Point count of the variety of modules over the quantum plane over a finite field

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Background

Given a field $\mathbb{F}$ and $n \geq 0$, define the $n$-th **commuting variety** over $F$ as

$$K_{1,n}(\mathbb{F}) := \{(A, B) \in \text{Mat}_n(\mathbb{F}) \times \text{Mat}_n(\mathbb{F}) : AB = BA\}.$$ 

(The meaning of the notation will be clear later.)

What’s known:

- When $\mathbb{F} = \mathbb{C}$, the commuting variety $K_{1,n}(\mathbb{C})$ is a complex algebraic variety. Motzkin and Taussky (1955) and Gerstenhaber (1961) showed that $K_{1,n}(\mathbb{C})$ is irreducible.

- When $\mathbb{F} = \mathbb{F}_q$, the finite field of $q$ elements, the set $K_{1,n}(\mathbb{F}_q)$ is a finite set. Feit and Fine (1960) gave its cardinality by the formula:

$$\sum_{n=0}^{\infty} \frac{|K_{1,n}(\mathbb{F}_q)|}{(q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})} x^n = \prod_{i,j \geq 1} \frac{1}{1 - x^i q^{2-j}}. \quad (1)$$
A quantum deformation of the commuting variety has also been considered. Let \( \zeta \) be a nonzero element of \( \mathbb{F} \), define the \( n \)-th \( \zeta \)-commuting variety as

\[
K_{\zeta,n}(\mathbb{F}) := \{(A, B) \in \text{Mat}_n(\mathbb{F}) \times \text{Mat}_n(\mathbb{F}) : AB = \zeta BA\}.
\]

If \( \zeta = 1 \), then it simply becomes the commuting variety, hence the notation \( K_{1,n} \) for the commuting variety.

Efforts have been to spent to extend the work of Motzkin, Taussky and Gerstenhaber to the \( \zeta \)-commuting variety:

- Chen and Wang (2018) described the irreducible components of the anti-commuting variety \( K_{-1,n}(\mathbb{C}) \). There are more than one, unlike the \( \zeta = 1 \) case.
- Chen and Lu (2019) further extended the above result to general \( \zeta \).
Main result

We give a direct generalization of Feit–Fine’s formula.

**Theorem 1 (H., 2021).**

Let $\zeta$ be a nonzero element of $\mathbb{F}_q$, and let $m$ be the smallest positive integer such that $\zeta^m = 1$. Then

$$
\sum_{n=0}^{\infty} \frac{|K_{\zeta,n}(\mathbb{F}_q)|}{(q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})} x^n = \prod_{i=1}^{\infty} F_m(x^i; q),
$$

where

$$
F_m(x; q) := \frac{1 - x^m}{(1 - x)(1 - x^m q)} \cdot \frac{1}{(1 - x)(1 - x q^{-1})(1 - x q^{-2}) \cdots}.
$$

When $\zeta = 1$, we have $m = 1$, so $F_1(x^i; q) = \prod_{j \geq 1} \frac{1}{1 - x^i q^{2-j}}$ and we recover Feit–Fine.
The commuting variety $K_{1,n}(\mathbb{F})$ parametrizes and classifies finite-$\mathbb{F}$-dimensional modules over the polynomial ring $\mathbb{F}[X, Y]$. So $K_{1,n}(\mathbb{F})$ is also called the variety of modules over $\mathbb{F}[X, Y]$. To specify an $\mathbb{F}[X, Y]$-module with underlying space $\mathbb{F}^n$, it suffices to specify the $x$-action $A : \mathbb{F}^n \rightarrow \mathbb{F}^n$ and the $y$-action $B : \mathbb{F}^n \rightarrow \mathbb{F}^n$ under the constraint $AB = BA$. This constraint is because $x$ and $y$ commute in $\mathbb{F}[X, Y]$.

Similarly, the $\zeta$-commuting variety parametrizes finite-$\mathbb{F}$-dimensional modules over the associative algebra $\mathbb{F}\{X, Y\}/(XY - \zeta YX)$. This algebra is called the quantum plane, and is considered as a quantum deformation of $\mathbb{F}[X, Y]$. 
Remarks on Theorem 1

The cardinality of $K_{\zeta,n}(\mathbb{F}_q)$ depends only on the order of $\zeta$ as a root of unity of $\mathbb{F}_q$. This is expected.

The denominator $(q^n - 1)(q^n - q)\ldots(q^n - q^{n-1})$ is precisely the size of $\text{GL}_n(\mathbb{F}_q)$. This is the natural denominator in this type of generating function. In fact, the coefficient $|K_{\zeta,n}(\mathbb{F}_q)|/|\text{GL}_n(\mathbb{F}_q)|$ is the number of $n$-dimensional modules over the quantum plane up to isomorphism, each measured with a weight of $1/(\text{size of automorphism group})$.

Bavula (1997) classified simple modules over the quantum plane; Theorem 1 should encode statistical information about this classification.
Main result: further breakdown

We now state a refinement of Theorem 1. Let

\[ U_{\zeta,n}(\mathbb{F}_q) := \{(A, B) \in \text{GL}_n(\mathbb{F}_q) \times \text{Mat}_n(\mathbb{F}_q) : AB = \zeta BA\}, \]

and

\[ N_{\zeta,n}(\mathbb{F}_q) := \{(A, B) \in \text{Nilp}_n(\mathbb{F}_q) \times \text{Mat}_n(\mathbb{F}_q) : AB = \zeta BA\}. \]

It turns out that the varieties \( U_{\zeta,n}(\mathbb{F}_q) \) and \( N_{\zeta,n}(\mathbb{F}_q) \) are building blocks of \( K_{\zeta,n}(\mathbb{F}_q) \), in the sense that

\[
\sum_{n=0}^{\infty} \frac{|K_{\zeta,n}(\mathbb{F}_q)|}{|\text{GL}_n(\mathbb{F}_q)|} x^n = \left( \sum_{n=0}^{\infty} \frac{|U_{\zeta,n}(\mathbb{F}_q)|}{|\text{GL}_n(\mathbb{F}_q)|} x^n \right) \left( \sum_{n=0}^{\infty} \frac{|N_{\zeta,n}(\mathbb{F}_q)|}{|\text{GL}_n(\mathbb{F}_q)|} x^n \right)
\]

Recall that the left-hand side is the content of Theorem 1.
Main result: further breakdown

Theorem 2 (H., 2021)

Let $m$ be the order of $\zeta$. Then

$$\sum_{n=0}^{\infty} \frac{|U_{\zeta,n}(\mathbb{F}_q)|}{|\text{GL}_n(\mathbb{F}_q)|} x^n = \prod_{i=1}^{\infty} G_m(x^i; q),$$

where

$$G_m(x; q) := \frac{1 - x^m}{(1 - x)(1 - x^m q)}.$$

$$\sum_{n=0}^{\infty} \frac{|N_{\zeta,n}(\mathbb{F}_q)|}{|\text{GL}_n(\mathbb{F}_q)|} x^n = \prod_{i=1}^{\infty} H(x^i; q),$$

where

$$H(x; q) := \frac{1}{(1 - x)(1 - xq^{-1})(1 - xq^{-2}) \ldots}.$$
Remarks on Theorem 2

Theorem 2 can be interpreted as that in the formula

\[ F_m(x; q) := \frac{1 - x^m}{(1 - x)(1 - x^m q)} \cdot \frac{1}{(1 - x)(1 - xq^{-1})(1 - xq^{-2}) \ldots} \]

related to the count of \( \{(A, B) : AB = \zeta BA\} \), the factor

\[ \frac{1 - x^m}{(1 - x)(1 - x^m q)} \]

is the contribution of invertible \( A \), while the factor

\[ \frac{1}{(1 - x)(1 - xq^{-1})(1 - xq^{-2}) \ldots} \]

is the contribution of nilpotent \( A \).

Note that the latter does not depend on \( m \), so \( |N_{\zeta,n}(\mathbb{F}_q)| \) does not depend on \( \zeta \).
Ideas of proof: decomposition

- Given $A, B \in \text{Mat}_n(\mathbb{F}_q)$ such that $AB = \zeta BA$, by Fitting’s lemma, there is a unique direct sum decomposition $\mathbb{F}_q^n = V \oplus W$ such that $A(V) \subseteq V, A(W) \subseteq W$, $A|_V$ is invertible, and $A|_W$ is nilpotent.

- It turns out that $B$ must satisfy $B(V) \subseteq V, B(W) \subseteq W$. All we need in the proof is that $\zeta \neq 0$.

- This allows $K_{\zeta,n}(\mathbb{F}_q)$ to be “decomposed” into $U_{\zeta,n}(\mathbb{F}_q)$ (requiring invertible $A$) and $N_{\zeta,n}(\mathbb{F}_q)$ (requiring nilpotent $A$), in the sense of

$$
\sum_{n=0}^{\infty} \frac{|K_{\zeta,n}(\mathbb{F}_q)|}{|\text{GL}_n(\mathbb{F}_q)|} x^n = \left( \sum_{n=0}^{\infty} \frac{|U_{\zeta,n}(\mathbb{F}_q)|}{|\text{GL}_n(\mathbb{F}_q)|} x^n \right) \left( \sum_{n=0}^{\infty} \frac{|N_{\zeta,n}(\mathbb{F}_q)|}{|\text{GL}_n(\mathbb{F}_q)|} x^n \right)
$$
Ideas of proof: nilpotent part

- To compute $|N_{\zeta,n}(F_q)| = |\{(A, B) : AB = \zeta BA, A \text{ nilp}\}|$, we first fix $A$ and count the number of $B$.
- The number of $B$ only depends on the similarity class of $A$, so we may assume $A$ is in the Jordan canonical form.
- The general form of $B$ can then be determined entry-wise.
- In particular, the number of $B$ does not depend on $\zeta$ (even if $\zeta = 0$).
Ideas of proof: invertible part

- To compute $|U_{\zeta,n}(\mathbb{F}_q)| = |\{(A, B) : AB = \zeta BA, A \text{ invertible}\}|$, we first fix $B$ and count the number of $A$. (Opposite to the nilpotent case!!)

- Not every $B$ contributes. In order for the number of $A$ to be nonzero, we must have that $B$ is similar to $\zeta B$ (by the definition of similarity).

- Using the standard orbit-stabilizer argument, it suffices to count the number of similarity classes of $B$ such that $B$ is similar to $\zeta B$.

- **This is where $m$, the order of $\zeta$, matters.** The similarity class corresponds to a finite sequence $(g_1, g_2, \ldots)$ of monic polynomials over $\mathbb{F}_q$ such that $g_i$ divides $g_{i+1}$. Requiring $B$ to be similar to $\zeta B$ is equivalent to requiring every $g_i$ in the sequence of polynomials associated to $B$ to be of the following form: $t^d + c_1 t^{d-m} + c_2 t^{d-2m} + \ldots$
The numbers $|U_{\zeta,n}(\mathbb{F}_q)|$ and $|N_{\zeta,n}(\mathbb{F}_q)|$ look like two independent building blocks of $|K_{\zeta,n}(\mathbb{F}_q)|$.

However, in the commutative case $\zeta = 1$, the data of $|U_{1,n}(\mathbb{F}_q)|$ for all $n$ and $|N_{1,n}(\mathbb{F}_q)|$ for all $n$ recover each other.

The reason is from (commutative) algebraic geometry. This idea was used in an alternative proof of Feit–Fine formula given by Bryan and Morrison (2015). I will explain it in the next few slides.

For general $\zeta$, the number $|N_{\zeta,n}(\mathbb{F}_q)|$ does not seem to determine $|U_{\zeta,n}(\mathbb{F}_q)|$ because the former does not depend on $\zeta$ while the latter does.
We sketch their alternative proof of Feit–Fine’s formula.

- Compute \(|\{(A, B) \in \text{Nilp}_n(\mathbb{F}_q) \times \text{GL}_n(\mathbb{F}_q) : AB = BA\}|\) using the orbit-stabilizer argument.

- Use the notion of "power structure" due to Gusein-Zade, Luengo and Melle-Hernandez to recover \(|K_{1,n}(\mathbb{F}_q)|\) for all \(n\) from \(|\{(A, B) \in \text{Nilp}_n(\mathbb{F}_q) \times \text{GL}_n(\mathbb{F}_q) : AB = BA\}|\) for all \(n\). QED.

Remarks:

- The power structure is in the language of Grothendieck ring of complex varieties, but I will explain its consequence on point counting over finite fields in an elementary way.

- The point is that any one of \(|U_{1,n}(\mathbb{F}_q)|, |N_{1,n}(\mathbb{F}_q)|\) or their variants determines the rest. No variant is special; the one chosen by Bryan and Morrison is just the easiest to compute.
The reason why these quantities determine each other is that any of these counts and \( \nu_{n,q} := |\{(A, B) \in \text{Nilp}_n(\mathbb{F}_q) \times \text{Nilp}_n(\mathbb{F}_q) : AB = BA\}| \) determine each other; so \( \nu_{n,q} \) serves as a bridge to connect any two such quantities. I illustrate how \( |K_{1,n}(\mathbb{F}_q)| \) and \( \nu_{n,q} \) determine each other; other variants work the same way.

Recall that \( |K_{1,n}(\mathbb{F}_q)| \) “counts” (finite-\( \mathbb{F}_q \)-dimensional) modules over \( \mathbb{F}_q[X, Y] \). Such a module is determined by its localization at closed points of the affine plane \( \text{Spec} \mathbb{F}_q[X, Y] \). In other words, to classify \( \mathbb{F}_q[X, Y] \)-modules, it suffices to classify \( \mathbb{F}_q[X, Y] \)-modules supported at each given closed point.

On the other hand, \( \nu_{n,q} \) “counts” \( \mathbb{F}_q[X, Y] \)-modules supported at the origin (i.e., the maximal ideal \( (x, y) \)).
The key is that every closed point of $\text{Spec } \mathbb{F}_q[X, Y]$ “looks like” the origin, in the sense of Cohen’s structure theorem: for any maximal ideal $m$ of $\mathbb{F}_q[X, Y]$, the complete localization of $\mathbb{F}_q[X, Y]$ at $m$ is isomorphic to $\mathbb{F}[[X, Y]]$, where $\mathbb{F}$ is a finite extension of $\mathbb{F}_q$.

Hence $\nu_{n,q}$ determines $|K_{1,n}|$. In terms of formula, given

$$\sum_{n=0}^{\infty} \frac{\nu_{n,q}}{|\text{GL}_n(\mathbb{F}_q)|} x^n = \prod_{i, j \geq 1} \frac{1}{1 - x^i q^{-j}},$$

the geometric argument above shows that $\sum_{n=0}^{\infty} \frac{|K_{1,n}(\mathbb{F}_q)|}{|\text{GL}_n(\mathbb{F}_q)|} x^n$ is obtained by replacing $\frac{1}{1-x^i q^{-j}}$ by $Z_{\mathbb{F}_q[X,Y]}(x^i q^{-j})$, where $Z_{\mathbb{F}_q[X,Y]}$ is the Hasse–Weil zeta function of $\text{Spec } \mathbb{F}_q[X, Y]$. The reason why the Hasse–Weil zeta function appears is because its Euler product over all closed points.
Using $Z_{\mathbb{F}_q[X,Y]}(u) = 1/(1 - uq^2)$, we get

$$
\sum_{n=0}^{\infty} \frac{|K_{1,n}(\mathbb{F}_q)|}{|\text{GL}_n(\mathbb{F}_q)|} x^n = \prod_{i,j \geq 1} Z_{\mathbb{F}_q[X,Y]}(x^i q^{-j}) = \prod_{i,j \geq 1} \frac{1}{1 - x^i q^{2-j}},
$$

precisely the formula of Feit and Fine. This is how $\nu_{n,q}$ for all $n$ determines $|K_{1,n}(\mathbb{F}_q)|$ for all $n$.

In fact, the above process can be reversed, which is not surprising because $|K_{1,n}(\mathbb{F}_q)|$ is determined by $\nu_{n,q}$ alone, again thanks to the homogeneity of $\text{Spec} \mathbb{F}_q[X,Y]$. We can recover $\nu_{n,q}$ from $|K_{1,n}(\mathbb{F}_q)|$ as well.

This finishes the explanation why $K_{1,n}(\mathbb{F}_q)$ and $\nu_{n,q}$ determines each other.
Similarly, $N_{1,n}(\mathbb{F}_q)$ classifies $\mathbb{F}_q[X,Y]$-modules supported on the axis $X = 0$, while $U_{1,n}(\mathbb{F}_q)$ classifies $\mathbb{F}_q[X,Y]$-modules supported on the open set $X \neq 0$. Since each of these subsets consist of closed points that “look the same”, the same argument applies and we have

$$|N_{1,n}(\mathbb{F}_q)| \text{ for all } n \longleftrightarrow \nu_{n,q} \text{ for all } n \longleftrightarrow |U_{1,n}(\mathbb{F}_q)| \text{ for all } n,$$

where $\longleftrightarrow$ means “determines each other”.

We note that the notion of localization plays a key role; if we could not break down to closed points, the numbers $|N_{1,n}(\mathbb{F}_q)|$ and $|U_{1,n}(\mathbb{F}_q)|$ would not have been related because $X = 0$ and $X \neq 0$ are disjoint! Of course, another necessary ingredient is Cohen’s structure theorem: closed points look the same everywhere.
Noncommutative case?

Question

Can we find a geometric connection between $|N_{\zeta, n}(F_q)|$ and $|U_{\zeta, n}(F_q)|$, similar to the $\zeta = 1$ case explained before?

Recall that $m$ is the order of $\zeta$, and

$$\sum_{n=0}^{\infty} \frac{|N_{\zeta, n}(F_q)|}{|GL_n(F_q)|} x^n \sim \frac{1}{(1 - x)(1 - xq^{-1})(1 - xq^{-2}) \ldots}.$$  

$$\sum_{n=0}^{\infty} \frac{|U_{\zeta, n}(F_q)|}{|GL_n(F_q)|} x^n \sim \frac{1 - x^m}{(1 - x)(1 - x^mq)};$$

It seems impossible that $|N_{\zeta, n}(F_q)|$ determines $|U_{\zeta, n}(F_q)|$, since the former doesn’t depend on $m$ while the latter does. However, it is still possible that $|U_{\zeta, n}(F_q)|$ can be recovered from $|N_{\zeta, n}(F_q)|$ together with the geometry of the quantum plane $XY = \zeta YX$ (which depends on $m$).
We extend a formula that counts matrix pairs $AB = BA$ to the case $AB = \zeta BA$ where $\zeta$ is nonzero. The answer depends on the order of $\zeta$ as a root of unity.

The count of $AB = \zeta BA$ encodes statistical information about modules over the quantum plane.

The count in question has two seemingly independent building blocks that turn out to be interdependent in the $\zeta = 1$ case, using ingredients from (commutative) algebraic geometry. I hope that the study of a possible interdependence in the case of general $\zeta$ will inspire interesting noncommutative geometry.