(\pi/2)\sqrt{L/\mu}$ iterations
- We can easily design a decreasing sequence ρ_k that reaches $\rho^\star$ every time we restart the algorithm! Back