Fast splitting-based ordered-subsets X-ray CT image reconstruction

Hung Nien and Jeffrey A. Fessler

University of Michigan, Ann Arbor

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Acknowledgements

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- Sinogram data are provided by GE Healthcare

Outline

1. Background
2. Related works
3. Proposed algorithm
4. Experimental results
5. Conclusions and future work

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1 Background

2 Related works

3 Proposed algorithm

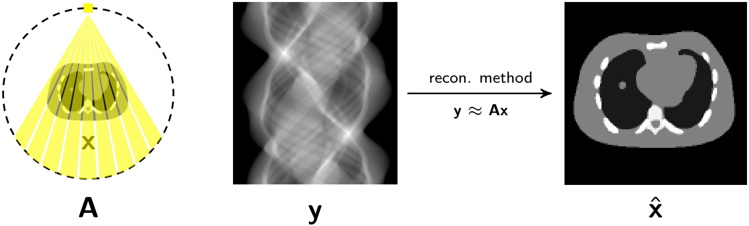
4 Experimental results

5 Conclusions and future work

X-ray CT image reconstruction

Computed tomography

An imaging technique that combines a series of X-ray projections taken from many different angles and computer processing (i.e., reconstruction methods) to create cross-sectional images of the bones and soft tissues inside to the human body.



Radiation dose reduction for X-ray CT

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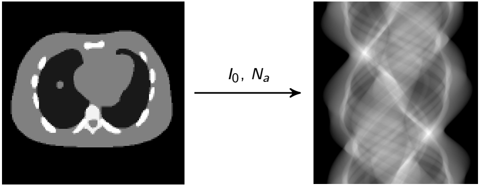
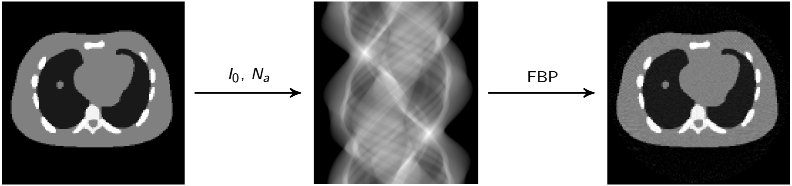


Radiation dose reduction for X-ray CT



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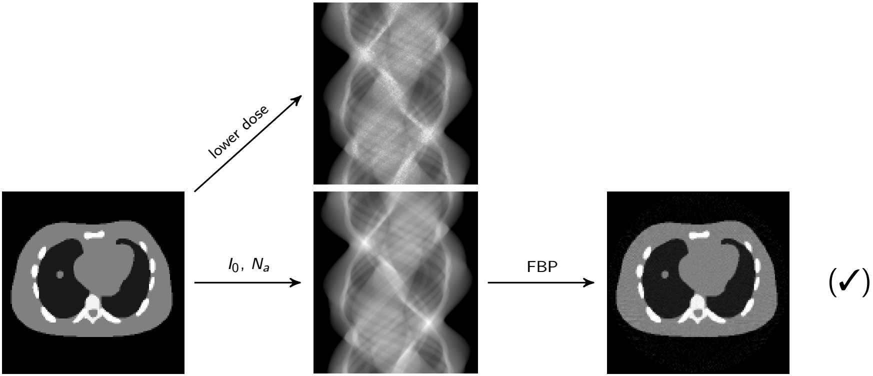
Radiation dose reduction for X-ray CT



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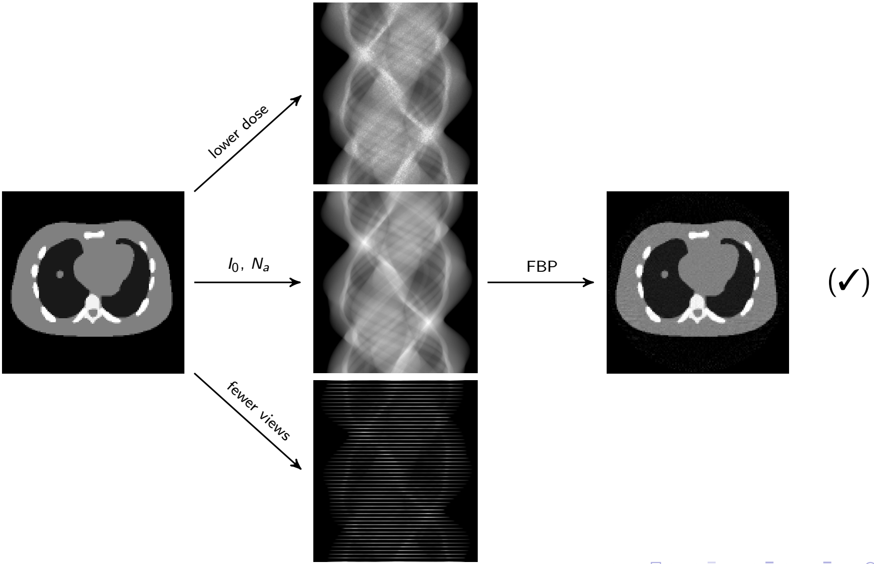

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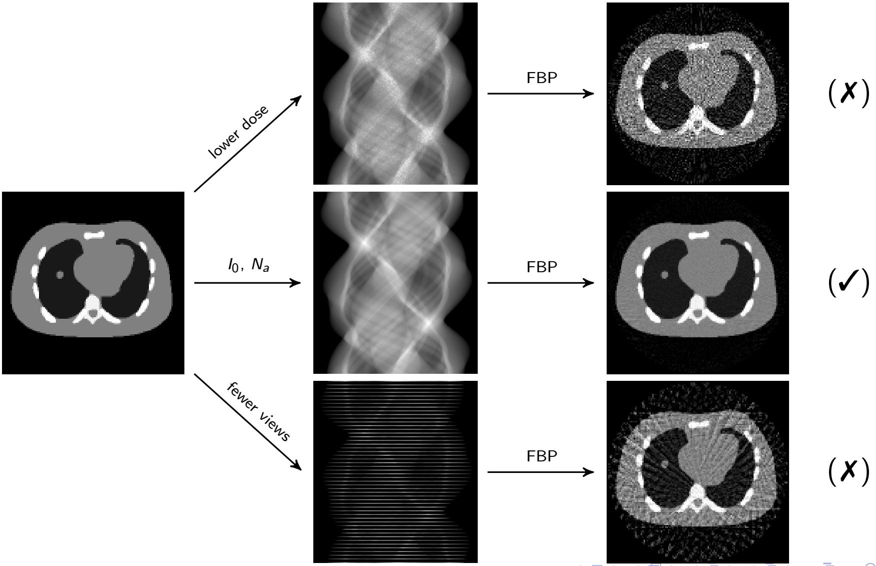

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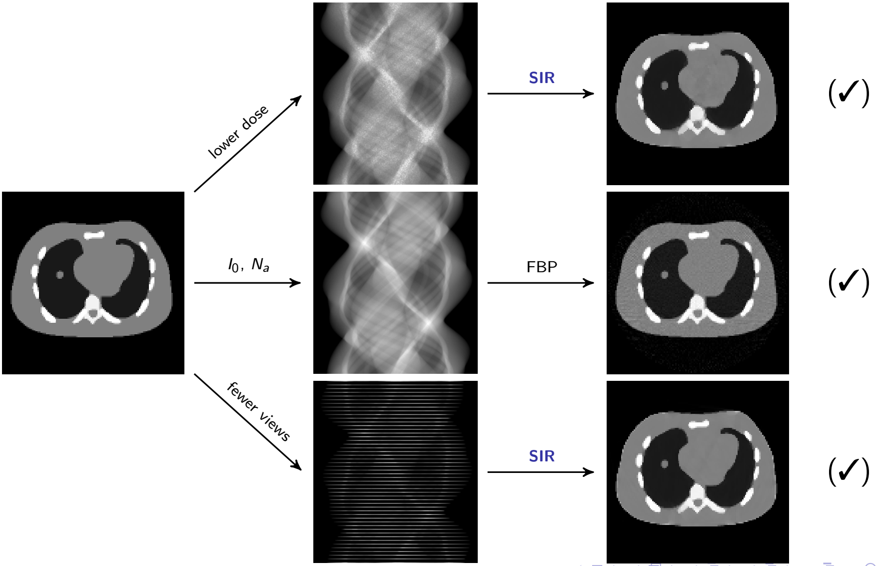

Statistical image reconstruction of X-ray CT

We focus on SIR methods minimizing a PWLS cost function:

$$\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}\in\Omega} \left\{ \frac{1}{2} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_{\mathbf{W}}^2 + \mathsf{R}(\mathbf{x}) \right\} ,$$

where $\mathbf{W}$ denotes the diagonal statistical weighting matrix, $\mathsf{R}(\mathbf{x}) \triangleq \Phi(\mathbf{C}\mathbf{x})$ is an edge-preserving regularizer, and Ω denotes some box constraint (e.g., the non-negativity constraint) on $\mathbf{x}$.
[J-B Thibault *et al.*, Med. Phys., Nov. 2007]

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Our goal

Find fast splitting-based algorithms solving statistical image reconstruction problems (with high-curvature or non-smooth regularizers) in reasonable reconstruction time

Outline

1. Background
2. Related works
3. Proposed algorithm
4. Experimental results
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Alternating direction method of multipliers (ADMM)

One way to accelerate SIR is to use splitting-based methods such as ADMM. Consider an equivalent (unconstrained) SIR formulation:

$$(\hat{\mathbf{x}}, \hat{\mathbf{u}}, \hat{\mathbf{v}}) \in \arg\min_{\mathbf{x},\mathbf{u},\mathbf{v}} \left\{ \frac{1}{2} \|\mathbf{y} - \mathbf{u}\|_{\mathbf{W}}^2 + \Phi(\mathbf{v}) \right\} \text{ s.t. } \mathbf{u} = \mathbf{A}\mathbf{x}, \mathbf{v} = \mathbf{C}\mathbf{x}$$

with the corresponding augmented Lagrangian:

$$\mathcal{L}_{\mathrm{A}}(\mathbf{x}, \mathbf{u}, \mathbf{v}, \mathbf{d}, \mathbf{e}; \rho, \eta) \triangleq \frac{1}{2} \|\mathbf{y} - \mathbf{u}\|_{\mathbf{W}}^2 + \Phi(\mathbf{v}) + \frac{\rho}{2} \|\mathbf{A}\mathbf{x} - \mathbf{u} - \mathbf{d}\|_2^2 + \frac{\eta}{2} \|\mathbf{C}\mathbf{x} - \mathbf{v} - \mathbf{e}\|_2^2 .$$

[M V Afonso *et al.*, IEEE T-IP, Mar. 2011]
[S Ramani & J A Fessler, IEEE T-MI, Mar. 2012]

Alternating direction method of multipliers (ADMM)

The ADMM iteration is:

1. $\mathbf{x}^{+} = \arg\min_{\mathbf{x}} \left\{ \frac{\rho}{2} \|\mathbf{A}\mathbf{x} - \mathbf{u} - \mathbf{d}\|_2^2 + \frac{\eta}{2} \|\mathbf{C}\mathbf{x} - \mathbf{v} - \mathbf{e}\|_2^2 \right\}$
2. $\mathbf{u}^{+} = \arg\min_{\mathbf{u}} \left\{ \frac{1}{2} \|\mathbf{y} - \mathbf{u}\|_{\mathbf{W}}^2 + \frac{\rho}{2} \|\mathbf{A}\mathbf{x}^{+} - \mathbf{u} - \mathbf{d}\|_2^2 \right\}$
3. $\mathbf{v}^{+} = \arg\min_{\mathbf{v}} \left\{ \Phi(\mathbf{v}) + \frac{\eta}{2} \|\mathbf{C}\mathbf{x}^{+} - \mathbf{v} - \mathbf{e}\|_2^2 \right\}$
4. $\mathbf{d}^{+} = \mathbf{d} - \mathbf{A}\mathbf{x}^{+} + \mathbf{u}^{+}$
5. $\mathbf{e}^{+} = \mathbf{e} - \mathbf{C}\mathbf{x}^{+} + \mathbf{v}^{+}$

[S Ramani & J A Fessler, IEEE T-MI, Mar. 2012]

Alternating direction method of multipliers (ADMM)

The ADMM iteration is:

1. $\mathbf{x}^{+} = \left(\rho\mathbf{A}'\mathbf{A} + \eta\mathbf{C}'\mathbf{C}\right)^{-1}\left(\rho\,\mathbf{A}'\left(\mathbf{u}+\mathbf{d}\right) + \eta\,\mathbf{C}'\left(\mathbf{v}+\mathbf{e}\right)\right)$
2. $\mathbf{u}^{+} = \arg\min_{\mathbf{u}} \left\{\frac{1}{2}\|\mathbf{y}-\mathbf{u}\|_{\mathbf{W}}^{2} + \frac{\rho}{2}\|\mathbf{A}\mathbf{x}^{+}-\mathbf{u}-\mathbf{d}\|_{2}^{2}\right\}$
3. $\mathbf{v}^{+} = \arg\min_{\mathbf{v}} \left\{\Phi(\mathbf{v}) + \frac{\eta}{2}\|\mathbf{C}\mathbf{x}^{+}-\mathbf{v}-\mathbf{e}\|_{2}^{2}\right\}$
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The ADMM iteration is:

1. $\mathbf{x}^{+} = \left(\rho \mathbf{A}'\mathbf{A} + \eta \mathbf{C}'\mathbf{C}\right)^{-1} \left(\rho \, \mathbf{A}' \left(\mathbf{u} + \mathbf{d}\right) + \eta \, \mathbf{C}' \left(\mathbf{v} + \mathbf{e}\right)\right)$
2. $\mathbf{u}^{+} = \left(\mathbf{W} + \rho \mathbf{I}\right)^{-1} \left(\mathbf{W}\mathbf{y} + \rho \left(\mathbf{A}\mathbf{x}^{+} - \mathbf{d}\right)\right)$
3. $\mathbf{v}^{+} = \arg\min_{\mathbf{v}} \left\{ \Phi(\mathbf{v}) + \frac{\eta}{2} \left\| \mathbf{C}\mathbf{x}^{+} - \mathbf{v} - \mathbf{e} \right\|_2^2 \right\}$
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2. $\mathbf{u}^{+} = (\mathbf{W} + \rho\mathbf{I})^{-1}(\mathbf{W}\mathbf{y} + \rho(\mathbf{A}\mathbf{x}^{+} - \mathbf{d}))$
3. $\mathbf{v}^{+} = \mathrm{shrink}_{\eta^{-1}\Phi}(\mathbf{C}\mathbf{x}^{+} - \mathbf{e})$
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2. $\mathbf{u}^{+} = (\mathbf{W} + \rho \mathbf{I})^{-1} (\mathbf{W}\mathbf{y} + \rho \, (\mathbf{A}\mathbf{x}^{+} - \mathbf{d}))$
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[S Ramani & J A Fessler, IEEE T-MI, Mar. 2012]

By introducing the auxiliary variable **u**, the image update is uncorrelated with the statistical weighting **W**, so it can be solved efficiently using PCG with an appropriate circulant preconditioner in 2D (but not 3D) CT.

Linearized AL method (LALM)

To solve the problem of iterative inner updates (with $\mathbf{A}$ and $\mathbf{A}'$), we use a technique called linearization, thus leading to a linearized AL method. Consider an alternative equivalent SIR formulation:

$$(\hat{\mathbf{x}}, \hat{\mathbf{u}}) \in \arg\min_{\mathbf{x}\in\Omega,\mathbf{u}} \left\{ \frac{1}{2} \|\mathbf{y} - \mathbf{u}\|_{\mathbf{W}}^2 + \mathsf{R}(\mathbf{x}) \right\} \text{ s.t. } \mathbf{u} = \mathbf{A}\mathbf{x}$$

with the corresponding augmented Lagrangian:

$$\mathcal{L}_{\mathsf{A}}(\mathbf{x}, \mathbf{u}, \mathbf{d}; \rho) \triangleq \frac{1}{2} \|\mathbf{y} - \mathbf{u}\|_{\mathbf{W}}^2 + \mathsf{R}(\mathbf{x}) + \iota_{\Omega}(\mathbf{x}) + \frac{\rho}{2} \|\mathbf{A}\mathbf{x} - \mathbf{u} - \mathbf{d}\|_{\mathbf{W}}^2 \ .$$

[HN & J A Fessler, Fully 3D, 2013]
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$$(\hat{\mathbf{x}}, \hat{\mathbf{u}}) \in \arg\min_{\mathbf{x}\in\Omega,\mathbf{u}} \left\{ \frac{1}{2} \|\mathbf{y} - \mathbf{u}\|_{\mathbf{W}}^2 + \mathrm{R}(\mathbf{x}) \right\} \text{ s.t. } \mathbf{u} = \mathbf{A}\mathbf{x}$$

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[HN & J A Fessler, Fully 3D, 2013]
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Linearized AL method (LALM)

The LALM iteration is:

1. $\mathbf{s}^{+} = \rho \nabla \ell(\mathbf{x}) + (1-\rho)\,\mathbf{g}$
2. $\mathbf{x}^{+} = \arg\min_{\mathbf{z}\in\Omega} \left\{ \frac{1}{2} \left\| \mathbf{z} - \left(\mathbf{x} - \left(\rho \mathbf{D}_{\mathrm{L}}\right)^{-1} \mathbf{s}^{+}\right) \right\|_{\rho \mathbf{D}_{\mathrm{L}}}^{2} + \mathsf{R}(\mathbf{z}) \right\}$
3. $\mathbf{g}^{+} = \frac{\rho}{\rho+1} \nabla \ell(\mathbf{x}^{+}) + \frac{1}{\rho+1} \mathbf{g}$

[HN & J A Fessler, SPIE MI, 2014]

Linearized AL method (LALM)

The inexact LALM iteration ($n = 1$) for smooth regularization is:

1. $\mathbf{s}^+ = \rho \nabla \ell(\mathbf{x}) + (1-\rho)\,\mathbf{g}$
2. $\mathbf{x}^+ = \left[\mathbf{x} - (\rho \mathbf{D}_{\mathrm{L}} + \mathbf{D}_{\mathrm{R}})^{-1} (\mathbf{s}^+ + \nabla \mathrm{R}(\mathbf{x}))\right]_{\Omega}$
3. $\mathbf{g}^+ = \frac{\rho}{\rho+1} \nabla \ell(\mathbf{x}^+) + \frac{1}{\rho+1}\mathbf{g}$

[HN & J A Fessler, SPIE MI, 2014]

Linearized AL method (LALM)

The LALM iteration is:

1. $\mathbf{s}^+ = \rho \nabla \ell(\mathbf{x}) + (1-\rho)\,\mathbf{g}$
2. $\mathbf{x}^+ = \arg\min_{\mathbf{z}\in\Omega} \left\{ \frac{1}{2} \left\| \mathbf{z} - \left(\mathbf{x} - \left(\rho \mathbf{D}_{\mathrm{L}}\right)^{-1} \mathbf{s}^+\right) \right\|_{\rho \mathbf{D}_{\mathrm{L}}}^2 + \mathrm{R}(\mathbf{z}) \right\}$
3. $\mathbf{g}^+ = \frac{\rho}{\rho+1} \nabla \ell(\mathbf{x}^+) + \frac{1}{\rho+1} \mathbf{g}$

[HN & J A Fessler, SPIE MI, 2014]

Linearized AL method (LALM)

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3. $\mathbf{g}^{+} = \frac{\rho}{\rho+1} \nabla \ell(\mathbf{x}^{+}) + \frac{1}{\rho+1} \mathbf{g}$

[HN & J A Fessler, SPIE MI, 2014]

Linearized AL method with OS acceleration (OS-LALM)

The OS-LALM iteration is:

1. $\mathbf{s}^{+} = \rho M \nabla \ell_m(\mathbf{x}) + (1-\rho)\,\mathbf{g}$
2. $\mathbf{x}^{+} = \arg\min_{\mathbf{z}\in\Omega} \left\{ \frac{1}{2} \left\| \mathbf{z} - \left(\mathbf{x} - \left(\rho \mathbf{D}_{\mathsf{L}}\right)^{-1} \mathbf{s}^{+}\right) \right\|_{\rho \mathbf{D}_{\mathsf{L}}}^{2} + \mathsf{R}(\mathbf{z}) \right\}$
3. $\mathbf{g}^{+} = \frac{\rho}{\rho+1} M \nabla \ell_{m^{+}=m+1}(\mathbf{x}^{+}) + \frac{1}{\rho+1}\mathbf{g}$

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[HN & J A Fessler, SPIE MI, 2014]

This method works well in typical CT scans in which the regularizer is smooth, and the loss function dominates the cost function. However, when the regularizer is non-smooth, we inevitably have to solve image updates iteratively (without $\mathbf{A}$ and $\mathbf{A}'$)!

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OS-LALM with additional variable splits

We propose to solve the equivalent SIR formulation:

$$(\hat{\mathbf{x}}, \hat{\mathbf{u}}, \hat{\mathbf{v}}) \in \arg \min_{\mathbf{x}\in\Omega,\mathbf{u},\mathbf{v}} \left\{ \frac{1}{2} \|\mathbf{y} - \mathbf{u}\|_{\mathbf{W}}^2 + \Phi(\mathbf{v}) \right\} \text{ s.t. } \mathbf{u} = \mathbf{A}\mathbf{x}, \mathbf{v} = \mathbf{C}\mathbf{x}$$

using OS-LALM with the corresponding augmented Lagrangian:

$$\mathcal{L}_{\mathsf{A}}(\mathbf{x}, \mathbf{u}, \mathbf{v}, \mathbf{d}, \mathbf{e}; \rho, \eta) \triangleq$$
$$\frac{1}{2} \|\mathbf{y} - \mathbf{u}\|_{\mathbf{W}}^2 + \Phi(\mathbf{v}) + \iota_{\Omega}(\mathbf{x}) + \frac{\rho}{2} \|\mathbf{A}\mathbf{x} - \mathbf{u} - \mathbf{d}\|_{\mathbf{W}}^2 + \frac{\eta}{2} \|\mathbf{C}\mathbf{x} - \mathbf{v} - \mathbf{e}\|_2^2 .$$

OS-LALM with additional variable splits

The proposed split OS-LALM iteration is:

1. $\mathbf{s}^{+} = \rho M \nabla \ell_m(\mathbf{x}) + (1-\rho)\,\mathbf{g}$
2. $\boldsymbol{\sigma}^{+} = \eta \mathbf{C}' \left(\mathbf{C}\mathbf{x} - \mathbf{v} - \mathbf{e}\right)$
3. $\mathbf{x}^{+} = \left[\mathbf{x} - \left(\rho \mathbf{D}_{\mathrm{L}} + \eta \mathbf{D}_{\mathrm{P}}\right)^{-1} \left(\mathbf{s}^{+} + \boldsymbol{\sigma}^{+}\right)\right]_{\Omega}$
4. $\mathbf{g}^{+} = \frac{\rho}{\rho+1} M \nabla \ell_{m^{+}=m+1}(\mathbf{x}^{+}) + \frac{1}{\rho+1}\mathbf{g}$
5. $\mathbf{v}^{+} = \mathrm{shrink}_{\eta^{-1}\Phi}(\mathbf{C}\mathbf{x}^{+} - \mathbf{e})$
6. $\mathbf{e}^{+} = \mathbf{e} - \mathbf{C}\mathbf{x}^{+} + \mathbf{v}^{+}$

where $\mathbf{D}_{\mathrm{P}} \triangleq \mathrm{diag}\left\{|\mathbf{C}|' \, |\mathbf{C}| \, \mathbf{1}\right\}$

OS-LALM with additional variable splits

The proposed split OS-LALM iteration is:

1. $\mathbf{s}^+ = \rho M \nabla \ell_m(\mathbf{x}) + (1-\rho)\,\mathbf{g}$ [search direction due to ℓ]
2. $\boldsymbol{\sigma}^+ = \eta \mathbf{C}' \left(\mathbf{C}\mathbf{x} - \mathbf{v} - \mathbf{e}\right)$
3. $\mathbf{x}^+ = \left[\mathbf{x} - \left(\rho \mathbf{D}_{\mathrm{L}} + \eta \mathbf{D}_{\mathrm{P}}\right)^{-1} \left(\mathbf{s}^+ + \boldsymbol{\sigma}^+\right)\right]_{\Omega}$
4. $\mathbf{g}^+ = \frac{\rho}{\rho+1} M \nabla \ell_{m^+ = m+1}(\mathbf{x}^+) + \frac{1}{\rho+1}\mathbf{g}$
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3. $\mathbf{x}^{+} = \left[\mathbf{x} - \left(\rho \mathbf{D}_{\mathrm{L}} + \eta \mathbf{D}_{\mathrm{P}}\right)^{-1} \left(\mathbf{s}^{+} + \boldsymbol{\sigma}^{+}\right)\right]_{\Omega}$
4. $\mathbf{g}^{+} = \frac{\rho}{\rho+1} M \nabla \ell_{m^{+}=m+1}(\mathbf{x}^{+}) + \frac{1}{\rho+1}\mathbf{g}$
5. $\mathbf{v}^{+} = \mathrm{shrink}_{\eta^{-1}\Phi}(\mathbf{C}\mathbf{x}^{+} - \mathbf{e})$
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3. $\mathbf{x}^+ = \left[\mathbf{x} - \left(\rho \mathbf{D}_{\mathrm{L}} + \eta \mathbf{D}_{\mathrm{P}}\right)^{-1} \left(\mathbf{s}^+ + \boldsymbol{\sigma}^+\right)\right]_{\Omega}$ [ρ, η-adjustable step sizes]
4. $\mathbf{g}^+ = \frac{\rho}{\rho+1} M \nabla \ell_{m^+ = m+1}(\mathbf{x}^+) + \frac{1}{\rho+1}\mathbf{g}$
5. $\mathbf{v}^+ = \mathrm{shrink}_{\eta^{-1}\Phi}(\mathbf{C}\mathbf{x}^+ - \mathbf{e})$
6. $\mathbf{e}^+ = \mathbf{e} - \mathbf{C}\mathbf{x}^+ + \mathbf{v}^+$

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1. $\mathbf{s}^+ = \rho M \nabla \ell_m(\mathbf{x}) + (1-\rho)\,\mathbf{g}$ [search direction due to ℓ]
2. $\boldsymbol{\sigma}^+ = \eta \mathbf{C}' \left(\mathbf{Cx} - \mathbf{v} - \mathbf{e}\right)$ [search direction due to R]
3. $\mathbf{x}^+ = \left[\mathbf{x} - \left(\rho \mathbf{D}_\mathsf{L} + \eta \mathbf{D}_\mathsf{P}\right)^{-1} \left(\mathbf{s}^+ + \boldsymbol{\sigma}^+\right)\right]_\Omega$ [ρ, η-adjustable step sizes]
4. $\mathbf{g}^+ = \frac{\rho}{\rho+1} M \nabla \ell_{m^+=m+1}(\mathbf{x}^+) + \frac{1}{\rho+1}\mathbf{g}$ [split gradient update]
5. $\mathbf{v}^+ = \mathrm{shrink}_{\eta^{-1}\Phi}(\mathbf{Cx}^+ - \mathbf{e})$
6. $\mathbf{e}^+ = \mathbf{e} - \mathbf{Cx}^+ + \mathbf{v}^+$

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5. $\mathbf{v}^{+} = \mathrm{shrink}_{\eta^{-1}\Phi}(\mathbf{C}\mathbf{x}^{+} - \mathbf{e})$ [split variable shrinkage]
6. $\mathbf{e}^{+} = \mathbf{e} - \mathbf{C}\mathbf{x}^{+} + \mathbf{v}^{+}$

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3. $\mathbf{x}^+ = \left[\mathbf{x} - \left(\rho \mathbf{D}_{\mathrm{L}} + \eta \mathbf{D}_{\mathrm{P}}\right)^{-1} \left(\mathbf{s}^+ + \boldsymbol{\sigma}^+\right)\right]_{\Omega}$ [ρ, η-adjustable step sizes]
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5. $\mathbf{v}^+ = \mathrm{shrink}_{\eta^{-1}\Phi}(\mathbf{C}\mathbf{x}^+ - \mathbf{e})$ [split variable shrinkage]
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where $\mathbf{D}_{\mathrm{P}} \triangleq \mathrm{diag}\left\{|\mathbf{C}|' \, |\mathbf{C}| \, \mathbf{1}\right\}$

OS-LALM with additional variable splits

The proposed split OS-LALM iteration is:

1. $\mathbf{s}^{+} = \rho M \nabla \ell_m(\mathbf{x}) + (1-\rho)\,\mathbf{g}$ [search direction due to ℓ]
2. $\boldsymbol{\sigma}^{+} = \eta \mathbf{C}' (\mathbf{Cx} - \mathbf{v} - \mathbf{e})$ [search direction due to R]
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All updates can be solved exactly and non-iteratively. The most expensive operations are the forward/back-projection and (perhaps) shrinkage.

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2. Related works
3. Proposed algorithm
4. Experimental results
5. Conclusions and future work

Chest region axial scan (sparse-view)

Specification

- Image size: $512 \times 512 \times 122$
- Sinogram size: $888 \times 64 \times 81$ (about 12.6% views are used)
- Non-smooth anisotropic TV regularization with 3 and 13 directions

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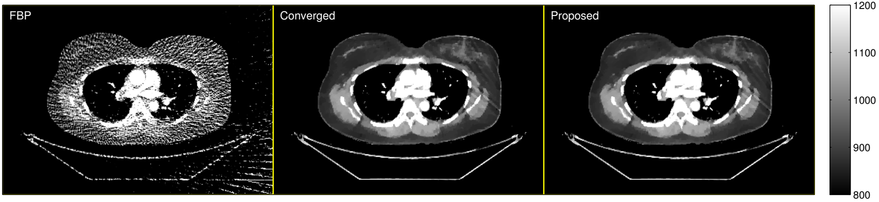

Figure: Chest [3 nbrs]: the initial FBP image (left), the reference reconstruction (middle), and the reconstructed image using split OS-LALM (right).

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Effect of continuation and OS acceleration

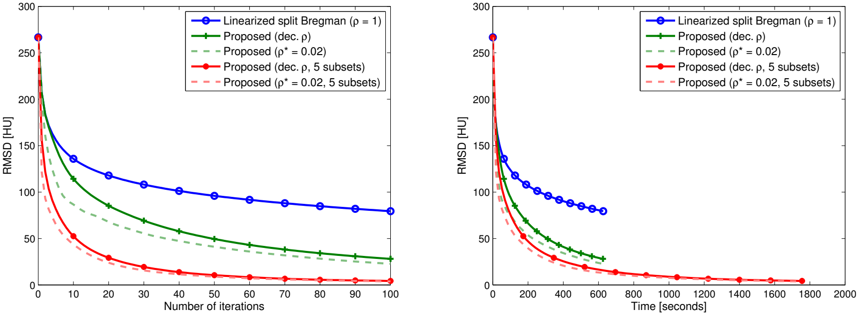


Figure: Chest [3 nbrs]: RMS differences as a function of iteration (left) and time (right) using split OS-LALM with different continuation scheme and number of subsets ($\eta = 433.33$).

Effect of large split variables

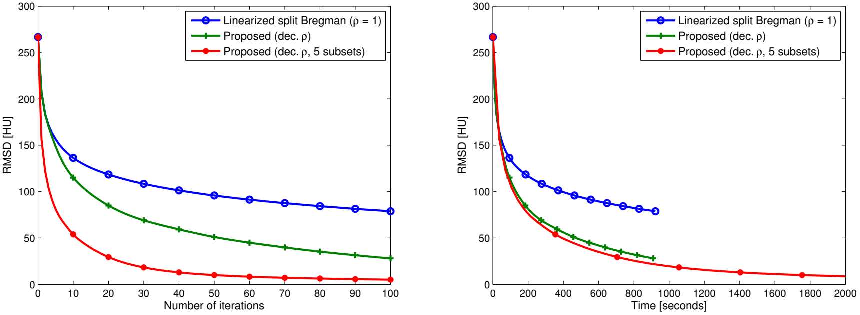


Figure: Chest [13 nbrs]: RMS differences as a function of iteration (left) and time (right) using split OS-LALM with different continuation scheme and number of subsets ($\eta = 100$).

Why constrained?

Figure: Chest [3 nbrs]: the initial FBP image (left), the reference reconstruction w/ constraint (middle), and the reference reconstruction w/o constraint (right).

Constrained vs. unconstrained

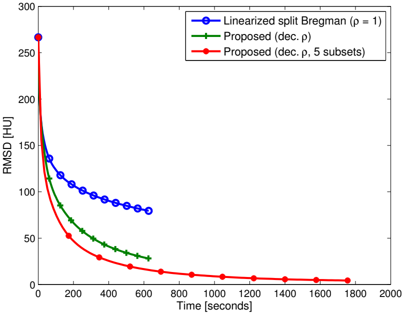
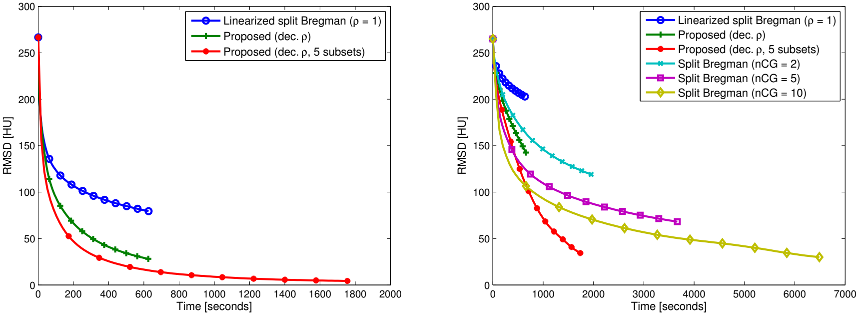


Figure: Chest [3 nbrs]: RMS differences as a function of time when solving the constrained (left) and unconstrained (right) formulations using split OS-LALM ($\eta_{\text{con}} = 433.33$ and $\eta_{\text{uncon}} = 1000$).

Constrained vs. unconstrained



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Last but not least

Summary of results:

- Split OS-LALM is a splitting-based OS algorithm for solving PWLS problems with general composite convex regularizers
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Summary of results:

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List of future work:

- Penalty parameter selection of η
- OS-LALM + GPU-based GCD denoising [M G McGaffin & J A Fessler, SPIE EI, 2014]
- Low-memory variant for the "lots-of-split" scenario

THANK YOU!

any question?

Deterministic downward continuation

Based on a second-order recursive system analysis, we proposed to decrease ρ as:

$$\rho_k = \begin{cases} 1 & \text{, if } k = 0 \\ \frac{\pi}{k+1}\sqrt{1 - \left(\frac{\pi}{2k+2}\right)^2} & \text{, otherwise}. \end{cases}$$

This sequence decreases a little bit faster than $1/k$ and exhibits remarkable acceleration in low-dose X-ray CT image reconstruction. [HN & J A Fessler, SPIE MI, 2014]