Combining Augmented Lagrangian Method with Ordered Subsets for X-Ray CT Reconstruction

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INTRODUCTION

The augmented Lagrangian (AL) method [1] has drawn more attention recently due to its scalability, simplicity, and fast convergence. One variation of the AL method is to precondition the ℓ_2 penalty term in the augmented Lagrangian by some positive definite matrix. For example, when choosing a diagonal matrix, we penalize each entry of the split variable differently, so we can have *larger* steps for those entries that are still far from the solution by *decreasing* the penalty.

In statistical X-ray CT image reconstruction, the image reconstruction is usually formulated as a PWLS problem, and the ordered-subset (OS) algorithm [5] can be used to accelerate the gradient descent method about M times by grouping the projections into M ordered subsets and updating the image incrementally using the M subset gradients.

In this paper, we propose to combine the AL method with OS, where a diagonal preconditioner is used in the AL method so that the inner minimization problem is another statistically weighted CT problem, and we solve it using the standard OS algorithm. Although the OS algorithm approaches some limit cycle eventually because of its incremental gradient descent structure, the M-time acceleration of solving a PWLS problem is still very promising for the AL method with *inexact* updates.

METHOD

We define the statistically weighted CT problem as follows:

$$\hat{\mathbf{x}} \in \underset{\mathbf{x}\in\Omega}{\operatorname{argmin}} \left\{ \tfrac{1}{2} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_{\mathbf{W}}^2 + \mathsf{R}(\mathbf{x}) \right\}, \quad (1)$$

where $\mathbf{y}$ is the noisy sinogram, $\mathbf{A}$ is the system matrix of a CT scanner, $\mathbf{W}$ is a diagonal weighting matrix accounting for measurement variance, R is an edge-preserving regularizer, and Ω is the nonnegativity constraint on the solution. Instead of solving (1) directly, we will solve an equivalent constrained minimization problem:

$$(\hat{\mathbf{x}}, \hat{\mathbf{u}}) \in \underset{\mathbf{x}\in\Omega, \mathbf{u}}{\operatorname{argmin}} \left\{ \tfrac{1}{2} \|\mathbf{y} - \mathbf{u}\|_{\mathbf{W}}^2 + \mathsf{R}(\mathbf{x}) \right\} \text{ s.t. } \mathbf{u} = \mathbf{A}\mathbf{x}, \quad (2)$$

using the (alternating direction) AL method, which alternatingly minimizes the augmented Lagrangian

$$\tfrac{1}{2} \|\mathbf{y} - \mathbf{u}\|_{\mathbf{W}}^2 + \mathsf{R}(\mathbf{x}) + \iota_\Omega(\mathbf{x}) + \tfrac{1}{2} \|\mathbf{A}\mathbf{x} - \mathbf{u} - \mathbf{d}\|_{\mathbf{G}}^2 \quad (3)$$

with respect to $\mathbf{x}$ and $\mathbf{u}$, followed by a gradient ascent of $\mathbf{d}$, where ι_Ω is the characteristic function of Ω, $\mathbf{d}$ is the scaled dual variable of $\mathbf{u}$, and $\mathbf{G}$ is a positive definite preconditioning matrix. The conventional choice of $\mathbf{G}$ is $\mathbf{G} = \eta\mathbf{I}$ with $\eta > 0$. The general AL iterates are as follows:

$$\begin{cases} \mathbf{x}^{(j+1)} \in \underset{\mathbf{x}\in\Omega}{\operatorname{argmin}} \left\{ \tfrac{1}{2} \left\| \left(\mathbf{u}^{(j)} + \mathbf{d}^{(j)}\right) - \mathbf{A}\mathbf{x} \right\|_{\mathbf{G}}^2 + \mathsf{R}(\mathbf{x}) \right\} \\ \mathbf{u}^{(j+1)} = (\mathbf{W} + \mathbf{G})^{-1} \left(\mathbf{W}\mathbf{y} + \mathbf{G}\left(\mathbf{A}\mathbf{x}^{(j+1)} - \mathbf{d}^{(j)}\right)\right) \\ \mathbf{d}^{(j+1)} = \mathbf{d}^{(j)} - \mathbf{A}\mathbf{x}^{(j+1)} + \mathbf{u}^{(j+1)}. \end{cases} \quad (4)$$

To find a "good" preconditioner $\mathbf{G}$, we analyze a simpler CT problem with a quadratic regularizer and no box constraint. We can show that no matter what regularization parameter is used, $\mathbf{G} = \mathbf{W}$ leads to a quite fast, with rate ½, convergence of the split variable $\mathbf{u}$. Inspired by this, we focus on the diagonal preconditioner $\mathbf{G} = \eta\mathbf{W}$ with $\eta > 0$, and the AL iterates become

$$\begin{cases} \mathbf{x}^{(j+1)} \in \underset{\mathbf{x}\in\Omega}{\operatorname{argmin}} \left\{ \tfrac{1}{2} \left\| \left(\mathbf{u}^{(j)} + \mathbf{d}^{(j)}\right) - \mathbf{A}\mathbf{x} \right\|_{\mathbf{W}}^2 + \eta^{-1}\mathsf{R}(\mathbf{x}) \right\} \\ \mathbf{u}^{(j+1)} = \frac{1}{1+\eta} \left(\mathbf{y} + \eta\left(\mathbf{A}\mathbf{x}^{(j+1)} - \mathbf{d}^{(j)}\right)\right) \\ \mathbf{d}^{(j+1)} = \mathbf{d}^{(j)} - \mathbf{A}\mathbf{x}^{(j+1)} + \mathbf{u}^{(j+1)}. \end{cases} \quad (5)$$

As can be seen in (5), the inner minimization is another CT problem with an *updated* sinogram and a *scaled* regularizer, and it can be solved using OS algorithms, e.g., [5]. Finally, we have our proposed AL-OS algorithm:

$$\begin{cases} \mathbf{x}^{(j+1)} = \mathcal{OS}_M^1\left(\mathbf{x}^{(j)}; \tfrac{1}{2} \left\| \mathbf{z}^{(j)} - \mathbf{A}\mathbf{x} \right\|_{\mathbf{W}}^2 + \eta^{-1}\mathsf{R}(\mathbf{x}), \Omega \right) \\ \mathbf{u}^{(j+1)} = \begin{cases} \mathbf{u}^{(j)} & \text{, if } \operatorname{mod}(j+1, P) \neq 0 \\ \frac{1}{1+\eta}\left(\mathbf{u}^{(j)} + \eta\mathbf{A}\mathbf{x}^{(j+1)}\right), & \text{otherwise}, \end{cases} \end{cases} \quad (6)$$

METHOD (CONT'D)

where the $\mathbf{x}$-update is just one complete OS iteration (with M subsets) of the inner CT problem, $\mathbf{z}^{(j)} = \eta^{-1}\mathbf{y} + (1-\eta^{-1})\mathbf{u}^{(j)}$ is the updated sinogram, and P is the update period. Note that $\mathbf{d}$ does not appear in (6) because of $\eta\mathbf{d}^{(j+1)} = \mathbf{y} - \mathbf{u}^{(j+1)}$ from (5).

To summarize, the proposed AL-OS algorithm solves the CT problem in (1) by solving a sequence of inner CT problems inexactly with warm start. There are two tuning parameters: the AL penalty parameter η and split variable update period P. When $\eta = 1$, the proposed AL-OS algorithm reduces to the OS algorithm. When $\eta < 1$, the inner CT problems become more regularized, and the step size of $\mathbf{u}$ is *larger* due to the *smaller* penalty in (3). The value of P determines the number of OS iterations we used to solve the inner CT problems. Typically, larger P leads to lower overhead for computing extra forward projections and higher accuracy of the inner CT problems.

RESULT

We evaluate our proposed AL-OS algorithm using a patient helical CT scan and investigate the effects of the update period and the AL penalty parameter with different values of P and η. The test sinogram is of size **888**x**32**x**7146** with pitch **0.5312**, and the image size is **512**x**512**x**109**. Lastly, since the test helical CT scan contains gain fluctuations [7], we include blind gain correction [8] in all of our reconstruction algorithms. With this correction, the statistical weighting matrix $\mathbf{W}$ and the preconditioning matrix $\mathbf{G}$ are "diagonal-plus-rank-one" rather than pure diagonal matrices, which is a simple extension of the proposed diagonal preconditioned AL method. The naming conventions in our experiment are as follows: **OS-SQS-*M*** denotes the standard OS algorithm [5] with M subsets, and **AL-OS-*M*-*η*-*P*** denotes the proposed AL-OS algorithm with M subsets, the AL penalty parameter η, and the update period P.

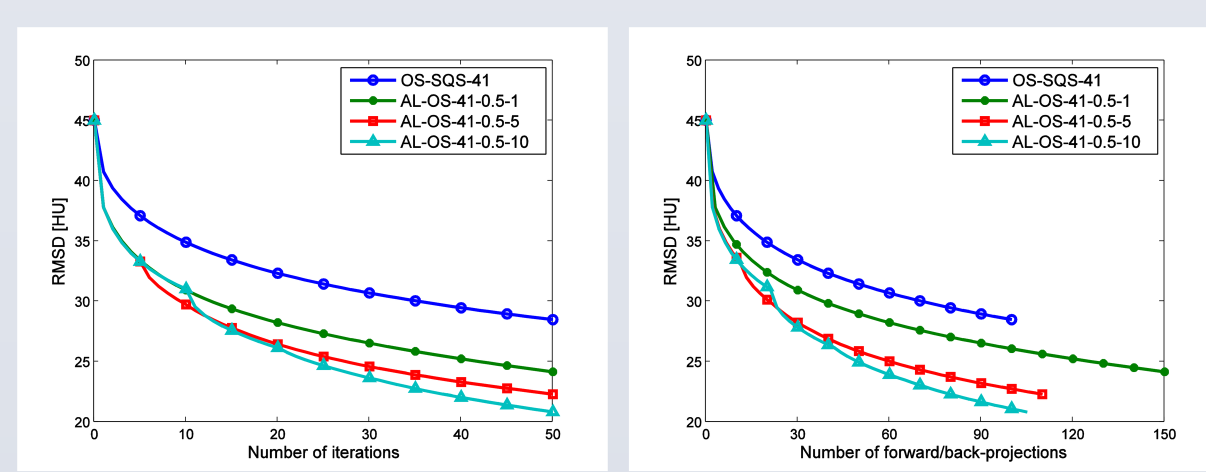

Figure 1. RMS differences between the reconstructed image and the converged reconstruction with different values of the update period P.

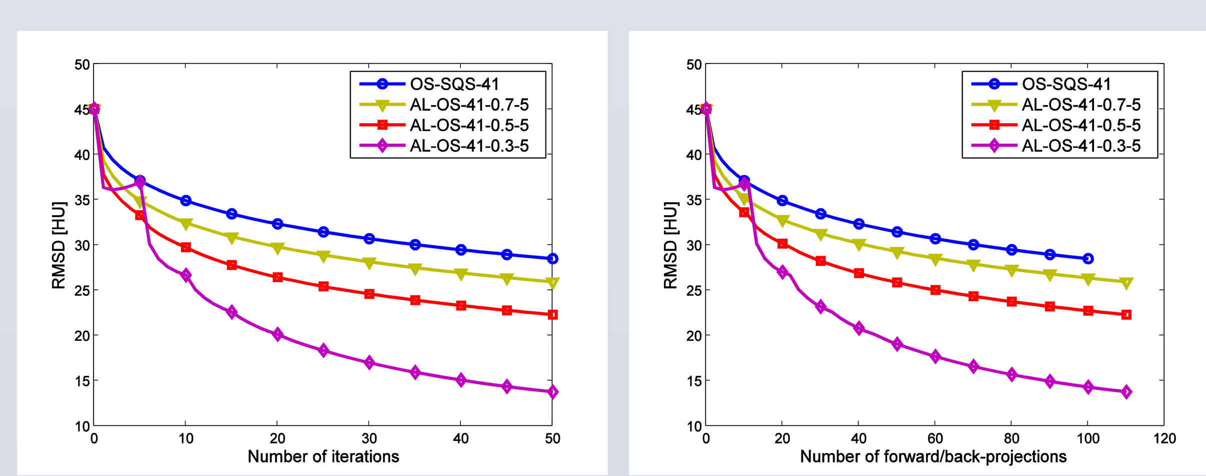

Figure 2. RMS differences between the reconstructed image and the converged reconstruction with different values of the AL penalty parameter η.

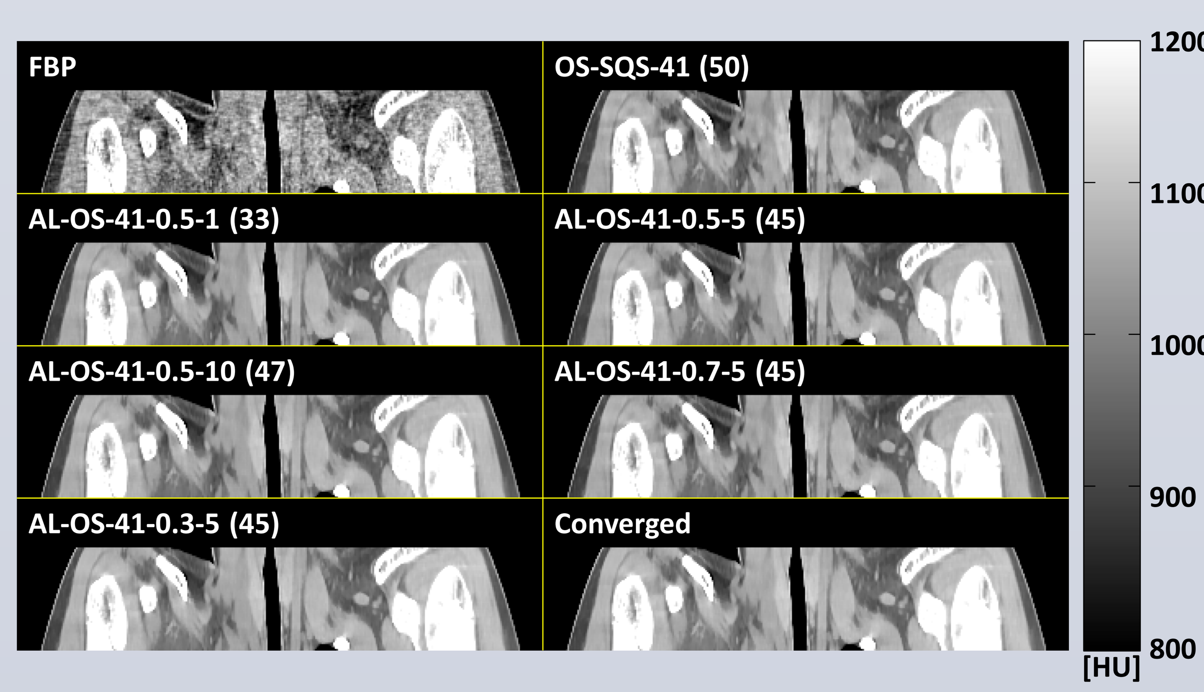

Figure 3. Cropped images from the central coronal plane of the reconstructed patient helical CT scan. Numbers in parentheses show the number of iterations of each algorithm so that the total number of forward/back-projections is approximately 100. The results of our proposed AL-OS algorithm with different values of η and P are less noisy and closer to the converged reconstruction.

RESULT (CONT'D)

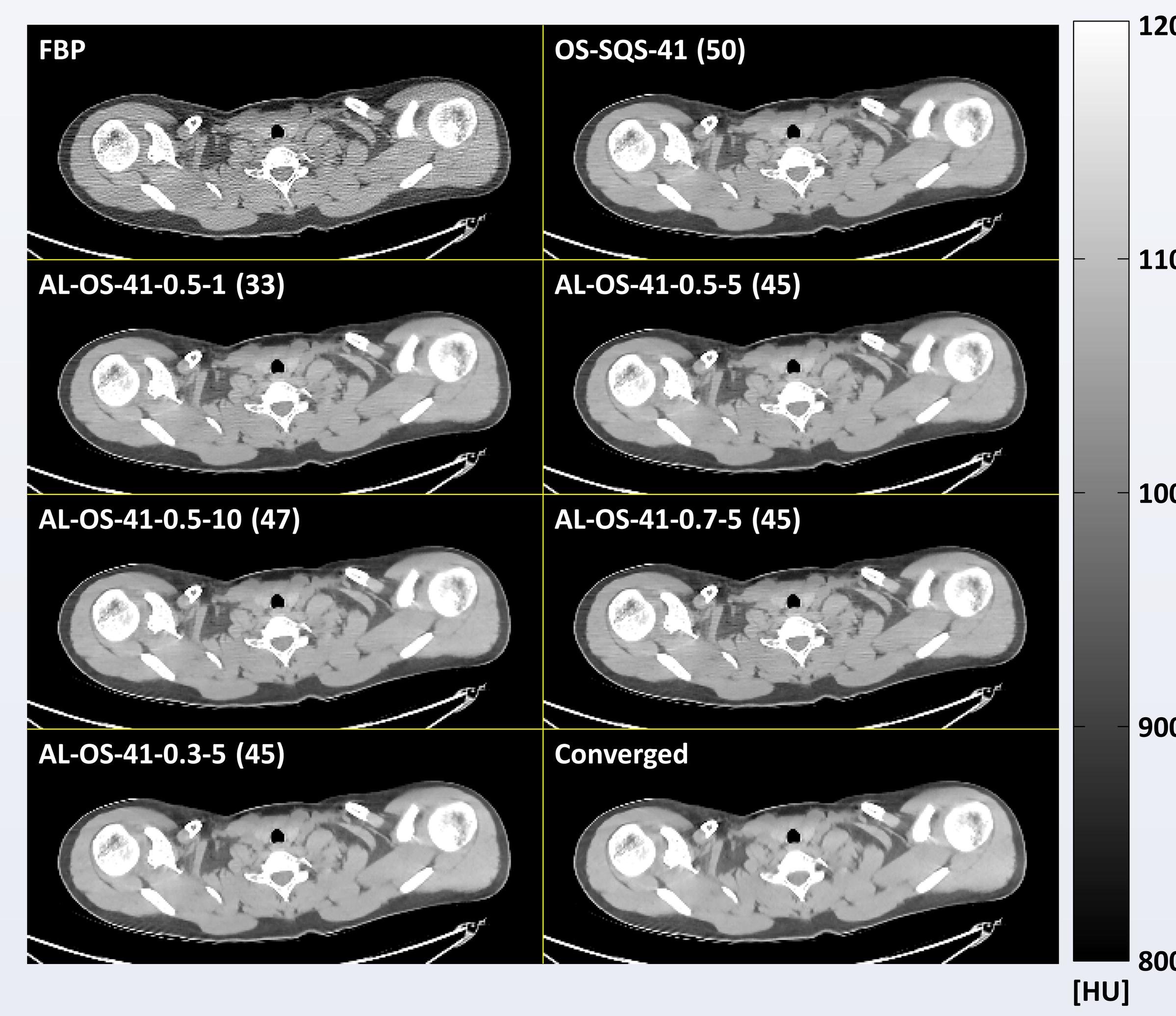

Figure 4. Cropped images from the central transaxial plane of the reconstructed patient helical CT scan. Numbers in parentheses show the number of iterations of each algorithm so that the total number of forward/back-projections is approximately 100. The **AL-OS-41-0.3-5** result after 45 iterations is very similar to the converged image, whereas the other images exhibit residual streak artifacts for the same computation time.

CONCLUSION

In this paper, we proposed to combine the AL method with OS for solving X-ray CT image reconstruction problems. Inspired by the convergence rate analysis of the AL method for quadratic regularized CT problems, we focused on a diagonal preconditioning matrix $\mathbf{G}$ that is proportional to the statistical weighting matrix $\mathbf{W}$. Experimental results show that the proposed AL-OS algorithm accelerates the standard OS algorithm remarkably. As can be seen from the convergence rate curves, smaller η leads to faster but *non-monotone* convergence. One possible future work is to consider using continuation, e.g., a decreasing sequence of η from 1 to 0, in the proposed AL-OS algorithm for further acceleration.

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