Splitting-Based Statistical X-Ray CT Image Reconstruction with **Blind Gain Correction**

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*Supported in part by NIH grant R01-HL-098686 and by an equipment donation from Intel.

Outline

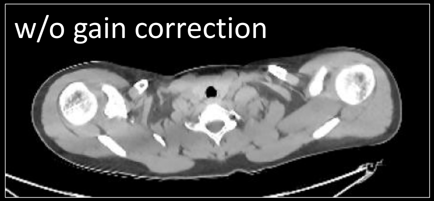
I. Description of purpose

$$(\hat{\mathbf{x}}, \hat{\mathbf{g}}) \in \underset{\mathbf{x},\mathbf{g}}{\operatorname{argmin}} \left\{ \Psi(\mathbf{x}, \mathbf{g}) \triangleq \frac{1}{2} \|\mathbf{y} - \mathbf{A}\mathbf{x} - \mathbf{g} \otimes \mathbf{1}\|_{\mathbf{W}}^2 + \mathsf{R}(\mathbf{x}) \right\}$$

II. Method

$$\hat{\mathbf{x}} \in \underset{\mathbf{x}}{\operatorname{argmin}} \left\{ \Psi(\mathbf{x}) \triangleq \frac{1}{2} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_{\tilde{\mathbf{W}}}^2 + \mathsf{R}(\mathbf{x}) \right\}$$

III. Results



IV. Conclusion

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Gain fluctuations in CT scans

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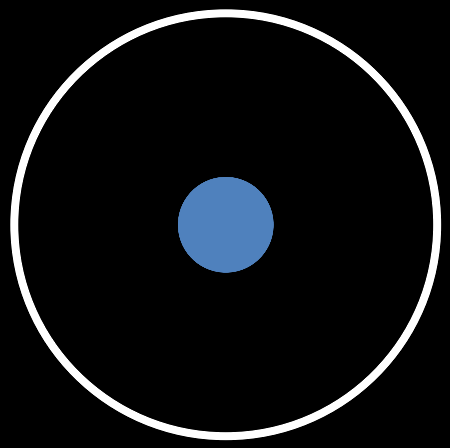
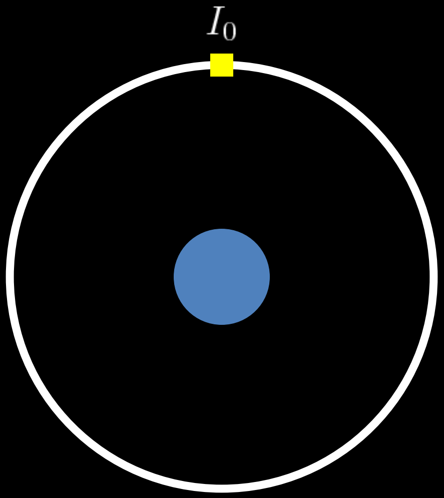


Gain fluctuations in CT scans



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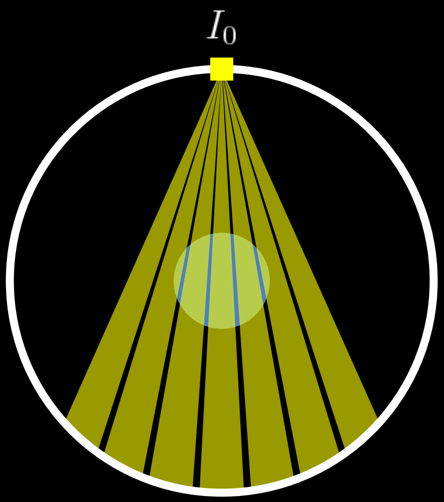
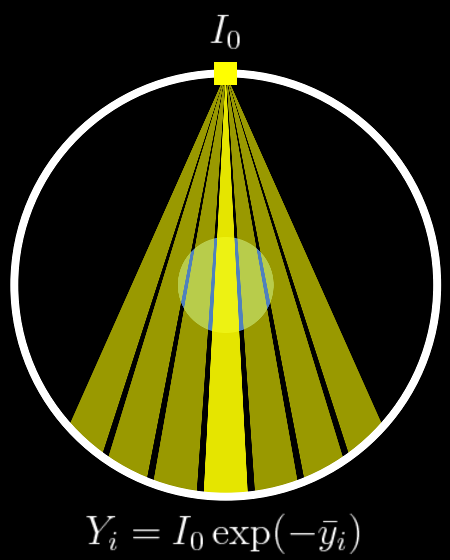
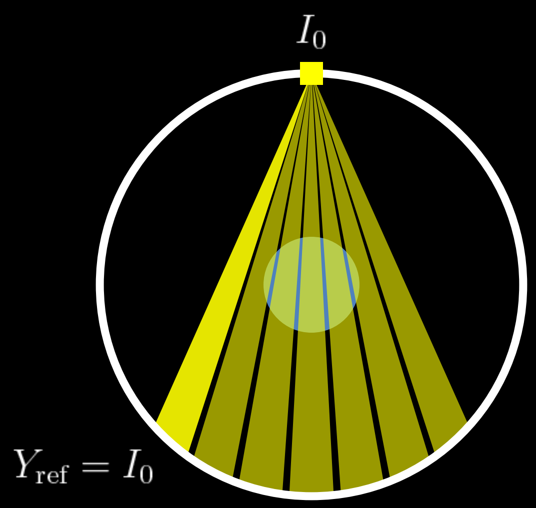


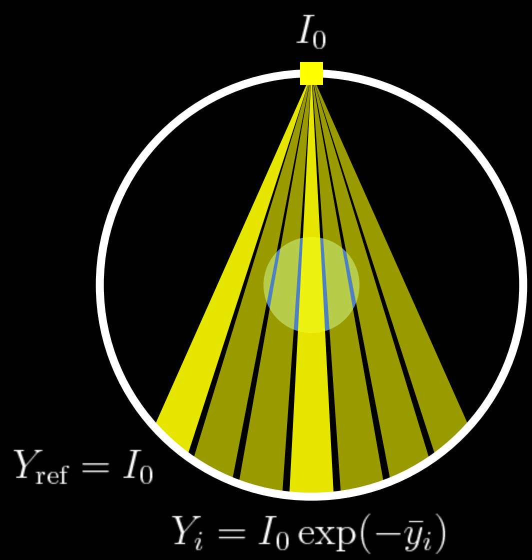
Gain fluctuations in CT scans



Gain fluctuations in CT scans

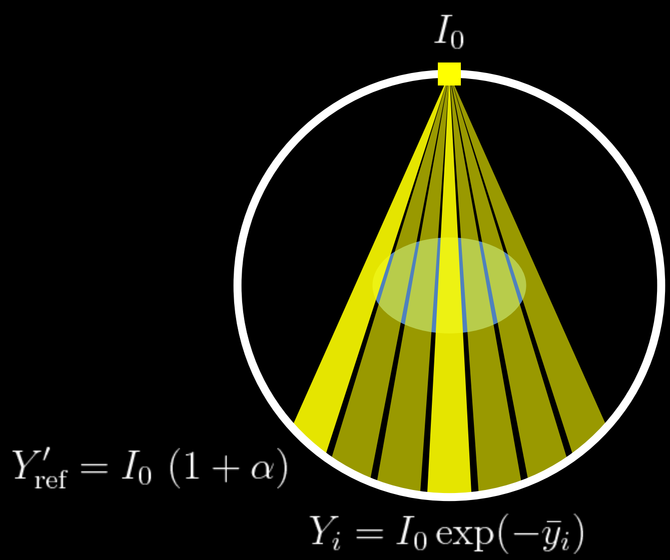


Gain fluctuations in CT scans



$$\bar{y}_i = \ln\left(\frac{Y_{\text{ref}}}{Y_i}\right) = \ln\left(\frac{I_0}{Y_i}\right)$$

Gain fluctuations in CT scans



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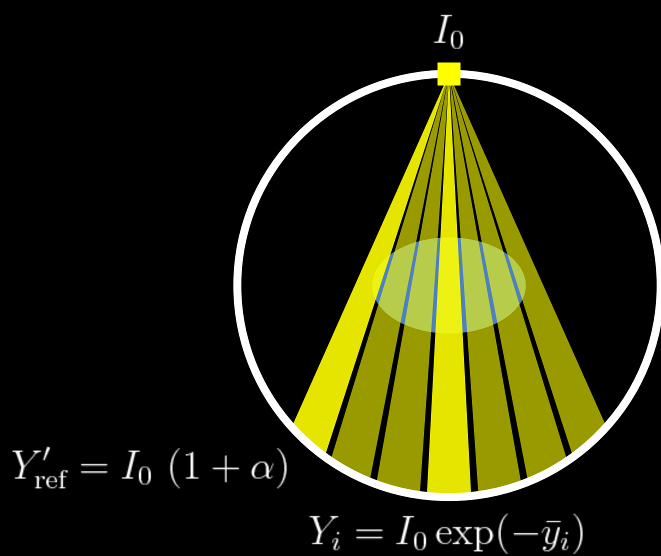

$$y_i = \ln\left(\frac{Y'_{\text{ref}}}{Y_i}\right) = \ln\left(\frac{I_0}{Y_i}\right) + \ln(1+\alpha) = \bar{y}_i + \bar{\alpha}$$

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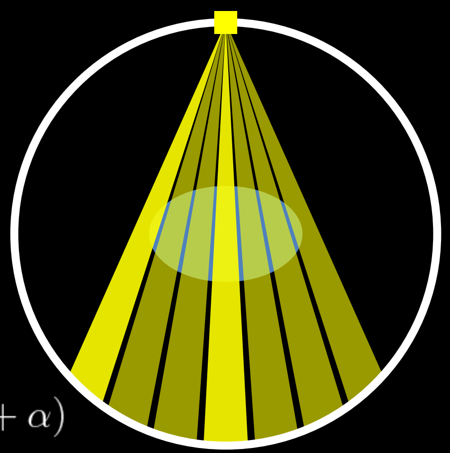

$$y_i = \ln\left(\frac{Y'_{\text{ref}}}{Y_i}\right) = \ln\left(\frac{I_0}{Y_i}\right) + \ln(1+\alpha) = \bar{y}_i + \bar{\alpha}$$

$$\boxed{[\mathbf{Ax}]_i \triangleq \bar{y}_i = y_i - \bar{\alpha}}$$

Correction of gain fluctuation

Correction of gain fluctuation

Without gain correction:

$$\hat{\mathbf{x}} \in \underset{\mathbf{x}}{\operatorname{argmin}} \left\{ \Psi(\mathbf{x}) \triangleq -\ln L(\mathbf{y};\mathbf{x}) + \mathsf{R}(\mathbf{x}) \right\}$$

Correction of gain fluctuation

Without gain correction (statistical PWLS formulation):

$$\hat{\mathbf{x}} \in \underset{\mathbf{x}}{\operatorname{argmin}} \left\{ \Psi(\mathbf{x}) \triangleq \frac{1}{2} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_{\mathbf{W}}^2 + \mathsf{R}(\mathbf{x}) \right\}$$

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Statistical weighting accounts for ***measurement variance***

Correction of gain fluctuation

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With gain correction (statistical PWLS formulation):

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With gain correction (statistical PWLS formulation):

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J-B. Thibault, Z. Yu, K. Sauer, C. Bouman, and J. Hsieh. **Correction of gain fluctuations in iterative tomographic image reconstruction**. In *Proc. Intl. Mtg. on Fully 3D Image Recon. in Rad. and Nuc. Med*, pages 112–115, 2007.

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The ***unknown*** correction term

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GOAL:

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GOAL:

Find an ***equivalent PWLS formulation*** so that we can ***jointly (but implicitly) reconstruct*** **x** and **g** using ***existing methods*** such as **CG**, **ADMM**, and **OS** that guarantee convergence*!

J-B. Thibault, Z. Yu, K. Sauer, C. Bouman, and J. Hsieh. **Correction of gain fluctuations in iterative tomographic image reconstruction**. In *Proc. Intl. Mtg. on Fully 3D Image Recon. in Rad. and Nuc. Med*, pages 112–115, 2007.

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$$\hat{\mathbf{x}} \in \operatorname*{argmin}_{\mathbf{x}} \left\{ \Psi(\mathbf{x}) \triangleq \frac{1}{2} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|^2_{\tilde{\mathbf{W}}} + \mathsf{R}(\mathbf{x}) \right\}$$

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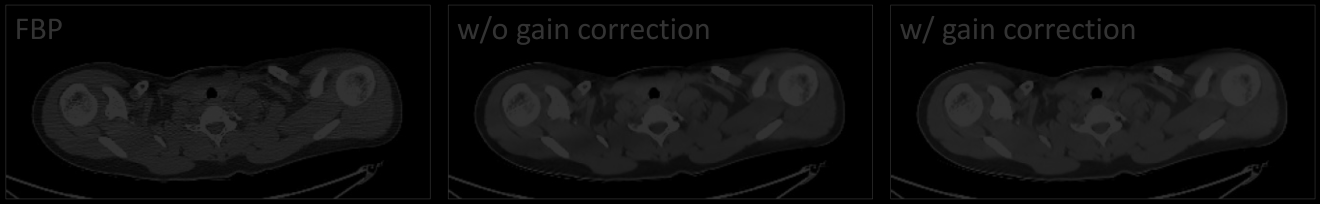

IV. Conclusion

Single-view analysis

Single-view analysis

$$(\hat{\mathbf{x}}, \hat{\mathbf{g}}) \in \underset{\mathbf{x},\mathbf{g}}{\operatorname{argmin}} \left\{ \sum_{k=1}^{K} \frac{1}{2} \left\| \mathbf{y}_k - \mathbf{A}_k \mathbf{x} - g_k \mathbf{1} \right\|_{\mathbf{W}_k}^2 + \mathsf{R}(\mathbf{x}) \right\}$$

Single-view analysis

$$(\hat{\mathbf{x}}, \hat{\mathbf{g}}) \in \underset{\mathbf{x},\mathbf{g}}{\operatorname{argmin}} \left\{ \sum_{k=1}^{K} \frac{1}{2} \|\mathbf{y}_k - \mathbf{A}_k \mathbf{x} - g_k \mathbf{1}\|^2_{\mathbf{W}_k} + \mathsf{R}(\mathbf{x}) \right\}$$

$$= \underset{\mathbf{x}}{\operatorname{argmin}} \left\{ \min_{\mathbf{g}} \left\{ \sum_{k=1}^{K} \frac{1}{2} \|\mathbf{y}_k - \mathbf{A}_k \mathbf{x} - g_k \mathbf{1}\|^2_{\mathbf{W}_k} \right\} + \mathsf{R}(\mathbf{x}) \right\}$$

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$$= \underset{\mathbf{x}}{\operatorname{argmin}} \left\{ \sum_{k=1}^{K} \frac{1}{2} \underline{\min_{g_k} \left\{ \left\| \mathbf{y}_k - \mathbf{A}_k \mathbf{x} - g_k \mathbf{1} \right\|_{\mathbf{W}_k}^2 \right\}} + \mathsf{R}(\mathbf{x}) \right\}$$

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$$\hat{g}_k(\mathbf{x}) = \frac{\mathbf{1}^\top \mathbf{W}_k (\mathbf{y}_k - \mathbf{A}_k \mathbf{x})}{\mathbf{1}^\top \mathbf{W}_k \mathbf{1}}$$

$$\Rightarrow \|\mathbf{y}_k - \mathbf{A}_k \mathbf{x} - \hat{g}_k(\mathbf{x})\, \mathbf{1}\|^2_{\mathbf{W}_k} = \|\mathbf{y}_k - \mathbf{A}_k \mathbf{x}\|^2_{\tilde{\mathbf{W}}_k} ,$$

$$\text{where } \tilde{\mathbf{W}}_k = \mathbf{W}_k - \frac{\mathbf{W}_k \mathbf{1}\mathbf{1}^\top \mathbf{W}_k}{\mathbf{1}^\top \mathbf{W}_k \mathbf{1}}$$

Single-view analysis

$$(\hat{\mathbf{x}}, \hat{\mathbf{g}}) \in \underset{\mathbf{x},\mathbf{g}}{\operatorname{argmin}} \left\{ \sum_{k=1}^{K} \frac{1}{2} \|\mathbf{y}_k - \mathbf{A}_k \mathbf{x} - g_k \mathbf{1}\|_{\mathbf{W}_k}^2 + \mathsf{R}(\mathbf{x}) \right\}$$

$$= \underset{\mathbf{x}}{\operatorname{argmin}} \left\{ \min_{\mathbf{g}} \left\{ \sum_{k=1}^{K} \frac{1}{2} \|\mathbf{y}_k - \mathbf{A}_k \mathbf{x} - g_k \mathbf{1}\|_{\mathbf{W}_k}^2 \right\} + \mathsf{R}(\mathbf{x}) \right\}$$

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$$= \underset{\mathbf{x}}{\operatorname{argmin}} \left\{ \sum_{k=1}^{K} \frac{1}{2} \|\mathbf{y}_k - \mathbf{A}_k \mathbf{x}\|_{\tilde{\mathbf{W}}_k}^2 + \mathsf{R}(\mathbf{x}) \right\}$$

$$\hat{g}_k(\mathbf{x}) = \frac{\mathbf{1}^\top \mathbf{W}_k (\mathbf{y}_k - \mathbf{A}_k \mathbf{x})}{\mathbf{1}^\top \mathbf{W}_k \mathbf{1}}$$

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where $\tilde{\mathbf{W}}_k = \mathbf{W}_k - \frac{\mathbf{W}_k \mathbf{1}\mathbf{1}^\top \mathbf{W}_k}{\mathbf{1}^\top \mathbf{W}_k \mathbf{1}}$

Proposed equivalent formulation

$$\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}} \left\{ \Psi(\mathbf{x}) \triangleq \frac{1}{2} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_{\tilde{\mathbf{W}}}^2 + \mathsf{R}(\mathbf{x}) \right\}$$

Proposed equivalent formulation

$$\hat{\mathbf{x}} \in \underset{\mathbf{x}}{\operatorname{argmin}} \left\{ \Psi(\mathbf{x}) \triangleq \frac{1}{2} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_{\tilde{\mathbf{W}}}^2 + \mathsf{R}(\mathbf{x}) \right\}$$

$$\boxed{\tilde{\mathbf{W}} \triangleq \operatorname{diag}\left\{ \tilde{\mathbf{W}}_1, \tilde{\mathbf{W}}_2, \ldots, \tilde{\mathbf{W}}_K \right\}}$$

Proposed equivalent formulation

$$\hat{\mathbf{x}} \in \underset{\mathbf{x}}{\operatorname{argmin}} \left\{ \Psi(\mathbf{x}) \triangleq \frac{1}{2} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_{\tilde{\mathbf{W}}}^2 + \mathsf{R}(\mathbf{x}) \right\}$$

$$\Updownarrow$$

$$(\hat{\mathbf{x}}, \hat{\mathbf{u}}, \hat{\mathbf{v}}) \in \underset{\mathbf{x},\mathbf{u},\mathbf{v}}{\operatorname{argmin}} \left\{ \frac{1}{2} \|\mathbf{y} - \mathbf{u}\|_{\tilde{\mathbf{W}}}^2 + \Phi(\mathbf{v}) \right\} \text{ s.t. } \mathbf{u} = \mathbf{A}\mathbf{x}, \mathbf{v} = \mathbf{C}\mathbf{x}$$

S. Ramani and J. A. Fessler. **A splitting-based iterative algorithm for accelerated statistical X-ray CT reconstruction**. *IEEE Trans. Med. Imag.*, 31(3):677–88, March 2012

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$$\begin{cases} \mathbf{x}^{(j+1)} = \left(\rho\, \mathbf{A}^\top \mathbf{A} + \eta\, \mathbf{C}^\top \mathbf{C}\right)^{-1} \left(\rho\, \mathbf{A}^\top \left(\mathbf{u}^{(j)} + \mathbf{d}^{(j)}\right) + \eta\, \mathbf{C}^\top \left(\mathbf{v}^{(j)} + \mathbf{e}^{(j)}\right)\right) \\ \mathbf{u}^{(j+1)} = \left(\tilde{\mathbf{W}} + \rho\, \mathbf{I}\right)^{-1} \left(\tilde{\mathbf{W}}\mathbf{y} + \rho \left(\mathbf{A}\mathbf{x}^{(j+1)} - \mathbf{d}^{(j)}\right)\right) \\ \mathbf{v}^{(j+1)} \in \underset{\mathbf{v}}{\operatorname{argmin}} \left\{ \Phi(\mathbf{v}) + \frac{\eta}{2} \left\| \mathbf{v} - \left(\mathbf{C}\mathbf{x}^{(j+1)} - \mathbf{e}^{(j)}\right) \right\|^2 \right\} \\ \mathbf{d}^{(j+1)} = \mathbf{d}^{(j)} - \mathbf{A}\mathbf{x}^{(j+1)} + \mathbf{u}^{(j+1)} \\ \mathbf{e}^{(j+1)} = \mathbf{e}^{(j)} - \mathbf{C}\mathbf{x}^{(j+1)} + \mathbf{v}^{(j+1)} \end{cases}$$

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$$(\hat{\mathbf{x}}, \hat{\mathbf{u}}, \hat{\mathbf{v}}) \in \underset{\mathbf{x},\mathbf{u},\mathbf{v}}{\operatorname{argmin}} \left\{ \frac{1}{2} \|\mathbf{y} - \mathbf{u}\|^2_{\tilde{\mathbf{W}}} + \Phi(\mathbf{v}) \right\} \text{ s.t. } \mathbf{u} = \mathbf{A}\mathbf{x}, \mathbf{v} = \mathbf{C}\mathbf{x}$$

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Proposed equivalent formulation

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Solved exactly and efficiently using the ***Sherman-Morrison formula***

S. Ramani and J. A. Fessler. **A splitting-based iterative algorithm for accelerated statistical X-ray CT reconstruction.** *IEEE Trans. Med. Imag.*, 31(3):677–88, March 2012

Non-blind gain correction

$$(\hat{\mathbf{x}}, \hat{\mathbf{g}}) \in \underset{\mathbf{x},\mathbf{g}}{\operatorname{argmin}} \left\{ \Psi(\mathbf{x}, \mathbf{g}) \triangleq \frac{1}{2} \|\mathbf{y} - \mathbf{A}\mathbf{x} - \mathbf{g} \otimes \mathbf{1}\|_{\mathbf{W}}^2 + \mathsf{R}(\mathbf{x}) \right\}$$

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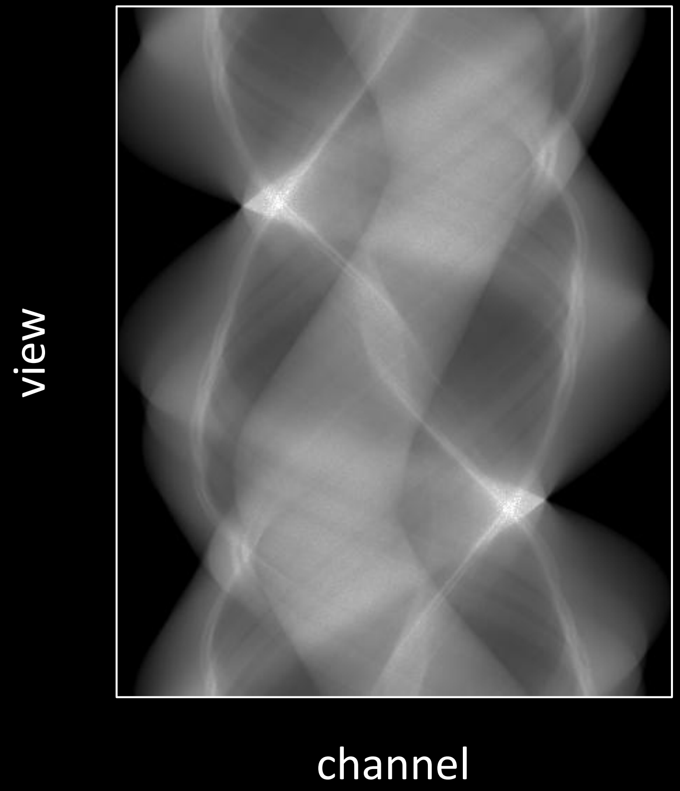
Correct ***all*** views
(i.e., ***blind*** gain correction)

Non-blind gain correction

$$(\hat{\mathbf{x}}, \hat{\mathbf{g}}) \in \underset{\mathbf{x},\mathbf{g}}{\operatorname{argmin}} \left\{ \Psi(\mathbf{x}, \mathbf{g}) \triangleq \frac{1}{2} \|\mathbf{y} - \mathbf{A}\mathbf{x} - \mathbf{g} \otimes \mathbf{1}\|_{\mathbf{W}}^2 + \mathsf{R}(\mathbf{x}) \right\}$$



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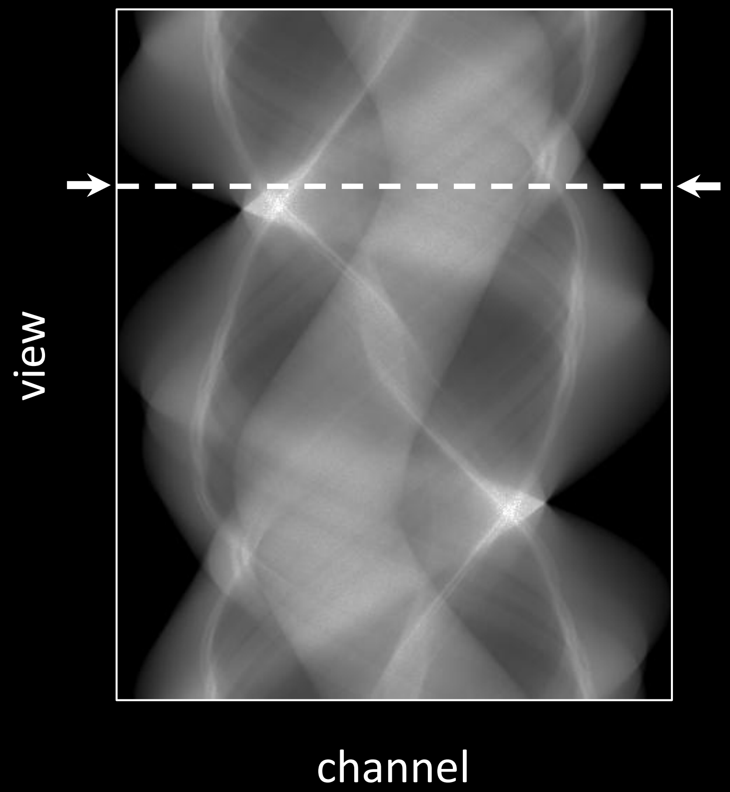


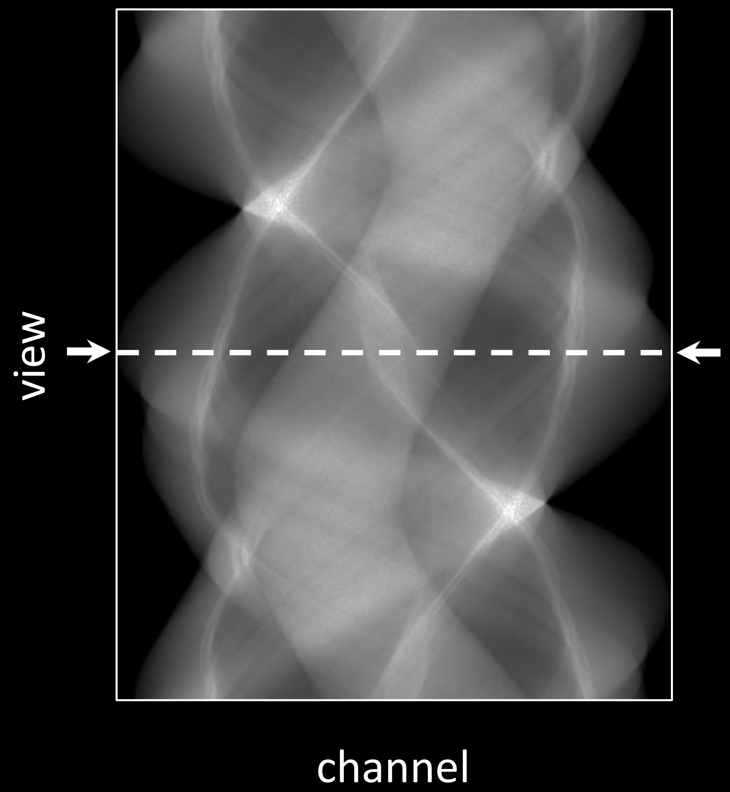
Object is ***inside*** FOV.

$\Longrightarrow$ Reference channel is possibly ***not blocked***.

$\Longrightarrow$ Gain fluctuation (g_k) is possibly ***zero***!!!

Non-blind gain correction

$$(\hat{\mathbf{x}}, \hat{\mathbf{g}}) \in \underset{\mathbf{x},\mathbf{g}}{\operatorname{argmin}} \left\{ \Psi(\mathbf{x}, \mathbf{g}) \triangleq \frac{1}{2} \|\mathbf{y} - \mathbf{A}\mathbf{x} - \mathbf{g} \otimes \mathbf{1}\|_{\mathbf{W}}^2 + \mathsf{R}(\mathbf{x}) \right\}$$



Object is ***outside*** FOV.

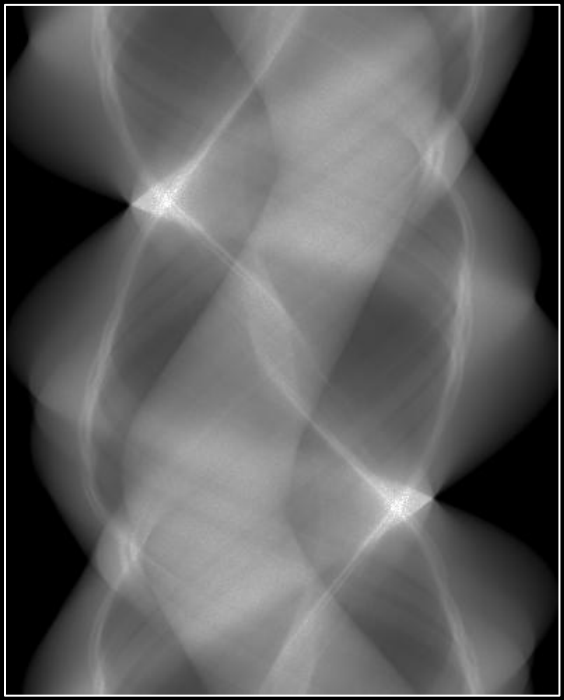
$\Rightarrow$ Reference channel is possibly ***blocked***.

$\Rightarrow$ Gain fluctuation (g_k) is possibly ***non-zero***!!!

Non-blind gain correction

$$(\hat{\mathbf{x}}, \hat{\mathbf{g}}) \in \underset{\mathbf{x},\mathbf{g}}{\operatorname{argmin}} \left\{ \Psi(\mathbf{x}, \mathbf{g}) \triangleq \frac{1}{2} \|\mathbf{y} - \mathbf{A}\mathbf{x} - \mathbf{g} \otimes \mathbf{1}\|_{\mathbf{W}}^2 + \mathsf{R}(\mathbf{x}) \right\}$$

$$\text{s.t.} \quad [\mathbf{g}]_k = 0 \text{ if } k \notin \mathcal{K}$$



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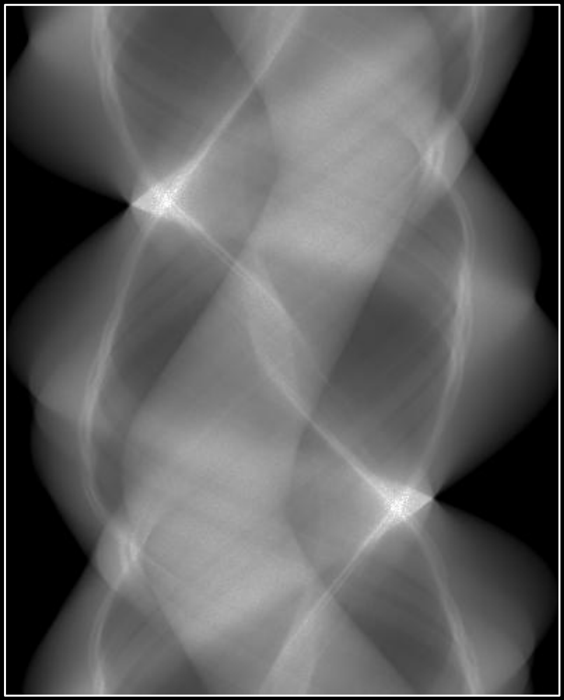
$$\text{s.t.} \quad [\mathbf{g}]_k = 0 \text{ if } k \notin \mathcal{K}$$

Correct ***some*** views
(i.e., ***non-blind*** gain correction)



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$\Longrightarrow$ Gain fluctuation (g_k) is possibly ***zero***!!!

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$$\text{s.t.} \quad [\mathbf{g}]_k = 0 \text{ if } k \notin \mathcal{K}$$

$\Longleftrightarrow$

$$\hat{\mathbf{x}} \in \underset{\mathbf{x}}{\operatorname{argmin}} \left\{ \Psi(\mathbf{x}) \triangleq \frac{1}{2} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_{\tilde{\mathbf{W}}}^2 + \mathsf{R}(\mathbf{x}) \right\},$$

where $\tilde{\mathbf{W}} \triangleq \operatorname{diag}\left\{ \tilde{\mathbf{W}}_1, \tilde{\mathbf{W}}_2, \ldots, \tilde{\mathbf{W}}_K \right\}$ and

$$\tilde{\mathbf{W}}_k = \begin{cases} \mathbf{W}_k & , \text{if } k \notin \mathcal{K} \\ \mathbf{W}_k - \dfrac{\mathbf{W}_k \mathbf{1}\mathbf{1}^\top \mathbf{W}_k}{\mathbf{1}^\top \mathbf{W}_k \mathbf{1}} & , \text{otherwise}. \end{cases}$$

Gain correction using OS-SQS

$$\hat{\mathbf{x}} \in \operatorname*{argmin}_{\mathbf{x}} \left\{ \Psi(\mathbf{x}) \triangleq \frac{1}{2} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|^2_{\tilde{\mathbf{W}}} + \mathsf{R}(\mathbf{x}) \right\} ,$$

where $\tilde{\mathbf{W}} \triangleq \operatorname{diag}\left\{ \tilde{\mathbf{W}}_1, \tilde{\mathbf{W}}_2, \ldots, \tilde{\mathbf{W}}_K \right\}$ and

$$\tilde{\mathbf{W}}_k \triangleq \mathbf{W}_k - \frac{\mathbf{W}_k \mathbf{1}\mathbf{1}^\top \mathbf{W}_k}{\mathbf{1}^\top \mathbf{W}_k \mathbf{1}} \cdot \mathbb{I}_{k \in \mathcal{K}} \text{ for } k = 1, 2, \ldots, K$$

1. $\nabla \Psi(\mathbf{x}) = \mathbf{A}^\top \tilde{\mathbf{W}} \left(\mathbf{A}\mathbf{x} - \mathbf{y} \right) + \nabla \mathsf{R}(\mathbf{x})$
2. $\nabla_\ell \Psi(\mathbf{x}) = L \cdot \mathbf{A}^\top_{(\ell)} \tilde{\mathbf{W}}_{(\ell)} \left(\mathbf{A}_{(\ell)} \mathbf{x} - \mathbf{y}_{(\ell)} \right) + \nabla \mathsf{R}(\mathbf{x})$,
   where $\tilde{\mathbf{W}}_{(\ell)} \triangleq \operatorname{diag}\left\{ \tilde{\mathbf{W}}_k \mid k \in \text{the } \ell\text{th ordered subsets} \right\}$
3. Diagonal majorizer $= \mathbf{D}_{\mathrm{WLS}} \triangleq \operatorname{diag}\{ \mathbf{A}^\top \mathbf{W} \mathbf{A} \mathbf{1} \}$

Outline

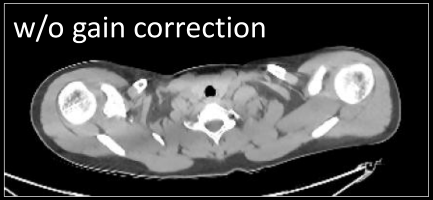
I. Description of purpose

$$(\hat{\mathbf{x}}, \hat{\mathbf{g}}) \in \operatorname*{argmin}_{\mathbf{x},\mathbf{g}} \left\{ \Psi(\mathbf{x}, \mathbf{g}) \triangleq \frac{1}{2} \|\mathbf{y} - \mathbf{A}\mathbf{x} - \mathbf{g} \otimes \mathbf{1}\|_{\mathbf{W}}^2 + \mathsf{R}(\mathbf{x}) \right\}$$

II. Method

$$\hat{\mathbf{x}} \in \operatorname*{argmin}_{\mathbf{x}} \left\{ \Psi(\mathbf{x}) \triangleq \frac{1}{2} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_{\tilde{\mathbf{W}}}^2 + \mathsf{R}(\mathbf{x}) \right\}$$

III. Results



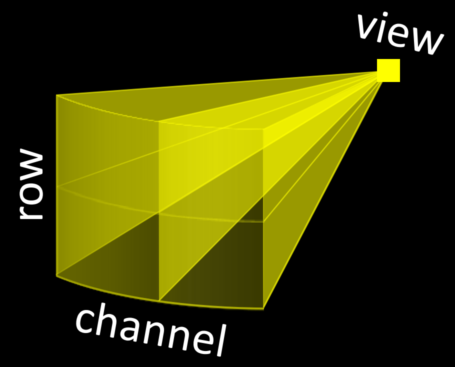
IV. Conclusion

Gain correction in 3D X-ray CT

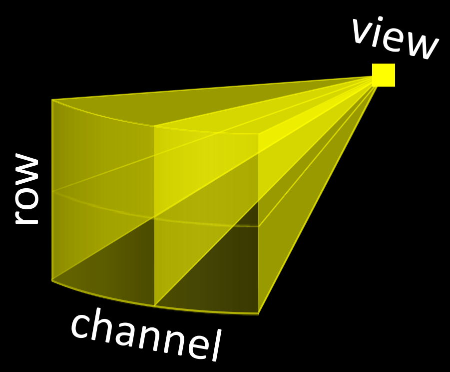
Gain correction in 3D X-ray CT

view

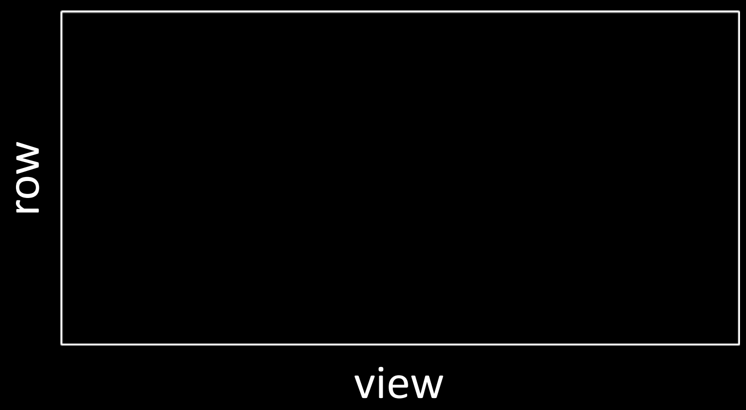
Gain correction in 3D X-ray CT



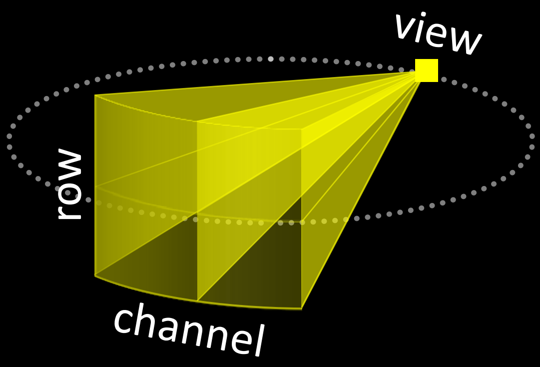
Gain correction in 3D X-ray CT



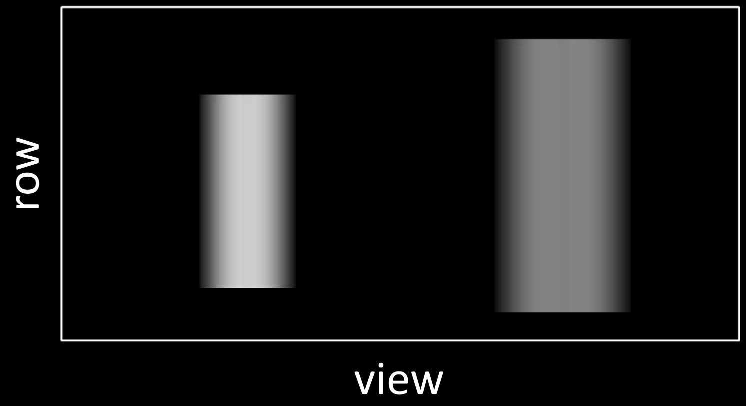
Gain fluctuations:



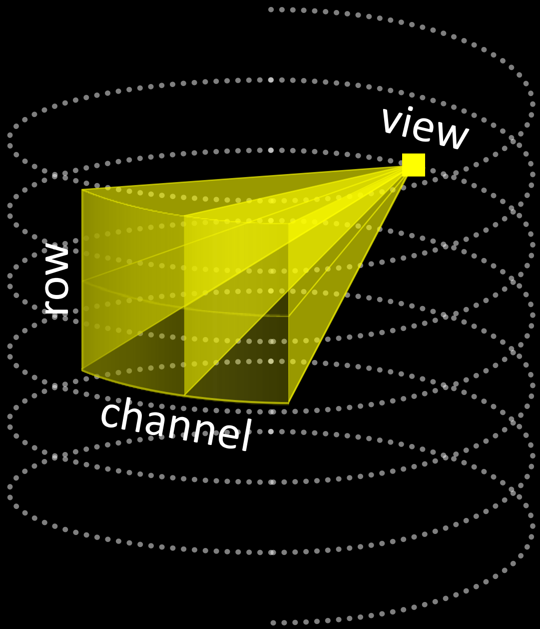
Gain correction in 3D X-ray CT



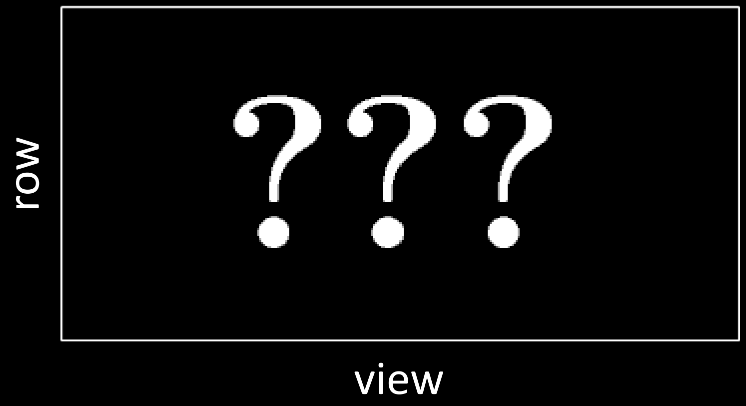
Gain fluctuations: **axial**



Gain correction in 3D X-ray CT



Gain fluctuations: **helical**



3D axial CT – using ADMM

3D axial CT – using ADMM

Image:

128 x **120** x **100** 30cm-diameter cylinder
with three bone-like inserts
Voxel size: **3.9062** x **3.9062** x **3.9062 mm^3**

Geometry:

128 x **120** x **144** 3D axial CT
Orbit: **360°**
Pitch = **0**
Detector channel spacing = **8 mm**
Detector row spacing = **8 mm**

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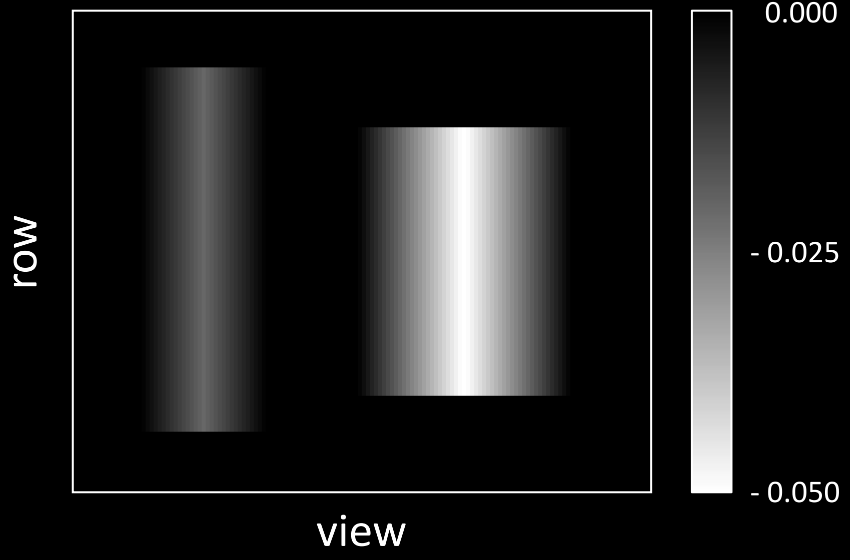
True gain fluctuations



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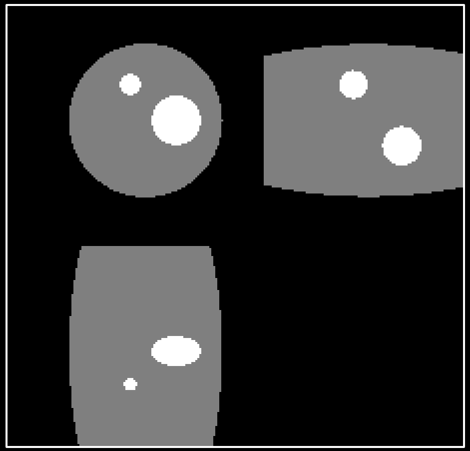
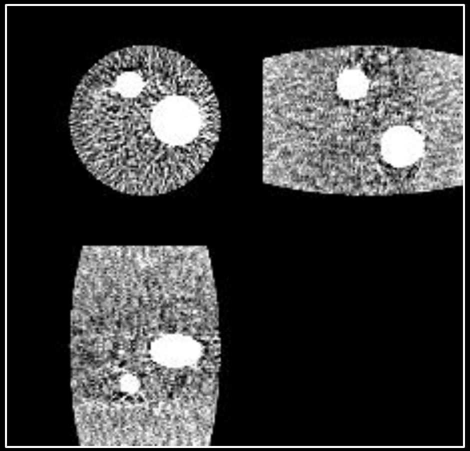
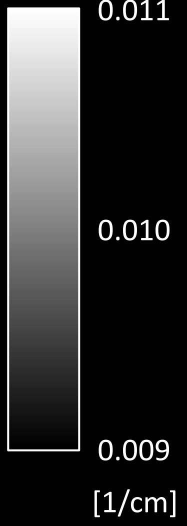


True image



FDK (initial image)



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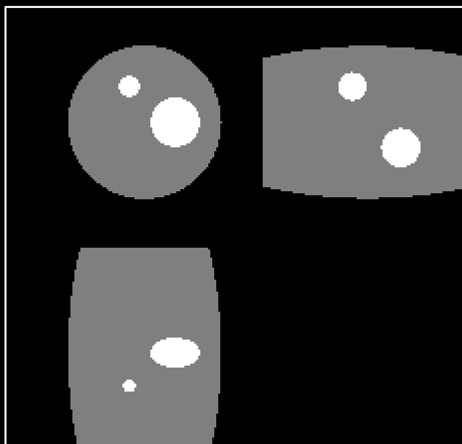

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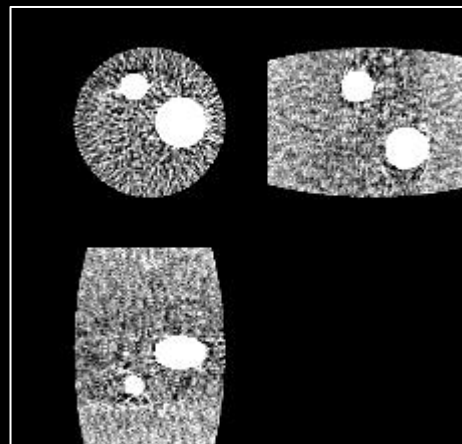

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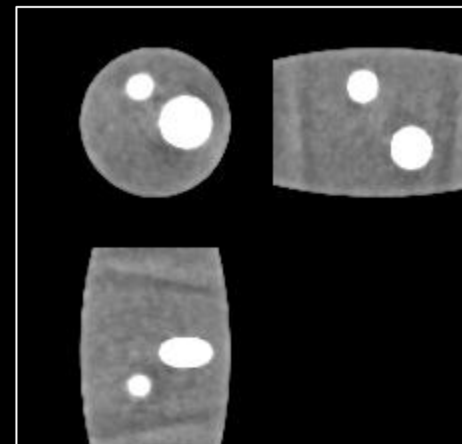
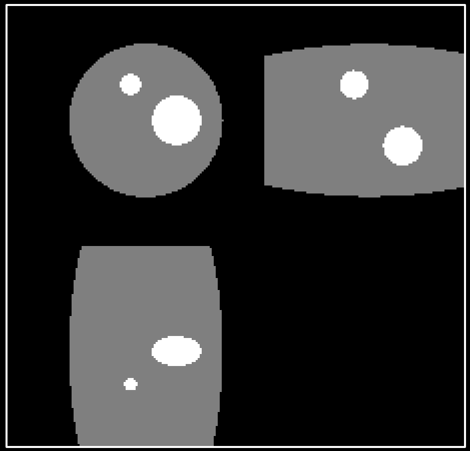
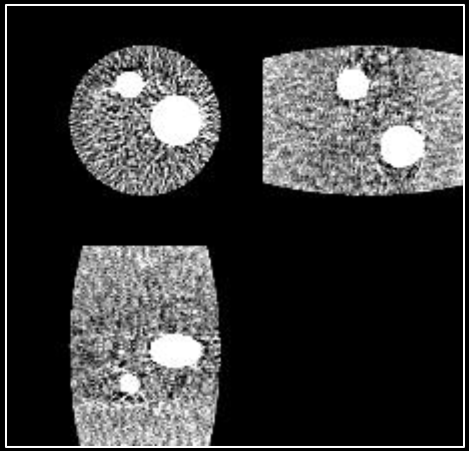


No gain correction

3D axial CT – using ADMM



True image



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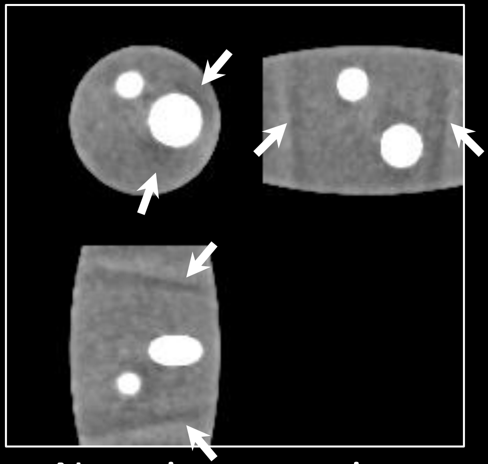

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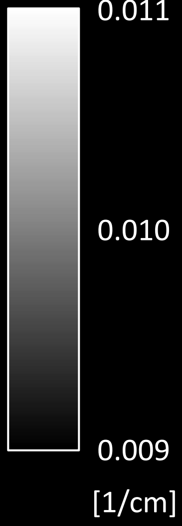

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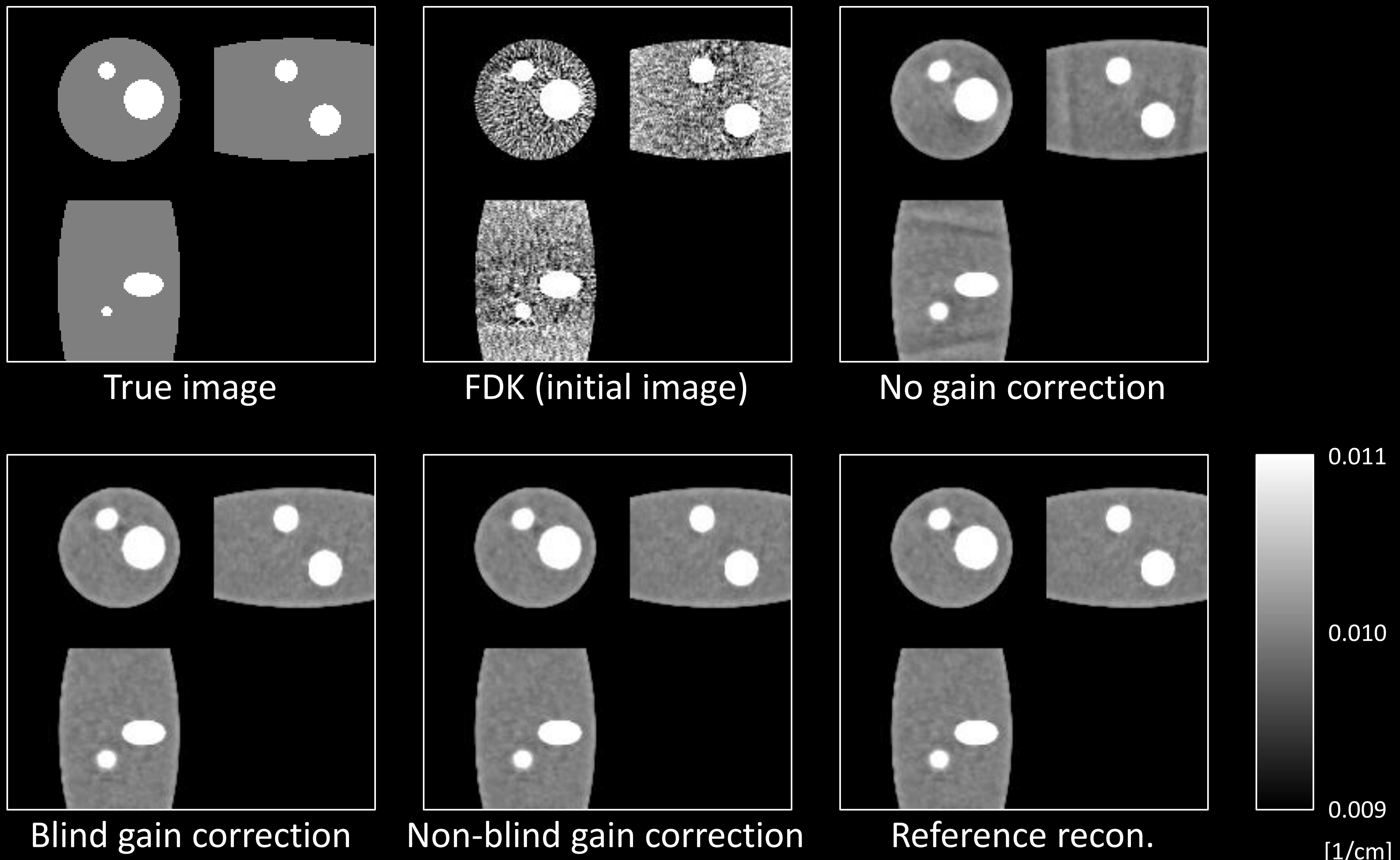

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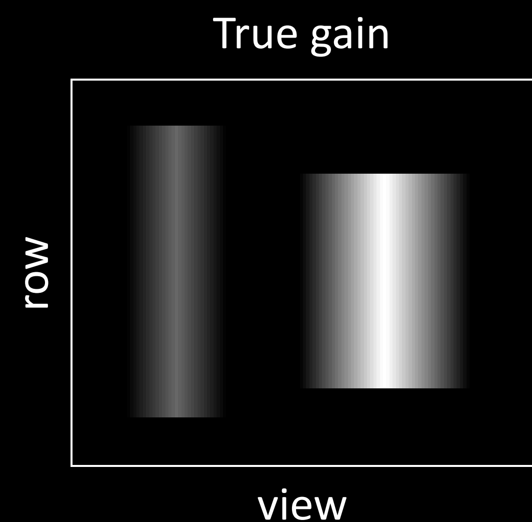

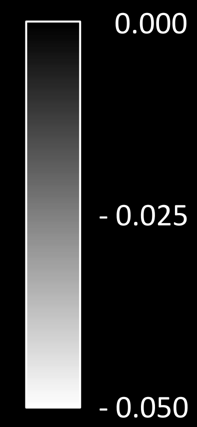

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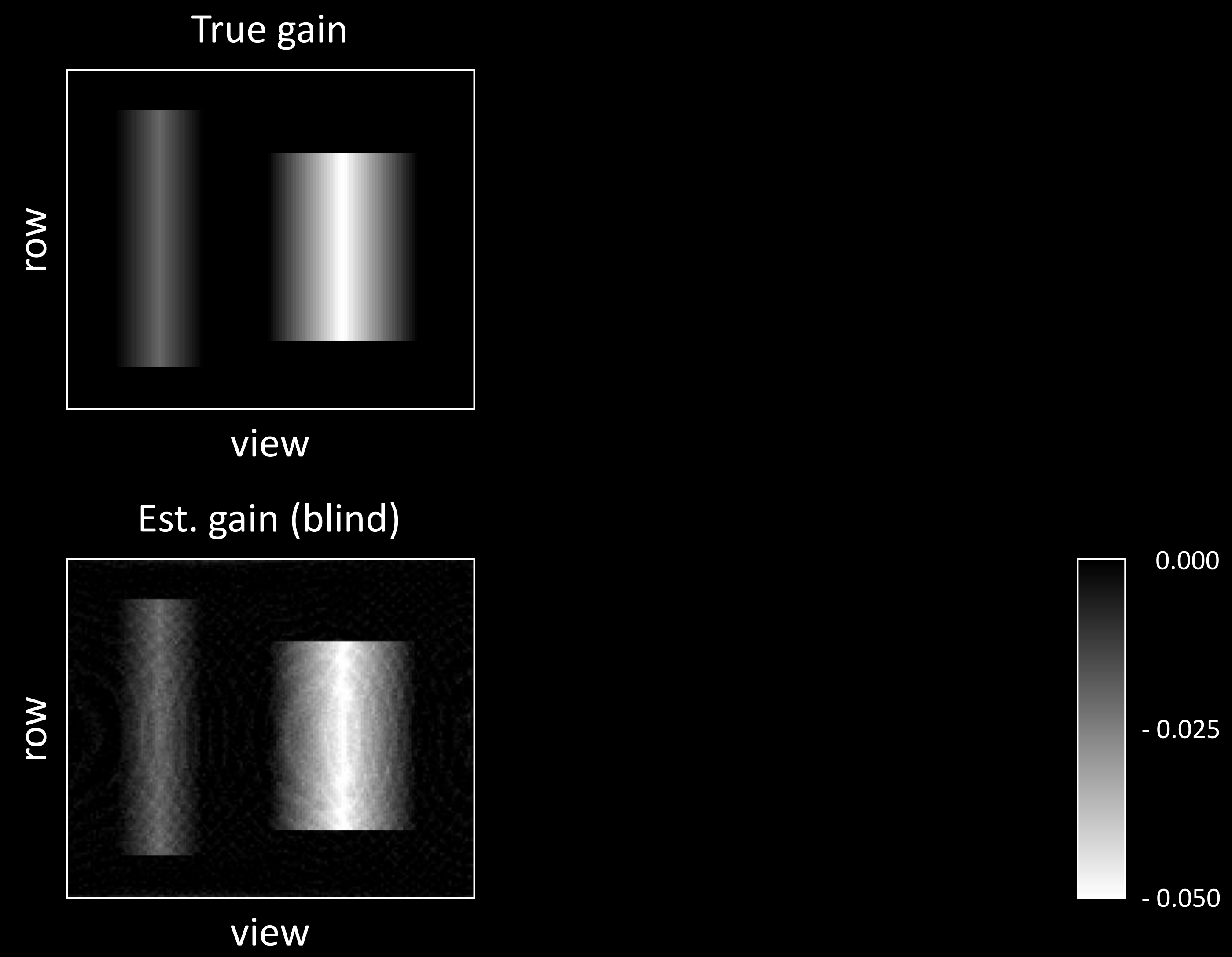

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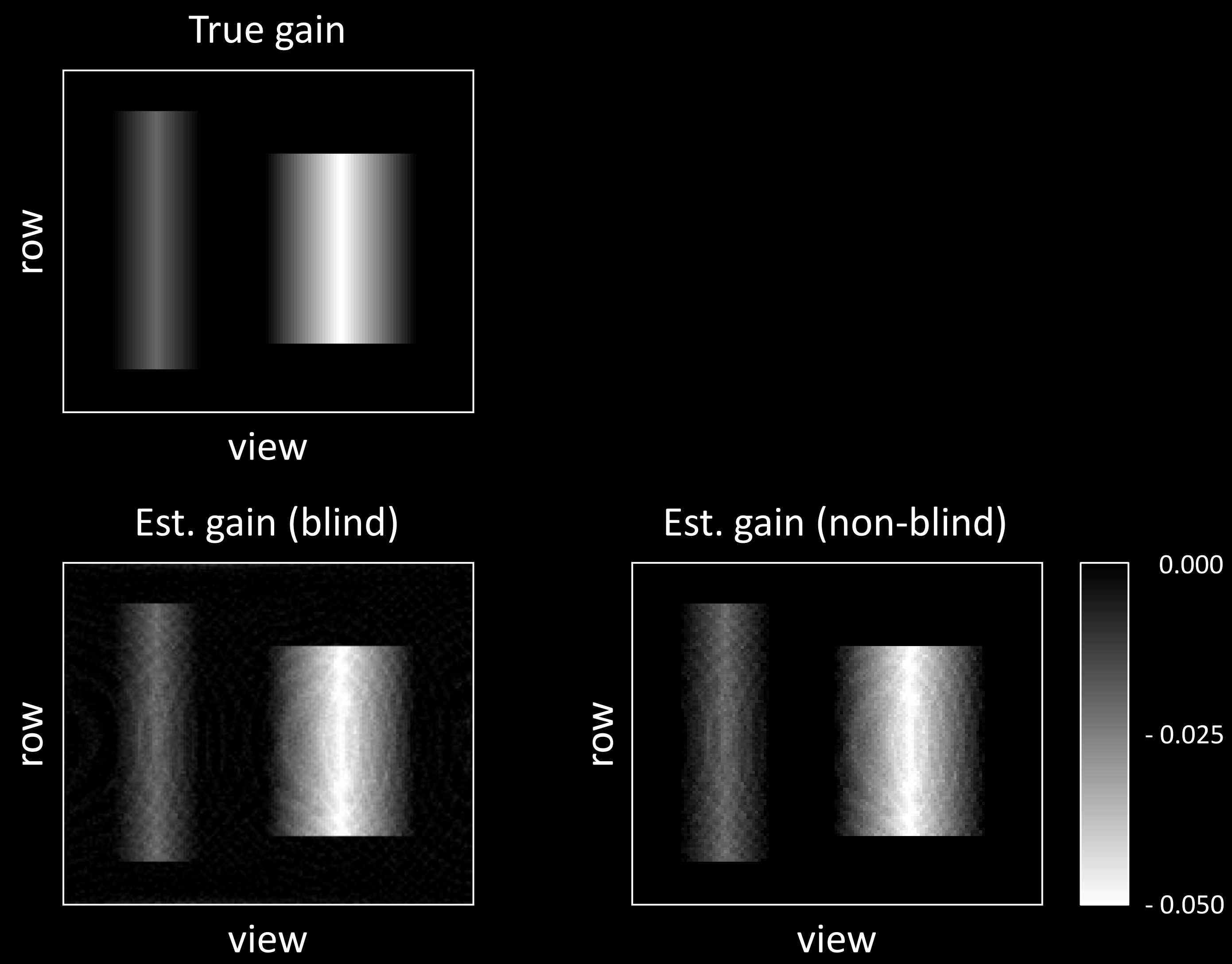

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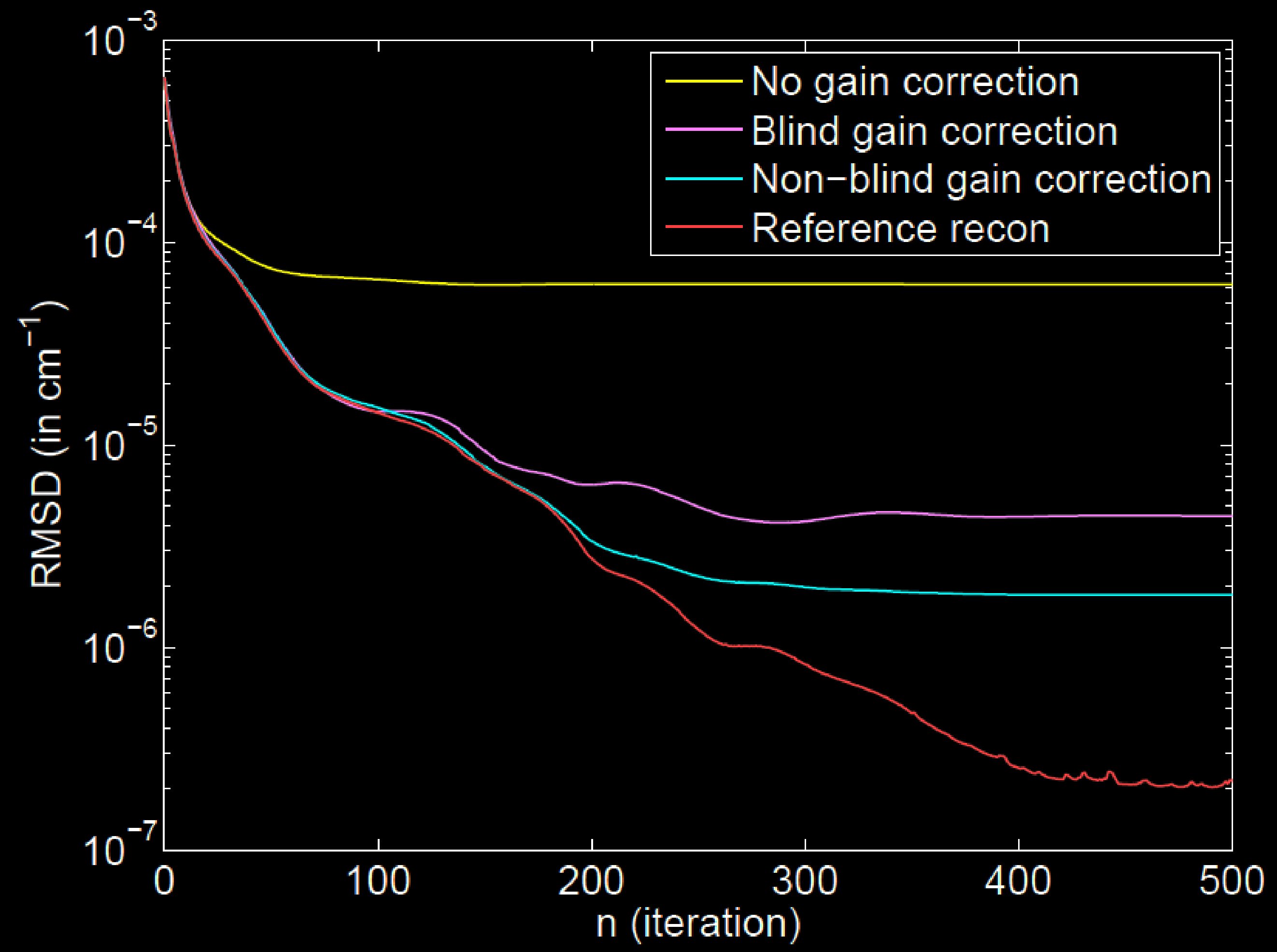

3D helical CT – using OS-SQS

3D helical CT – using OS-SQS

Image:

512 x **512** x **109** patient (shoulder)
Voxel size: **1.3695** x **1.3695** x **0.6250 mm^3**

Geometry:

888 x **32** x **7146** 3D helical CT
Orbit: **2614°**
Pitch = **0.5312**
Detector channel spacing = **1.0239 mm**
Detector row spacing = **1.0964 mm**

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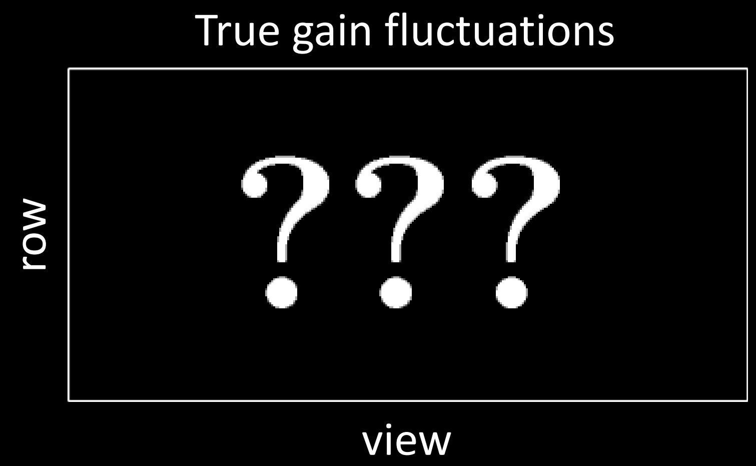
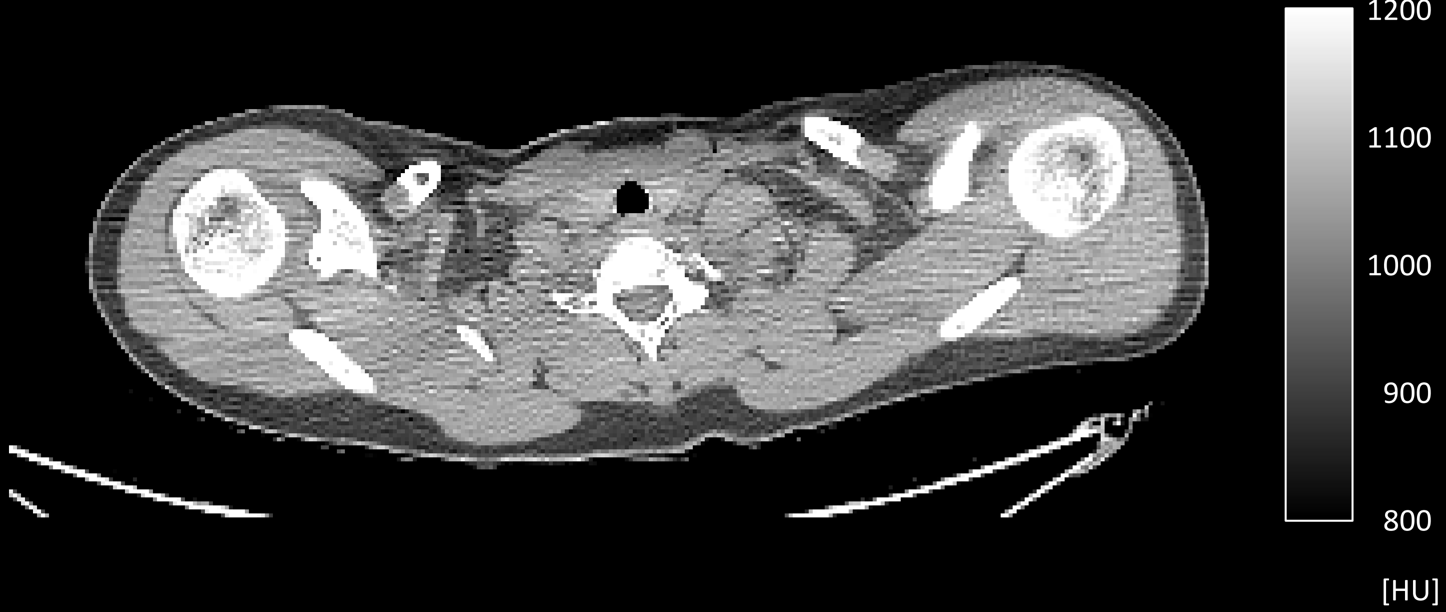


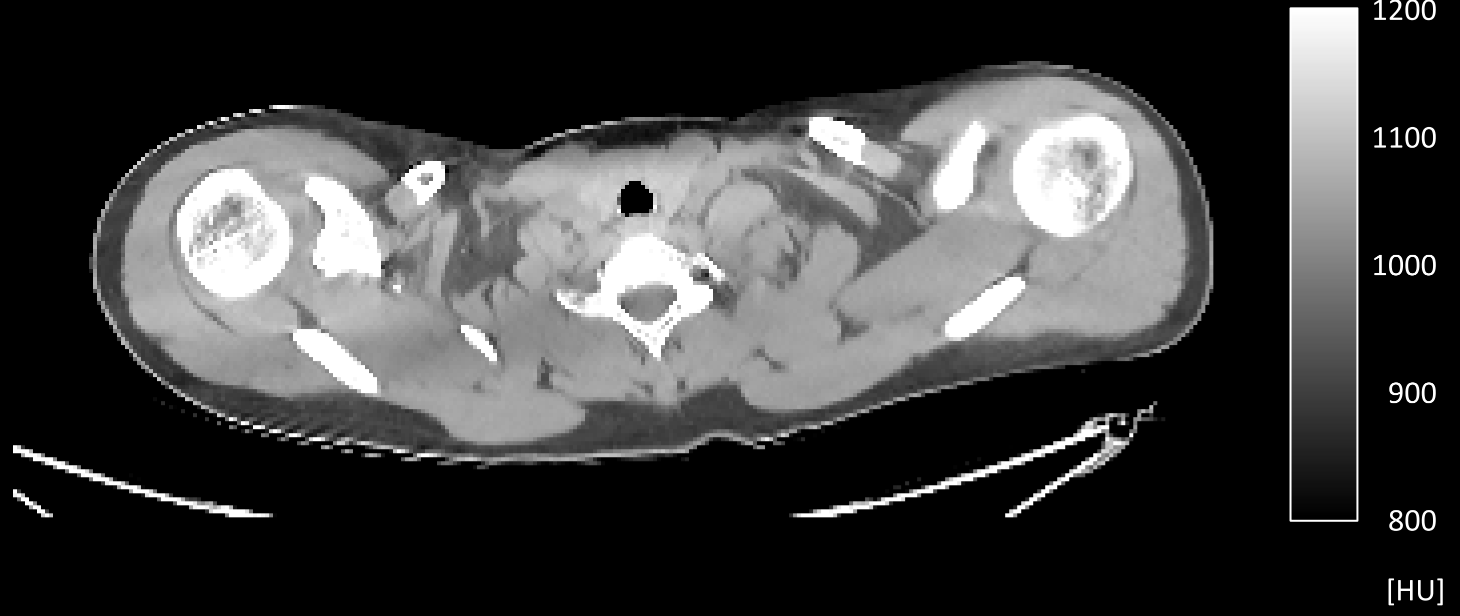
3D helical CT – using OS-SQS

Transaxial plane: **FDK** reconstruction



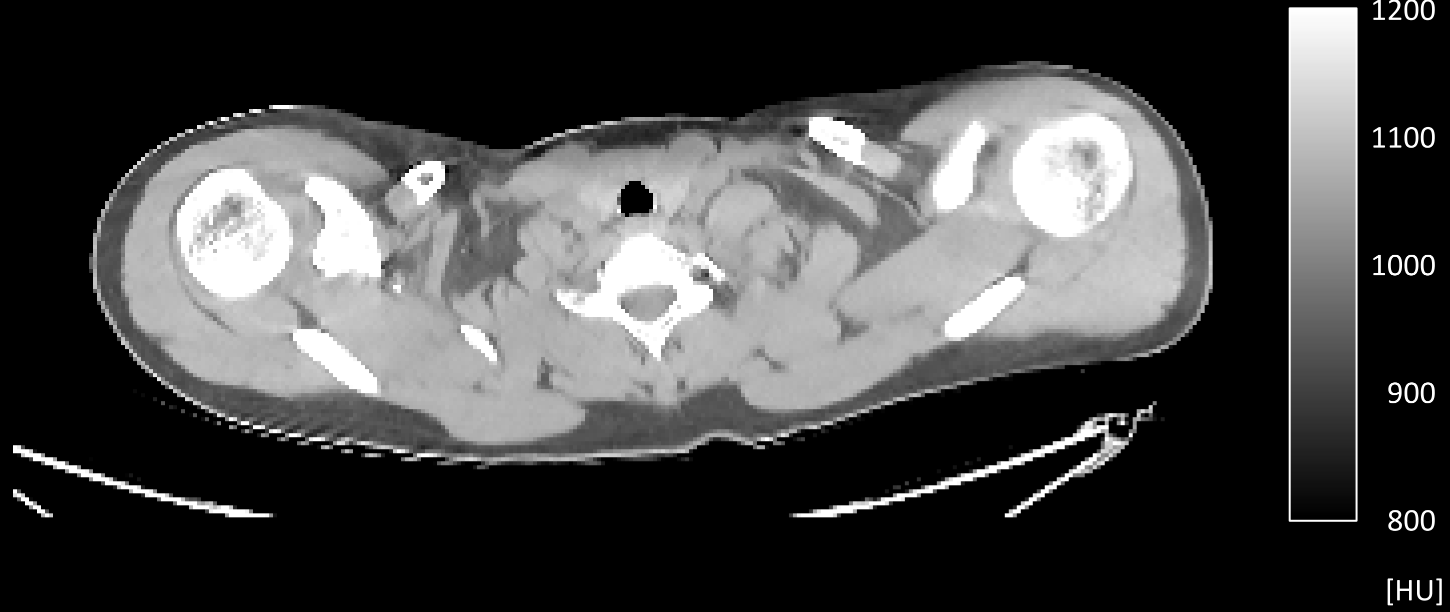
3D helical CT – using OS-SQS

Transaxial plane: **OS-SQS-41** reconstruction (w/o gain correction)



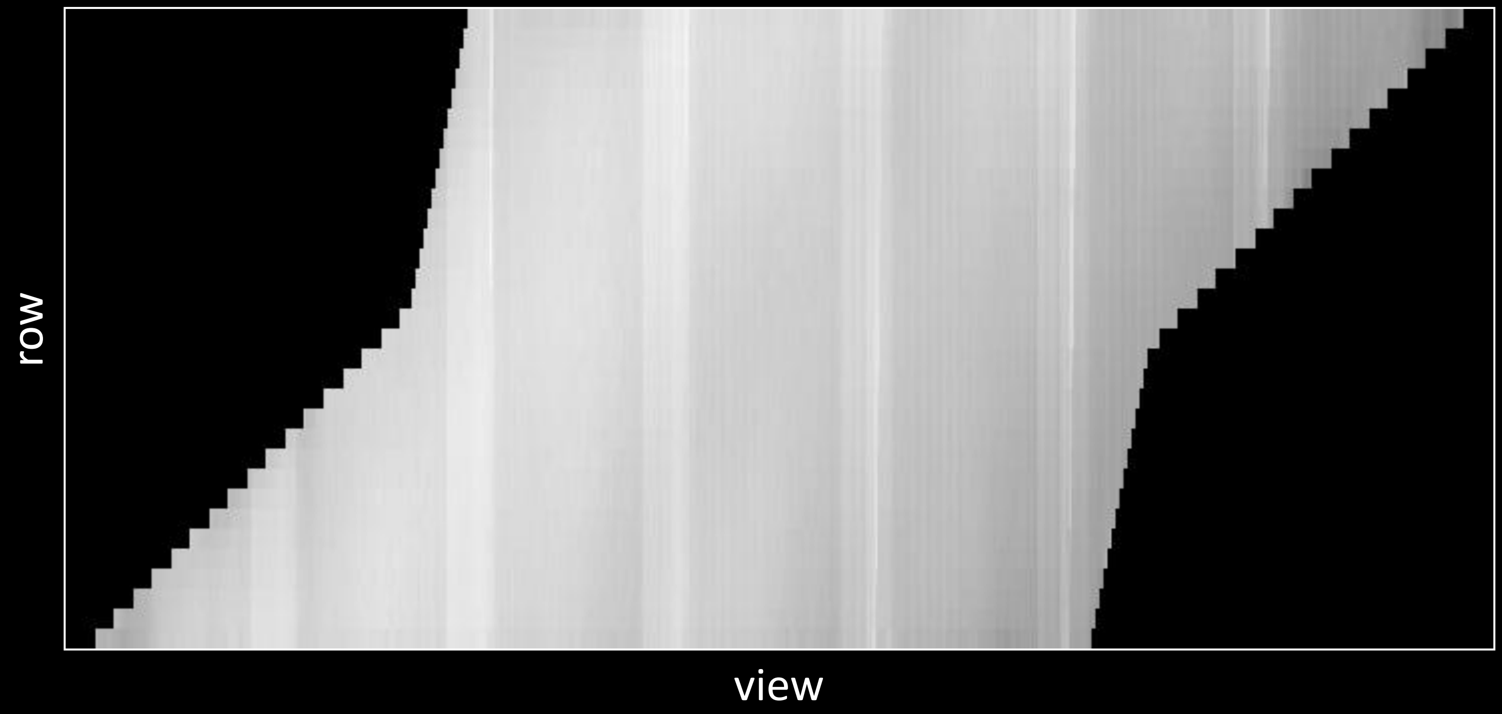
3D helical CT – using OS-SQS

Transaxial plane: **OS-SQS-41** reconstruction (w/ blind gain correction)



3D helical CT – using OS-SQS

Estimated gain fluctuations (blind gain correction)



3D helical CT – using OS-SQS

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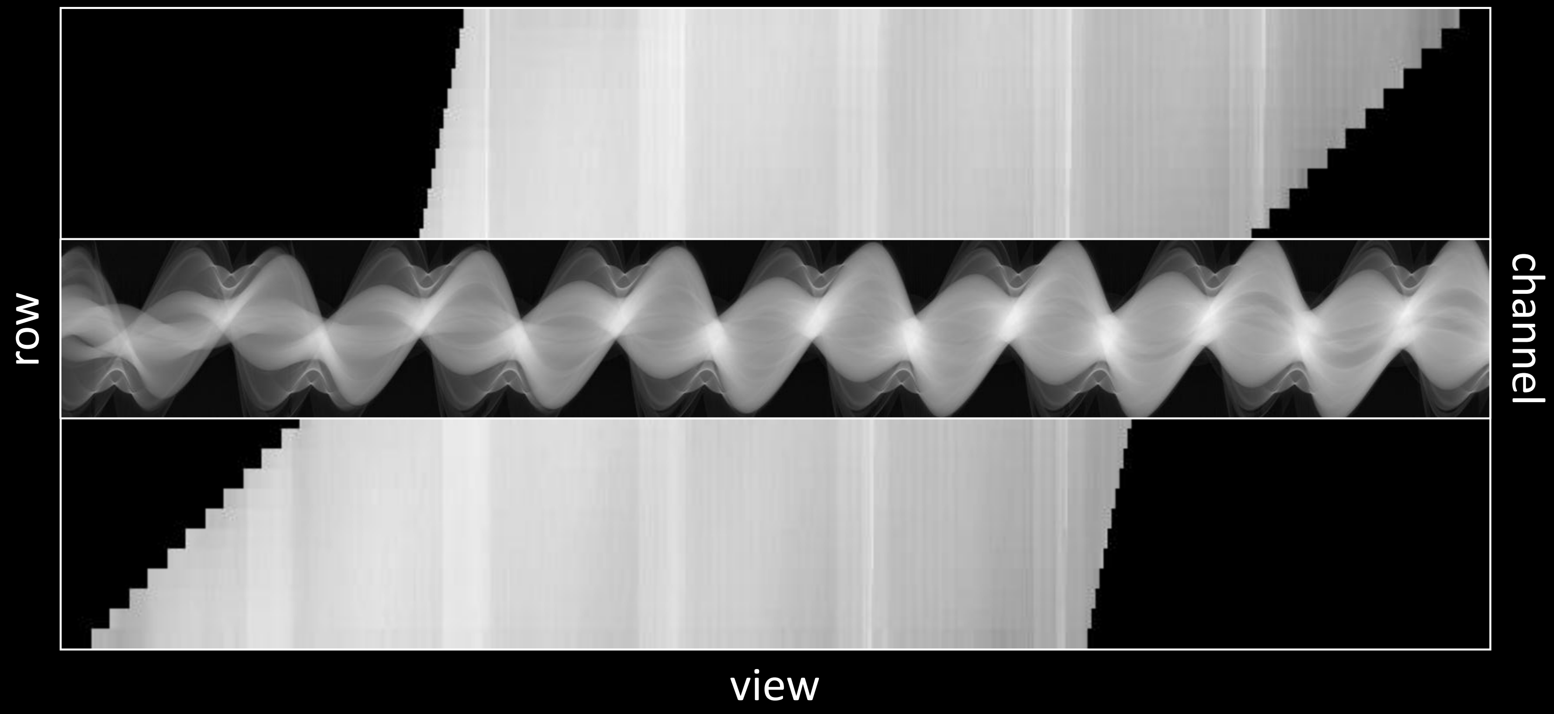

Outline

I. Description of purpose

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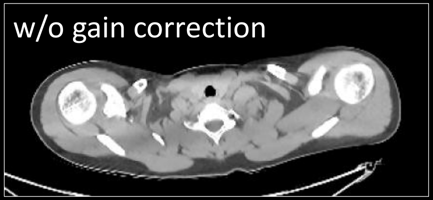
III. Results



IV. Conclusion

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Conclusion

- A **new variational formulation** for X-ray CT image reconstruction **with gain correction** is proposed
- Both **splitting-** and **OS-based** algorithms can be applied to the proposed formulation
- We've evaluated the proposed method with both **synthetic** and **real patient CT scans**
- Shading artifacts are **largely reduced** with **almost unchanged** complexity per iteration