COHOMOLOGY OF CONFIGURATION SPACES ON PUNCTURED VARIETIES

YIFENG HUANG

Abstract. Given a smooth quasiprojective variety Y over $\mathbb{C}$ that is not projective, consider its unordered configuration spaces $\mathrm{Conf}^n(Y)$ for $n \geq 0$. Remove a point P of Y and obtain a one-puncture $Y-P$ of Y. We give a decomposition formula that computes the singular cohomology groups of $\mathrm{Conf}^n(Y-P)$ in terms of those of $\mathrm{Conf}^m(Y)$ ($0 \leq m \leq n$), and prove it for several families of examples of Y, including the case where Y is obtained from a smooth projective variety by puncturing one or more points. This formula keeps track of the mixed Hodge structures of the cohomology groups as well. This result simultaneously implies a result of Kallel involving Betti numbers and a consequence of a combinatorial property of configuration spaces due to Vakil and Wood. We also obtain intermediate results involving ordered configuration spaces that potentially work for more examples of Y.

1. Introduction

In this paper, the word *variety* always means a smooth quasiprojective reduced variety over $\mathbb{C}$, so a compact variety means a projective variety. Let Y be a variety. Consider the *ordered configuration space*

$$F(Y,n) := Y^n - \bigcup_{i<j}\{(x_1,...,x_n) \in Y^n : x_i = x_j\}$$

and the *unordered configuration space*

$$\mathrm{Conf}^n(Y) := F(Y,n)/S_n,$$

where the symmetric group S_n acts by permuting coordinates. The singular (co)homology of configuration spaces has played an important role in the study of their topology and has a rich literature (to list a few, [1, 3, 20]).

Let Y be a smooth noncompact variety, and P be a point of Y. Consider the open subvariety $Y-P$ of Y obtained from removing the point P. Existing works suggest that puncturing one point out of a variety has a consistent effect on the singular cohomology groups of configuration spaces,

We thank Gilyoung Cheong, Mircea Mustaţă, Roberto Pagaria, Dan Petersen, Andrew Snowden, John Stembridge, Burt Totaro and Craig Westerland for helpful conversations and useful comments. We thank Christin Bibby and Philip Tosteson for informing the author about numerous crucial results and references. This work was done with the support of Research Training Grant (RTG): Number Theory and Representation Theory at the University of Michigan.

and this effect is usually called the “splitting theorem”; see [12, §1.2] for a brief history. In the case where Y is a smooth noncompact algebraic curve, Napolitano [15, Theorem 2] obtained a formula that relates the cohomology groups of $\mathrm{Conf}^n(Y-P)$ to those of $\mathrm{Conf}^m(Y)$ for $0 \le m \le n$:

$$H^i(\mathrm{Conf}^n(Y-P),\mathbb{Z}) \cong \bigoplus_{t=0}^{\infty} H^{i-t}(\mathrm{Conf}^{n-t}(Y),\mathbb{Z}) \tag{1.1}$$

where H^i denotes the i-th singular cohomology of the analytic topology, and $H^i(\mathrm{Conf}^j(Y),\mathbb{Z}) = 0$ for $j < 0$ and all i by convention. In particular, the right hand side is a finite sum.

Kallel [12, Theorem 1.5(4)] generalizes the above result to the case where Y is an *r-punctured variety*, namely, a smooth projective variety with $r \ge 1$ points removed, but in rational coefficients:

$$H^i(\mathrm{Conf}^n(Y-P),\mathbb{F}) \cong \bigoplus_{t=0}^{\infty} H^{i-(2d-1)t}(\mathrm{Conf}^{n-t}(Y),\mathbb{F}) \tag{1.2}$$

where $d = \dim_{\mathbb{C}} Y$. (This is a cleaned-up restatement of [12, Theorem 1.5(4)]; see Section Corollary 1.5 for a comparison.) The focus of this paper is to study the splitting formula for more examples of Y and in a stronger sense. We remark that none of the above results is true if Y is compact, even when $Y = \mathbb{P}^1$. In fact, the compact case satisfies a different formula [12, Theorem 1.5(3)] in $\mathbb{Z}/2\mathbb{Z}$ coefficients: if M is a closed connected manifold, then

$$\begin{aligned} &H^i(\mathrm{Conf}^n(M),\mathbb{Z}/2\mathbb{Z}) \\ &\quad\cong H^i(\mathrm{Conf}^n(M-P),\mathbb{Z}/2\mathbb{Z}) \oplus H^{i-\dim_{\mathbb{R}} M}(\mathrm{Conf}^{n-1}(M-P),\mathbb{Z}/2\mathbb{Z}). \end{aligned}$$

In the main theorem below, we will state a generalization of Kallel’s formula (1.2). Moreover, it states a splitting formula for the *ordered* configuration spaces as well, and the comparison of the singular cohomology groups also keeps track of a finer structure, namely the mixed Hodge structure due to Deligne. For an introduction to this topic, see for example [5] and [17]. An (integral or rational) *mixed Hodge structure* is a linear-algebraic construction, and Deligne equipped $H^i(X,\mathbb{Z})$ with a mixed Hodge structure for any complex variety X. To any mixed Hodge structure H, we can associate $\mathbb{C}$-vector spaces $H^{p,q}$ for $p,q \ge 0$, which are subquotients of $H \otimes \mathbb{C}$. The *mixed Hodge numbers* are $h^{p,q}(H) := \dim_{\mathbb{C}} H^{p,q}$. The mixed Hodge numbers of $H^i(X,\mathbb{Q})$ are simply called the mixed Hodge numbers of X, denoted by $h^{p,q;i}(X)$ (see [11]). These numbers are algebro-geometric invariants of X of particular interest, and they provide a refinement of the rational Betti numbers $h^i(X)$, in the sense that $h^i(X) = \sum_{p,q\ge 0} h^{p,q;i}(X)$. As far as we know, Theorem 1.1 below is the first result of this kind that keeps track of the mixed Hodge numbers.

We introduce the following notions in order to state the result in a stronger sense. Mixed Hodge structures form an abelian category, and we denote the Grothendieck group of mixed Hodge structures by $K_0(\mathrm{MHS})$. For example, if rational mixed Hodge structures H and H' are both an iterated extension of mixed Hodge structures $W_1, \ldots, W_m$ (i.e., H has a filtration whose successive quotients are W_i, and so does H'), then H and H' have the same class $[H] = [H']$ in $K_0(\mathrm{MHS})$, though they may not be isomorphic as mixed Hodge structures. We say that mixed Hodge structures H and H' are *virtually equivalent* if $[H] = [H'] \in K_0(\mathrm{MHS})$, and we denote this by $H \sim H'$. Virtually equivalent mixed Hodge structures have the same mixed Hodge numbers. The discussion can be repeated for the category of S_n-mixed Hodge structures (namely, mixed Hodge structures with an S_n action).

Theorem 1.1 (Main Result)**.** *If Y is a connected smooth noncompact complex variety of dimension d in one of the cases below, and P is a point of Y, then we have a virtual equivalence of mixed Hodge structures*

$$H^i(\mathrm{Conf}^n(Y - P), \mathbb{Q}) \sim \bigoplus_{t=0}^{\infty} H^{i-(2d-1)t}(\mathrm{Conf}^{n-t}(Y), \mathbb{Q})(-d \cdot t) \tag{1.3}$$

where $(-dt)$ denotes the Tate twist (raising the Hodge type by (dt, dt)). Moreover, we have an isomorphism of S_n representations over $\mathbb{Q}$ and an (S_n-equivariant) virtual equivalence of S_n-mixed Hodge structures

$$H^i(F(Y - P, n), \mathbb{Q}) \sim \bigoplus_{t=0}^{\infty} \mathrm{Ind}_{S_{n-t}}^{S_n} H^{i-(2d-1)t}(F(Y, n - t), \mathbb{Q})(-d \cdot t), \tag{1.4}$$

where $\mathrm{Ind}_{S_{n-t}}^{S_n}$ is the induction operator for group representations, and the symmetric groups act on the cohomology groups via the natural pullback maps. For explicit formulas about Betti numbers and mixed Hodge numbers, see Section 1.1. Here are the choices of Y:

1. *Y is a smooth projective variety with $r \geq 1$ points removed;*
2. *Y is the affine space $\mathbb{C}^d$ (or the complement of a union of hyperplanes in it), or the torus $(\mathbb{C}^*)^d$ (or the complement of a union of codimension-1 subtori in it), with $r \geq 0$ points removed;*
3. *Y is $\mathbb{P}^2 - C$ with $r \geq 0$ points removed, where C is a smooth plane curve in $\mathbb{P}^2$.*

We note that the case (1) above implies Kallel's formula (1.2) as a corollary, and upgrade it from a comparison of Betti numbers to a comparison of mixed Hodge numbers. It is worth pointing out that we actually get even more information in this case: we get explicit vector space isomorphisms whose interaction with the cup product is understood. See Remark 3.5(1).

As an application, one can combine Theorem 1.1(1) and a recent formula in [4] about mixed Hodge numbers of $\mathrm{Conf}^n(E - P)$, where $E - P$ is an elliptic curve minus one point, to obtain the mixed Hodge structure of $\mathrm{Conf}^n(E_r)$, where E_r is an r-punctured elliptic curve with $r > 0$. The author is also working on the mixed Hodge structure of $\mathrm{Conf}^n(C-P)$ where C is a smooth projective curve of genus $g \geq 2$ (work in progress), and this together with Theorem 1.1 can determine the mixed Hodge numbers of configuration spaces on *any* smooth noncompact curve over $\mathbb{C}$. It would then imply the formula [7, Proposition 3.5] about the Betti numbers of configuration spaces on r-punctured Riemann surfaces.

If one uses the Poincaré duality to convert the singular cohomology H^* to the compactly supported singular cohomology H_c^*, then Theorem 1.1 becomes

$$H_c^i(\mathrm{Conf}^n(Y-P),\mathbb{Q}) \sim \bigoplus_{t=0}^{\infty} H_c^{i-t}(\mathrm{Conf}^{n-t}(Y),\mathbb{Q}) \tag{1.5}$$

$$H_c^i(F(Y-P,n),\mathbb{Q}) \sim \bigoplus_{t=0}^{\infty} \mathrm{Ind}_{S_{n-t}}^{S_n} H_c^{i-t}(F(Y,n-t),\mathbb{Q}) \tag{1.6}$$

Note that the dimension of Y no longer appears in the formulas, and they are still virtual equivalences of mixed Hodge structures even though the Tate twist disappears. Extracting the mixed Hodge numbers of both sides, we obtain an identity between generating functions of *compactly supported mixed Hodge numbers* $h_c^{p,q;i}$ (see [11] for the definition):

$$\sum_{p,q,i,n\geq 0} h_c^{p,q;i}(\mathrm{Conf}^n(Y-P))x^p y^q u^i t^n = \frac{1}{1-ut}\sum_{p,q,i,n\geq 0} h_c^{p,q;i}(\mathrm{Conf}^n(Y))x^p y^q u^i t^n$$

The specialization of this identity obtained by substituting $u=-1$ is a consequence of a combinatorial result by Vakil and Wood [21, Proposition 5.9]. In a sense, (1.3) is the simplest upgrade of Kallel's Betti number result (1.2) that is compatible with [21, Proposition 5.9], supporting the philosophical principle of Occam's razor for Hodge structures mentioned in [21].

1.1. **Main result in elementary form.**

For clarity, we state several interesting elementary consequences of Theorem 1.1.

Corollary 1.2 (Betti numbers)**.** *Let Y satisfy the assumption in Theorem 1.1 and $d=\dim_{\mathbb{C}} Y$. Then we have a formula of rational Betti numbers*

$$h^i(\mathrm{Conf}^n(Y-P)) = \sum_{t=0}^{\infty} h^{i-(2d-1)t}(\mathrm{Conf}^{n-t}(Y)).$$

As S_n-representations over $\mathbb{Q}$, we have a (possibly noncanonical) isomorphism

$$H^i(F(Y-P,n),\mathbb{Q}) \cong \bigoplus_{t=0}^{\infty} \mathrm{Ind}_{S_{n-t}}^{S_n} H^{i-(2d-1)t}(F(Y,n-t),\mathbb{Q}).$$

Proof. These are (1.3) and (1.4), ignoring the mixed Hodge structure and the Tate twist. We get isomorphisms instead of virtual equivalences because S_n-representations over a field of characteristic zero are semisimple. □

Let $H^{p,q;i}(X)$ denote the (p,q)-piece of the mixed Hodge structure $H^i(X,\mathbb{Q})$. Recall that $H^{p,q;i}(X)$ is a subquotient of $H^i(X,\mathbb{C})$ and $h^{p,q;i}(X)=\dim_{\mathbb{C}} H^{p,q;i}(X)$. The construction of $H^{p,q;i}(X)$ is functorial, so that if a finite group G acts on X as variety isomorphisms, then G acts on $H^{p,q;i}(X)$ via pullback.

Corollary 1.3 (Mixed Hodge numbers)**.** *Let Y satisfy the assumption in Theorem 1.1 and $d=\dim_{\mathbb{C}} Y$. Then we have a formula of mixed Hodge numbers*

$$h^{p,q;i}(\mathrm{Conf}^n(Y-P))=\sum_{t=0}^{\infty} h^{p-dt,q-dt;i-(2d-1)t}(\mathrm{Conf}^{n-t}(Y)).$$

As S_n-representations over $\mathbb{C}$, we have a (possibly noncanonical) isomorphism

$$H^{p,q;i}(F(Y-P,n))\cong\bigoplus_{t=0}^{\infty}\mathrm{Ind}_{S_{n-t}}^{S_n} H^{p-dt,q-dt;i-(2d-1)t}(F(Y,n-t)).$$

Corollary 1.4 (Generating functions)**.** *Let Y satisfy the assumption in Theorem 1.1 and $d=\dim_{\mathbb{C}} Y$. Then*

$$\sum_{i,n\geq 0} h^i(\mathrm{Conf}^n(Y-P))u^i t^n=\frac{1}{1-u^{2d-1}t}\sum_{i,n\geq 0} h^i(\mathrm{Conf}^n(Y))u^i t^n$$

and

$$\sum_{p,q,i,n\geq 0} h^{p,q;i}(\mathrm{Conf}^n(Y-P))x^p y^q u^i t^n$$
$$=\frac{1}{1-(xy)^d u^{2d-1}t}\sum_{p,q,i,n\geq 0} h^{p,q;i}(\mathrm{Conf}^n(Y))x^p y^q u^i t^n.$$

Corollary 1.5 (Kallel's original statement)**.** *If M is a smooth compact complex variety of dimension d, and $\{P_1,\dots,P_k\}$ are distinct points of M, then*

$$H^i(\mathrm{Conf}^n(M-\{P_1,\dots,P_k\}),\mathbb{Q})$$
$$\cong\sum_{r=0}^{n} H^{i-(n-r)(2d-1)}(\mathrm{Conf}^r(M-P_1),\mathbb{Q})^{\oplus p(k-1,n-r)},$$

where $p(s,t)$ is the number of ordered partitions of t into s nonnegative integers.

Proof. Applying the first part of Corollary 1.4 to $Y=M-P_1, Y=M-\{P_1,P_2\},\dots,Y=M-\{P_1,\dots,P_{k-1}\}$ in order ($k-1$ times in total), all satisfying the case (1) of Theorem 1.1, we get

$$\sum_{i,n\geq 0} h^i(\mathrm{Conf}^n(M-\{P_1,\dots,P_k\}))u^it^n = \frac{1}{(1-u^{2d-1}t)^{k-1}} \sum_{i,n\geq 0} h^i(\mathrm{Conf}^n(M-P_1))u^it^n$$

The rest is a consequence of the binomial series

$$(1-x)^{-n} = \sum_{i=0}^{\infty} \binom{n+i-1}{i} x^i = \sum_{i=0}^{\infty} p(n,i)x^i.$$

□

1.2. **Technical ingredients.** We first notice that (1.4) of Theorem 1.1 implies (1.3) by taking S_n-invariants on both sides, where the S_n-invariant of the right hand side of (1.4) is computed using the Frobenius reciprocity (or by Shapiro's lemma [23, p. 171]). Thus it suffices to study the ordered configuration spaces.

We then use the spectral sequence described by Tosteson [19] (whose Poincaré dual also appears in [18]) to describe the cohomology of the ordered configuration spaces. We remark that to prove the case where $\dim_{\mathbb{C}} Y = 1$, it suffices to consider a well understood spectral sequence described by many authors (for example, [2, 8, 13]) in different forms. We refer the readers to [8, Theorem 3.1] for a statement and a discussion of this spectral sequence.

Theorem 1.1 relies on the following theorem about the spectral sequence, which can potentially be applied to settings more general than Theorem 1.1.

Theorem 1.6. *Let X be a connected smooth complex variety of dimension d, and $P_1, \dots, P_r$ $(r \geq 1)$ are distinct points of X. Write $Y = X - \{P_1, \dots, P_{r-1}\}$ and $P = P_r$. Then*

(1) *There are spectral sequences of mixed Hodge structures $E_1^{i,j}(Y-P,n) \implies H^{i-j}(F(Y-P,n),\mathbb{Z})$ and $E_1^{i,j}(Y,n) \implies H^{i-j}(F(Y,n),\mathbb{Z})$ for all n, whose first page is described in Section Proposition 2.1, such that there exists an isomorphism (see Lemma 3.1) of rational mixed Hodge structures*

$$\Phi : E_1^{i,j}(Y,n)_{\mathbb{Q}} \oplus \mathrm{Ind}_{S_{n-1}}^{S_n} E_1^{i-2d,j-1}(Y-P,n-1)_{\mathbb{Q}}(-d) \to E_1^{i,j}(Y-P,n)_{\mathbb{Q}}$$

that commutes with the S_n action, where $(\cdot)_{\mathbb{Q}}$ means tensoring with $\mathbb{Q}$.

(2) *If X is noncompact, then Φ preserves the first-page differential $d_1^{i,j} : E_1^{i,j} \to E_1^{i,j-1}$.*

(3) *If there is a rational number $1 \leq w \leq 2$ such that $H^i(X,\mathbb{Q})$ is pure of weight $w \cdot i$ for all i (which forces $H^i(X,\mathbb{Q}) = 0$ whenever $w \cdot i$ is not an integer), then both spectral sequences $E(Y-P,n)_{\mathbb{Q}}$ and $E(Y,n)_{\mathbb{Q}}$ degenerate at E_2, i.e., all higher-page differentials d_r $(r \geq 2)$ vanish.*

We remark that the actual assumption for Y in Theorem 1.1 is $Y = X - \{\text{zero or more points}\}$ where X satisfies the assumptions of Theorem 1.6(2)(3) above. These assumptions are satisfied in the three cases of Theorem 1.1, see Section 3.4. The cases listed in Theorem 1.1 is far from being exhaustive; for example, if X satisfies Theorem 1.6(2)(3), then so does its scheme-theoretic quotient by a finite group.

We give an overview of how Theorem 1.6 implies Theorem 1.1.

Sketch of Proof of Theorem 1.1. Let X, Y be as in Theorem 1.6 such that the assumptions for X in (2)(3) are satisfied.

By Theorem 1.6(1)(2), we have an isomorphism on the second page:

$$E_2^{i,j}(Y,n)_{\mathbb{Q}} \oplus \operatorname{Ind}_{S_{n-1}}^{S_n} E_2^{i-2d,j-1}(Y-P,n-1)_{\mathbb{Q}}(-d) \to E_2^{i,j}(Y-P,n)_{\mathbb{Q}}$$

Due to degeneracy, the E_2 page is the same as the E_∞ page. Taking iterated extensions of both sides over all i, j with $i - j = k$, we get

$$H^k(F(Y,n),\mathbb{Q}) \oplus \operatorname{Ind}_{S_{n-1}}^{S_n} H^{k-(2d-1)}(F(Y-P,n-1),\mathbb{Q}))(-d) \sim H^k(F(Y-P,n),\mathbb{Q})$$

This implies (1.4) by induction on n. □

We wonder if Theorem 1.1 can be generalized to more cases and/or strengthened to integer coefficients. One may also ask if we can compare the mixed Hodge structures in a sense stronger than the virtual equivalence, even isomorphism. In fact, something slightly better can be said in all three cases of Theorem 1.1, see Remark 3.5(2).

Conjecture 1.7. Theorem 1.1 still holds if Y is any connected smooth noncompact complex variety, and if the coefficient ring $\mathbb{Q}$ is replaced by $\mathbb{Z}$.

Remark 1.8.

(1) If we ignore the mixed Hodge structure, Theorem 1.1 is still new in (1.4) as well as the case (3). As a topological question, one may study a possible analogue of Theorem 1.6 (ignoring the mixed Hodge structure) in an attempt to extend Kallel's (1.2) to the case of orientable even-dimensional open manifolds that are not necessarily complex varieties.

(2) The same method also works in the setting of Galois representations given by the étale ℓ-adic cohomology. An analogue of Theorem 1.1 holds as an isomorphism of Galois representations up to semisimplification, which remembers Frobenius eigenvalues in particular.

1.3. **Organization of the paper.** The rest of the paper is focused on the proof of Theorem 1.6. Section 2 is devoted to Proposition 2.3, an explicit description of the spectral sequence used in Theorem 1.6. Section 3 is the core of the paper, where we prove Theorem 1.6 by computations based on Proposition 2.3. In Section 4, we remark on several possible approaches to further the main result.

Contents

2. Description of the spectral sequence

2.1. **Arrangements and Orlik–Solomon algebra.** We recall some preliminaries about hyperplane arrangements and the Orlik–Solomon algebra. For a reference, see for instance [16].

A *hyperplane arrangement* is a set $\mathcal{A} := \{Y_1, \dots, Y_h\}$ of complex hyperplanes in $\mathbb{C}^n$. A *stratum* F of $\mathcal{A}$ is the intersection of zero or hyperplanes in $\mathcal{A}$ (so $\mathbb{C}^n$ is also a stratum). The *lattice* $L(\mathcal{A})$ of $\mathcal{A}$ is the partially ordered set (poset) of strata of $\mathcal{A}$, ordered by inclusion. It has a top element $\hat{1} := \mathbb{C}^n$. In the rest of this section, all of the constructions associated to $\mathcal{A}$ are determined by the poset $L(\mathcal{A})$ alone.

The *rank* of a stratum F is its complex codimension in $\mathbb{C}^n$, or equivalently, the length of any maximal chain from F to $\hat{1}$. A subset S of $\mathcal{A}$ is called (1) *independent*, if $\bigcap S := \bigcap_{F\in S} F \neq \varnothing$ and $\operatorname{rk}(\bigcap S) = |S|$; (2) *dependent*, if $\bigcap S \neq \varnothing$ and $\operatorname{rk}(\bigcap S) < |S|$; (3) *vanishing*, if $\bigcap S = \varnothing$.

Define the algebra $B(\mathcal{A})$ to be the free graded-commutative $\mathbb{Z}$-algebra (i.e. the exterior algebra) generated by degree-one elements $e_Y, Y \in \mathcal{A}$. Let $\partial : B(\mathcal{A}) \to B(\mathcal{A})$ be the unique $\mathbb{Z}$-linear map on $B(\mathcal{A})$ that satisfies the graded Leibniz rule and $\partial e_Y = 1$ for $Y \in \mathcal{A}$. It has the property that $\partial^2 = 0$. For an ordered subset S of $\mathcal{A}$, we denote $e_S := \prod_{Y\in S} e_Y$, with $g_\varnothing = 1$ and $\bigcap \varnothing = X$ by convention. Define $I(\mathcal{A})$ to be the ideal $B(\mathcal{A})$ generated by e_S for S vanishing and $\partial(e_S)$ for S dependent. As a fact, it is enough to generate $I(\mathcal{A})$ using e_S for minimal vanishing sets S and $\partial(e_S)$ for minimal dependent sets S. Denote the image of $e_Y \in B(\mathcal{A})$ in $A(\mathcal{A})$ by g_Y. Define the *Orlik–Solomon algebra* of $\mathcal{A}$ (or of $L(\mathcal{A})$) as $A(\mathcal{A}) := B(\mathcal{A})/I(\mathcal{A})$. The differential ∂ on $B(\mathcal{A})$ descends to a differential $\partial : A(\mathcal{A}) \to A(\mathcal{A})$, making $A(\mathcal{A})$ a differential graded algebra.

More concretely, $A(\mathcal{A})$ is the graded-commutative $\mathbb{Z}$-algebra generated by degree-one elements g_Y, $Y \in \mathcal{A}$ with the following relations:

(1) $g_S = 0$ if S is vanishing;
(2) If $S = \{Y_{i_1}, \dots, Y_{i_k}\}$ is dependent, then

$$\sum_{j=1}^{k} (-1)^{j+1} g_{Y_{i_1}} \dots \widehat{g}_{Y_{i_j}} \dots g_{Y_{i_k}} = 0$$

where the notation $\widehat{g}_{Y_{i_j}}$ means skipping j-th factor in the product. Note that this implies that $g_S = 0$ for S dependent.

The *complement* of $\mathcal{A}$ is the complex variety $M(\mathcal{A}) := \mathbb{A}^n - \bigcup_{Y \in \mathcal{A}} Y$. The cohomology ring $H^*(M(\mathcal{A})) := \bigoplus_{i=0}^{\infty} H^i(M(\mathcal{A}), \mathbb{Z})$ is isomorphic to the graded algebra $A(\mathcal{A})$.

Let F be a stratum of $\mathcal{A}$. Consider the subarrangement $\mathcal{A}_F := \{Y \in \mathcal{A} : Y \supseteq F\}$, then we have $M(\mathcal{A}) \subseteq M(\mathcal{A}_F)$, and the pullback of the inclusion map gives a morphism of graded algebras $H^*(M(\mathcal{A}_F)) \to H^*(M(\mathcal{A}))$. Via the identification above, this is the ring homomorphism $A(\mathcal{A}_F) \to A(\mathcal{A})$ that sends g_Y to g_Y. If $F' \subseteq F$ are two strata of $\mathcal{A}$, then we have a natural map $A(\mathcal{A}_F) \to A(\mathcal{A}_{F'})$.

Let $A_F(\mathcal{A})$ be the abelian subgroup of $A(\mathcal{A})$ generated by g_S with $\bigcap S = F$. Then $A(\mathcal{A}) = \bigoplus_F A_F(\mathcal{A})$, where F ranges over all strata of $\mathcal{A}$. Also, $A_F(\mathcal{A})$ is the image of $A(\mathcal{A}_F)_{\operatorname{rk} F} \to A(\mathcal{A})_{\operatorname{rk} F}$, where $(\cdot)_d$ is taking the degree-d part.

2.2. **Lattice spectral sequence.** Consider a smooth complex variety V and a collection of smooth d-codimensional closed subvarieties $\mathcal{A} = \{Y_1, \dots, Y_h\}$. We say that $Y_1, \dots, Y_h$ intersect *like a hyperplane arrangement* if

(1) For any $S \subseteq \mathcal{A}$, the intersection $\bigcap S$ (if nonempty) is smooth and connected. Call such an intersection a *stratum* of $\mathcal{A}$.
(2) Every stratum has codimension a multiple of d.
(3) The poset of strata of $\mathcal{A}$ is isomorphic to the lattice of some hyperplane arrangement of a complex vector space. Denote this poset by $L(\mathcal{A})$, also called the *lattice* of $\mathcal{A}$.
(4) The set $\mathcal{A}$ is precisely the set of rank one strata in $L(\mathcal{A})$, and every stratum $F \in L(\mathcal{A})$ satisfies $\operatorname{codim}(F) = d \cdot \operatorname{rk}(F)$. (Recall the *rank* of a stratum F is the length of any maximal chain from F to $\hat{1} = V$.)

(The last two statements are actually redundant as they are consequences of the first two.)

We define $A(\mathcal{A})$ to be the Orlik–Solomon algebra associated to the lattice $L(\mathcal{A})$. Denote $M(\mathcal{A}) = V - \bigcup_{i=1}^{h} Y_i$.

Tosteson [19] describes a spectral sequence converging to $H^*(M(\mathcal{A}), \mathbb{Z})$ that works in a general setting. In the special case we describe above, the spectral sequence can be vastly simplified into a form similar to [8, Theorem 3.1].

Proposition 2.1. *Let V and $\mathcal{A}$ be as above. Then there is a spectral sequence of mixed Hodge structures $E_1^{i,j}(\mathcal{A}) \implies H^{i-j}(M(\mathcal{A}),\mathbb{Z})$ such that the abelian group $E_1(\mathcal{A}) := \bigoplus_{i,j\geq 0} E_1^{i,j}(\mathcal{A})$ bigraded by (i,j) is given by*

$$E_1(\mathcal{A}) = \bigoplus_{F\in L(\mathcal{A})} H^*(F,\mathbb{Z}) \otimes A_F(\mathcal{A})$$

where $H^i(F,\mathbb{Z})$ has bidegree $(i,0)$ and $A_F(\mathcal{A})$ has bidegree $(2\operatorname{codim}_{\mathbb{C}} F, \operatorname{rk} F)$ and Hodge type $(\operatorname{codim}_{\mathbb{C}} F, \operatorname{codim}_{\mathbb{C}} F)$.

The group $E_1(\mathcal{A})$ has a structure of graded-commutative algebra[1] induced from the algebra structure of $A(\mathcal{A}) = \bigoplus_F A_F(\mathcal{A})$ and the cup product on $H^*(F,\mathbb{Z})$.

The first-page differential map $d_1^{i,j} : E_1^{i,j} \to E_1^{i,j-1}$ is given by the Gysin map on the cohomology and the differential ∂ on $A(\mathcal{A})$. It makes $E_1(\mathcal{A})$ a differential graded algebra[2].

The spectral sequence is functorial in automorphisms of V that preserve $\mathcal{A}$.

Remark 2.2. In the case where $\mathcal{A}$ is the big diagonal arrangement that gives the ordered configuration space $F(X,n)$ of a d-dimensional variety X, the spectral sequence here is the same as [20] but with a degree shifting, so that $E_1^{2d,1}$ here corresponds to Totaro's $E_{2d}^{0,2d-1}$.

Proof. One can reuse Totaro's argument in [20] based on the result [10, pp. 237–239] of Goresky and MacPherson about arrangements of k-codimensional subspaces of $\mathbb{R}^n$ whose all strata have codimension a multiple of k (see the Remark after the proof of Lemma 3 in [20] for a discussion). This recognizes $E_1(\mathcal{A})$ as the E_{2d} page of the Leray spectral sequence of $F(X_r,n) \hookrightarrow X^n$, where a dga structure is present. See also [14, Lemma 3.1].

Alternatively, one can use the spectral sequence described in [19, Theorem 1.8]:

$$E_1^{i,j}(\mathcal{A}) = \bigoplus_{F\in L(\mathcal{A})} \widetilde{H}_{j-2}((F,\hat{1}); H^i(V, V-F;\mathbb{Z}))$$

where $\widetilde{H}_{j-2}((F,\hat{1}))$ is the reduced homology of the order complex of the poset $(F,\hat{1})$, with a special convention when $F=\hat{1}$. We refer the reader to [22] to a detailed account for these concepts.

---

[1] The direct summand $E_1^{i,j}$ is assigned the degree $i-j$.

[2] A *differential graded algebra* (dga) is a graded-commutative algebra equipped with a linear map d of degree 1 (namely, sending a degree-k element to a degree-$(k+1)$ element), called the differential, such that $d\circ d = 0$ and the graded Leibniz rule is satisfied.

Since $L(\mathcal{A})$ is isomorphic to the lattice of a hyperplane arrangement, we have

$$\widetilde{H}_{j-2}((F,\hat{1});\mathbb{Z}) = \begin{cases} \mathbb{Z}^{|\mu(F,\hat{1})|} = A_F(\mathcal{A}), & j = \operatorname{rk} F \\ 0 & j \neq \operatorname{rk} F \end{cases}$$

for all $F \in L(\mathcal{A})$ (including $F = \hat{1}$), where μ is the Möbius function of the lattice $L(\mathcal{A})$ (see [9, Theorem 4.1] and [16, §4.5]).

By the tubular neighborhood theorem and the excision theorem, we have $H^i(V, V-F;\mathbb{Z}) \cong H^i(\mathcal{N}_F, \mathcal{N}_F - F;\mathbb{Z})$, where $\mathcal{N}_F$ is the normal bundle of F in V. Since we are considering complex manifolds, the normal bundle has a canonical orientation, which gives a canonical Thom isomorphism

$$H^i(V, V-F;\mathbb{Z}) \cong H^{i-2\operatorname{codim}_{\mathbb{C}} F}(F,\mathbb{Z}).$$

Combining the above, we get $E_1^{i,j}(\mathcal{A}) = \bigoplus_{\operatorname{rk} F=j} H^{i-2\operatorname{codim}_{\mathbb{C}} F}(F,\mathbb{Z}) \otimes A_F(\mathcal{A})$ as required. For the compatibility with the mixed Hodge structure, see [18, §3.2]. The structure of differential graded algebra on $E_1(\mathcal{A})$ can be constructed using the functoriality of Tosteson's spectral sequence along the diagonal map $V \to V \times V$. □

2.3. **Arrangements arising from punctured varieties.** In this section, we provide a simplification of Proposition 2.1 in the following special setting. Consider a connected smooth complex variety X of dimension d and distinct points $P^1, \dots, P^r$ $(r \geq 1)$ of X. (We use superscripts for the points for future convenience.) Fix $n \geq 0$, and consider the arrangement $\mathcal{A}$ of X^n consisting of the following d-codimensional closed subvarieties:

$$\Delta_{ij} := \{(x_1, \dots, x_n) \in X^n : x_i = x_j\}$$
$$\Delta_i^s := \{(x_1, \dots, x_n) \in X^n : x_i = P^s\}$$

for $1 \leq i \neq j \leq n$ and $1 \leq s \leq r$. Then the complement $M(\mathcal{A})$ of the arrangement $\mathcal{A}$ is the ordered configuration space $F(X_r, n)$, where $X_r := X - \{P^1, \dots, P^r\}$.

The goal of this section is Proposition 2.3, an explicit description of the differential graded algebra $E_1(\mathcal{A})$ in terms of generators and relations. In the rest of this section, we assume every cohomology is in rational coefficients. The reason why we work in $\mathbb{Q}$ coefficients is Künneth's formula; see Remark 3.2.

We denote by $p_i : X^n \to X$ the projection map onto the i-th coordinate. Always denote $p_i^*(\alpha)$ by α_i. By Künneth's formula $H^*(X^n) \cong H^*(X)^{\otimes n}$, the cohomology ring $H^*(X^n)$ is generated by elements of the form α_i where $\alpha \in H^*(X)$ and $1 \leq i \leq n$. For $i \neq j$, let $p_{ij} : X^n \to X^2$ be the projection map $(x_1, ..., x_n) \mapsto (x_i, x_j)$. Let Δ be the diagonal of X^2. In the notation of above, we have

$$p_{ij}^{-1}(\Delta) = \Delta_{ij}$$
$$p_i^{-1}(P^s) = \Delta_i^s$$

To a smooth irreducible d-codimensional closed subvariety Z of any smooth variety Y, we associate a cohomology class $[Z] \in H^{2d}(Y)$ given by the image of the canonical generator $1 \in H^0(Z)$ under the Gysin map $H^*(Z) \to H^{*+2d}(Y)$. One way to define the Gysin map is using the Poincaré dual $H^i(Z) = H_c^{2(n-d)-i}(Z)^\vee \to H_c^{2n-(i+2d)}(Y)^\vee = H^{i+2d}(Y)$ of the pullback map $H^{2n-(i+2d)}(Y) \to H^{2(n-d)-i}(Z)$. The class of a closed subvariety satisfies the following property: if $f : X \to Y$ is a flat map of constant relative dimension, and D is an algebraic cycle of Y, then $f^*[D] = [f^{-1}(D)]$, where $f^{-1}(D)$ is the pullback.

Define $E_1(X_r, n) := E_1(\mathcal{A})$ and $E_1(X_r, n)_\mathbb{Q} := E_1(X_r, n) \otimes \mathbb{Q}$.

Proposition 2.3. *Let $\mathcal{A}$ be the arrangement above. Then the differential graded algebra $E_1(X_r, n)_\mathbb{Q} = E_1(\mathcal{A})_\mathbb{Q}$ is given by*

$$E_1(X_r, n)_\mathbb{Q} := \frac{H^*(X^n, \mathbb{Q})[g_{ij}, g_i^s : 1 \le i \ne j \le n, 1 \le s \le r]}{\text{(relations)}}$$

with g_{ij} and g_i^s having bidegree $(2d, 1)$ and Hodge type (d, d), subject to relations given by

$$g_{ij} = g_{ji} \tag{2.1}$$

$$g_{ij}g_{jk} + g_{jk}g_{ki} + g_{ki}g_{ij} = 0 \text{ for } i, j, k \text{ distinct} \tag{2.2}$$

$$g_{ij}\alpha_i = g_{ij}\alpha_j \text{ for } \alpha \in H^*(X) \tag{2.3}$$

$$g_i^s\alpha_i = 0 \text{ for } \alpha \in H^{\ge 1}(X) := \bigoplus_{p \ge 1} H^p(X) \tag{2.4}$$

$$g_i^s g_j^s - g_{ij}g_i^s + g_{ij}g_j^s = 0 \tag{2.5}$$

$$g_i^s g_i^t = 0 \text{ for } s \ne t \tag{2.6}$$

The differential $d := d_1$ on the E_1 page is determined by

$$dg_{ij} = p_{ij}^*[\Delta] \tag{2.7}$$

$$dg_i^s = [\Delta_i^s] \tag{2.8}$$

$$d|_{H^*(X^n)} = 0 \tag{2.9}$$

The symmetric group S_n acts on $E_1(X_r, n)_\mathbb{Q}$ by permuting lower indices.

Remark 2.4. When the generators are assigned an odd degree, the notation $\mathbb{Z}$[generators] does not mean a polynomial ring, but an exterior algebra.

2.4. **Proof of Proposition** 2.3. The statement and the proof idea of Proposition 2.3 is similar to Bibby's [2, Theorem 4.1], but we present a proof in full detail here due to the lack of a direct reference.

A stratum F of $\mathcal{A}$ can be uniquely indexed by a pair $(\chi, \sim)$ of a *coloring* function $\chi : \{1, \dots, n\} \to \{0, \dots, r\}$ and an equivalence relation $\sim$ on $\chi^{-1}(0)$, according to the rule

$$F_{(\chi,\sim)} = \{(x_1, \dots, x_n) \in X^n : x_i = P^{\chi(i)} \text{ if } \chi(i) \ne 0, \\ \text{and } x_i = x_j \text{ if } \chi(i) = \chi(j) = 0 \text{ and } i \sim j.\ \}$$

In other words, a coordinate that is colored s ($1 \le s \le r$) is required to take P^s as value, and the coordinates colored 0 have no such a requirement, but they must agree if they belong to the same block of the partition given by $\sim$. It is helpful to think of color 0 as "uncolored".

The arrangement $\mathcal{A}$ satisfies the following key property:

Lemma 2.5. *Let $F = F_{(\chi,\sim)}$ be a stratum of $\mathcal{A}$ described above. Then the pullback map $H^*(X^n) \to H^*(F)$ is surjective. Moreover, its kernel is the ideal generated by $p_i^*\alpha$ for $\alpha \in H^{\ge 1}(X), \chi(i) \neq 0$, and $p_i^*\alpha - p_j^*\alpha$ for $\alpha \in H^*(X), \chi(i) = \chi(j) = 0, i \sim j$.*

Proof. By Künneth formula, we can deal with each i with $\chi(i) \neq 0$ and each equivalence class of $\chi^{-1}(0)$ separately. It suffices to prove the lemma for the following cases:

(1) $n = 1, F = \Delta_1^s$, which is the point P^s. Then since $H^*(F) = H^0(F)$ and both X and F are connected, the kernel of $H^*(X) \to H^*(F)$ is $H^{\ge 1}(X)$.

(2) $n \ge 2, F = \{(x_1, \dots, x_n) : x_1 = \cdots = x_n\}$. Then $F \hookrightarrow X^n$ is isomorphic to the diagonal map $X \hookrightarrow X^n, x \mapsto (x, \dots, x)$, so the kernel of $H^*(X^n) \to H^*(F)$ is the same as the kernel of the multiplication map $H^*(X)^{\otimes n} \to H^*(X)$.

We recall a standard fact in commutative algebra, but now we state and prove an extension in the following setting. Let R be a commutative ring and A a graded-commutative R-algebra. The R-module $A^{\otimes n} := A \otimes_R \cdots \otimes_R A$ is a graded-commutative algebra such that (1) for $1 \le i \le n$, we have a degree-preserving algebra homomorphism $\theta_i : A \to A^{\otimes n}$ sending $a \in A$ to $1 \otimes \cdots \otimes a \otimes \cdots \otimes 1$ where a appears at the i-th factor; (2) $a_1 \otimes \cdots \otimes a_n = \theta_1(a_1) \dots \theta_n(a_n)$.

We claim that the kernel of the multiplication map $\mu_n : A^{\otimes n} \to A, a_1 \otimes \cdots \otimes a_n \mapsto a_1 \dots a_n$ is generated by $\theta_i(a) - \theta_j(a)$ for all $1 \le i, j \le n$ and $a \in A$. Lemma 2.5 follows from the claim by letting $A = H^*(X)$ and $R = \mathbb{Z}$.

It is obvious that $\theta_i(a) - \theta_j(a)$ is in the kernel. We shall prove the reverse inclusion by induction on n.

(a) Case $n = 2$. An element of $\ker \mu_2$ is of the form

$$\sum_{j=1}^{h} a_j \otimes b_j = \sum_{j=1}^{h} \theta_1(a_j)\theta_2(b_j)$$

such that $\sum a_j b_j = 0$ in A.
Then

$$\begin{aligned}\sum a_j \otimes b_j &= \sum \theta_1(a_j)\theta_2(b_j) - \theta_1(\sum a_j b_j)\\ &= \sum (\theta_1(a_j)\theta_2(b_j) - \theta_1(a_j)\theta_1(b_j))\\ &= \sum \theta_1(a_j)(\theta_2(b_j) - \theta_1(b_j))\end{aligned}$$

is in the ideal generated by $\theta_2(b_j) - \theta_1(b_j)$.

(b) Case $n > 2$. Decompose μ_n into two maps:

$$A^{\otimes(n-1)} \otimes_R A \xrightarrow{\mu_{n-1}\otimes 1} A \otimes_R A \xrightarrow{\mu_2} A$$

If $x \in \ker(\mu_n)$, then $(\mu_{n-1} \otimes 1)(x)$ must be in the kernel of μ_2. By the $n = 2$ case above, the kernel of μ_n is generated by preimages of $1 \otimes a - a \otimes 1, a \in A$ under $\mu_{n-1} \otimes 1$. One of its preimages is $\theta_n(a) - \theta_{n-1}(a)$, and all other preimages must differ from this one by an element of $\ker(\mu_{n-1} \otimes 1) = \ker(\mu_{n-1})\theta_n(A)$. By the induction hypothesis, the kernel of μ_n is contained in the ideal generated by $\theta_n(a) - \theta_{n-1}(a)$ and $\theta_i(a) - \theta_j(a), 1 \leq i, j \leq n-1, a \in A$.

□

Lemma 2.5 allows a simplication of Proposition 2.1.

Lemma 2.6. *Let $\mathcal{A}$ be an arrangement of d-codimensional subvarieties of a smooth variety V that intersect like a hyperplane arrangement. Assume in addition that for any stratum F of $\mathcal{A}$, the pullback map $H^*(V) \to H^*(F)$ is surjective with kernel I_F. Then*

$$E_1(\mathcal{A})_{\mathbb{Q}} \cong \frac{H^*(V) \otimes A(\mathcal{A})}{(I_F \cdot A_F(\mathcal{A}) : F \in L(\mathcal{A}))}$$

Proof. The graded algebra $E_1(\mathcal{A})$ in Proposition 2.1 is given by

$$\begin{aligned} \bigoplus_F H^*(F) \otimes A_F(\mathcal{A}) &= \bigoplus_F \frac{H^*(V)}{I_F} \otimes A_F(\mathcal{A}) \\ &= \frac{\bigoplus_F H^*(V) \otimes A_F(\mathcal{A})}{\sum_F I_F \cdot A_F(\mathcal{A})} \\ &= \frac{H^*(V) \otimes \bigoplus_F A_F(\mathcal{A})}{\sum_F I_F \cdot A_F(\mathcal{A})} \\ &= \frac{H^*(V) \otimes A(\mathcal{A})}{\sum_F I_F \cdot A_F(\mathcal{A})} \end{aligned}$$

□

We are now ready to compute the $E_1(X_r, n)$

Proof of Proposition 2.3. By Lemma 2.6, the arrangement $\mathcal{A} := \{\Delta_{ij}, \Delta_i^s : 1 \leq i \neq j \leq n, 1 \leq s \leq r\}$ has rational E_1 algebra

$$\begin{aligned} E_1(X_r, n)_{\mathbb{Q}} &= \frac{H^*(X^n) \otimes A(\mathcal{A})}{(I_F \cdot A_F(\mathcal{A}) : F)} \\ &= \frac{H^*(X^n) \otimes A(\mathcal{A})}{(I_F g_S : \bigcap S = F \text{ and } S \text{ is independent})} \end{aligned}$$

We denote the generators of $A(\mathcal{A})$ by $g_{ij} = g_{\Delta_{ij}}, g_i^s = g_{\Delta_i^s}$ (so that $g_{ij} = g_{ji}$ are the same generator). We work out the relation ideal $(I_F g_S : \bigcap S =$

F and S is independent) first. We claim that it is enough to use F of rank one. In other words, the relation ideal is equal to the ideal J generated by

(1) $g_{ij}(\alpha_i - \alpha_j)$;
(2) $g_i^s \alpha_i, \alpha \in H^{\geq 1}(X)$.

Let $F = F_{\chi,\sim}$ be a stratum of $\mathcal{A}$. We need to show that for any independent $S \subset \mathcal{A}$ such that $\bigcap S = F$, the ideal $g_S I_F$ is in the ideal J.

Such an S is classified by the following indirected graph, consisting of

(1) An (unrooted) spanning tree on each equivalence class of $\sim$ on $\chi^{-1}(0)$;
(2) A forest of rooted trees on $\chi^{-1}(s)$, for each $1 \leq s \leq r$.

The set S then consists of $\Delta_i^{\chi(i)}$ for each i that appears as a root and Δ_{ij} for each (i,j) that appears as an edge.

We observe that to generate $I_F = (\alpha_i - \alpha_j \ (i \sim j), \alpha_i \ (\chi(i) \neq 0))$, a part of the generators suffices: $\alpha_i - \alpha_j$ for $\Delta_{ij} \in S$ and α_i for $\Delta_i^s \in S, \alpha \in H^{\geq 1}(X)$. Indeed, this can be done by joining a path from i to j in the tree (if $i \sim j \in \chi^{-1}(0)$) or by joining a path from i to the root of the tree where i belongs (if $\chi(i) \neq 0$). But g_S multiplied by each of these special generators lies in J. This proves the claim and finishes the computation of $(I_F \cdot A_F(\mathcal{A}) : F \in L(\mathcal{A}))$.

It remains to compute a presentation of $A(\mathcal{A})$. Let $J(\mathcal{A})$ be the ideal of $B(\mathcal{A})$ generated by relations (2.2), (2.5) and (2.6) of Proposition 2.3. Claim that $J(\mathcal{A}) = I(\mathcal{A})$, the defining ideal for $A(\mathcal{A})$.

The ideal $I(\mathcal{A})$ is generated by e_S for minimal vanishing set e_S and ∂e_S for minimal dependent set S. These include

(1) $e_i^s e_j^t e_\gamma$, $s \neq t$;
(2) $\partial(e_{i_1 i_2} e_{i_2 i_3} \ldots e_{i_{h-1} i_h} e_{i_h i_1})$, with $h \geq 3$ and $i_1, \ldots, i_h$ are distinct;
(3) $\partial(e_i^s e_j^s e_\gamma)$,

where in both (1) and (3), $\gamma = (i = i_0 \to i_1 \to \cdots \to i_h = j)$ is a path joining i and j, and e_γ means $e_{i_0 i_1} e_{i_1 i_2} \ldots e_{i_{h-1} i_h}$. Here h is allowed to be 0, in which case $i = j$.

We need to show that these generators are in $J(\mathcal{A})$. This will require Lemma 2.7 below the end of this proof.

First, we prove that (2) is in $J(\mathcal{A})$ by induction on h. If $h = 3$, then (2) is just (2.2). If $h > 3$, we set $S_1 = \{\Delta_{i_1 i_2}, \Delta_{i_2 i_3}, \ldots, \Delta_{i_{h-2} i_{h-1}}\}, S_2 = \{\Delta_{i_{h-1} i_h}, \Delta_{i_h i_1}\}$ and $Y = \Delta_{i_1 i_{h-1}}$, then $\partial e_{S_1 \sqcup Y} \in J(\mathcal{A})$ by induction hypothesis, $\partial e_{S_2 \sqcup Y} \in J(\mathcal{A})$ by $h = 3$ case. Applying Lemma 2.7, we get $\partial e_{S_1 \sqcup S_2} \in J(\mathcal{A})$.

Next, we prove that (3) is in $J(\mathcal{A})$ by induction on h, the length of $\gamma = (i = i_0 \to i_1 \to \cdots \to i_h = j)$. The base case $h = 1$ is (2.5), and the induction step is proved similarly with $S_1 = \{\Delta_i^2, \Delta_{i_0 i_1}, \Delta_{i_1 i_2}, \ldots, \Delta_{i_{h-2} i_{h-1}}\}, S_2 = \{\Delta_{i_{h-1} i_h}, \Delta_j^s\}$ and $Y = \Delta_{i_{h-1}}^s$.

Finally, for (1), to prove that $e_i^s e_j^t e_\gamma \in J(\mathcal{A})$ for $s \neq t$, we note that

$$\partial(e_i^t e_j^t e_\gamma) = e_j^t e_\gamma - e_i^t \partial(e_i^t e_\gamma)$$

is just proved to be in $J(\mathcal{A})$ by case (3). Hence

$$e_j^t e_\gamma \equiv e_i^t \partial(e_i^t e_\gamma) \mod J(\mathcal{A}),$$

so

$$e_i^s e_j^t e_\gamma \equiv e_i^s e_i^t \partial(e_i^t e_\gamma) \mod J(\mathcal{A})$$

But $e_i^s e_i^t$ is just (2.6), so $e_i^s e_j^t e_\gamma \in J(\mathcal{A})$.

In summary, we have proved that

$$\begin{aligned} E_1(X_r, n)_{\mathbb{Q}} &= \frac{H^*(X^n) \otimes A(\mathcal{A})}{(I_F \cdot A_F(\mathcal{A}) : F)} \\ &= \frac{H^*(X^n) \otimes \mathbb{Q}[g_{ij}, g_i^s]/((2.1), (2.2), (2.5), (2.6))}{((2.3), (2.4))} \\ &= \frac{H^*(X^n)[g_{ij}, g_i^s]}{((2.1) \text{ through } (2.6))} \end{aligned}$$

This proves the description of $E_1(X_r, n)_{\mathbb{Q}}$ in Proposition 2.3. □

We now prove the following lemma required in the above proof of Proposition 2.3.

Lemma 2.7. *Let $\mathcal{A}$ be a hyperplane arrangement. Let $S_1, S_2 \subseteq \mathcal{A}$ be disjoint subsets and $Y \in \mathcal{A}$ be an element not in $S_1 \sqcup S_2$. Then inside the algebra $B(\mathcal{A})$, we have*

$$\partial e_{S_1 \sqcup S_2} \in (\partial e_{S_1 \sqcup \{Y\}}, \partial e_{S_2 \sqcup \{Y\}}) B(\mathcal{A})$$

Proof. Write $A = e_{S_1}$, $B = e_{S_2}$ and $e = e_Y$. We have

$$\begin{aligned} \partial(e\partial(AB)) &= (\partial e)(\partial(AB)) - e\partial^2(AB) \\ &= 1 \cdot \partial(AB) - 0 \\ &= \partial(AB) \end{aligned}$$

Thus

$$\begin{aligned} \partial(AB) &= \partial(e\partial(AB)) \\ &= \partial(e((\partial A)B \pm A\partial B)) \\ &= \pm\partial(eB\partial A) \pm \partial(eA\partial B) \\ &= \pm\partial(eB)\partial(A) \pm \partial(eA)\partial B \text{ (since } \partial^2 = 0) \\ &\in (\partial(eA), \partial(eB)) \end{aligned}$$

□

3. Proof of main results

Fix a connected smooth complex variety X of dimension d and $r \geq 1$ distinct points $P^1, \dots, P^r$ of X. Write $Y = X - \{P^0, P^1, \dots, P^{r-1}\}$ and $P = P^r$.

We will apply Proposition 2.3 to both X_r and X_{r-1}. Write $E_1(Y, n)_{\mathbb{Q}} := E_1(X_{r-1}, n)_{\mathbb{Q}}$ and $E_1(Y - P, n)_{\mathbb{Q}} := E_1(X_r, n)_{\mathbb{Q}}$. Note that both spectral sequences remember how Y or $Y - P$ are punctured from X, even though X does not appear in the notation.

We observe that

$$E_1(Y - P, n)_{\mathbb{Q}} = E_1(Y, n)_{\mathbb{Q}}[g_1^r, ..., g_n^r]/(\text{new relations})$$

where the new relations consist of

$$g_i^r g_j^r - g_{ij} g_i^r + g_{ij} g_j^r = 0 \text{ for } 1 \leq i, j \leq n \tag{3.1}$$

$$g_i^s g_i^r = 0 \text{ for } 1 \leq s \leq r - 1 \tag{3.2}$$

$$g_i^r \alpha_i = 0 \text{ for } \alpha \in H^p(X), p \geq 1 \tag{3.3}$$

In this proof, we will repetitively use the following elementary fact about quadratic algebras: let R be a graded-commutative ring with identity, and let $x_1, \dots, x_m$ be indeterminates of degree one. Consider the graded-commutative R-algebra A generated by $x_1, \dots, x_m$ with relations $x_i x_j = L_{ij}(x_1, \dots, x_m)$ for all $1 \leq i < j \leq m$, where $L_{ij}(x_1, \dots, x_m)$ is a left R-linear combination of $x_1, \dots, x_m$. Then A is isomorphic to $R\langle 1, x_1, \dots, x_m\rangle$ as a left R-module, where $R\langle 1, x_1, \dots, x_m\rangle$ denotes the free left R-module with basis $1, x_1, \dots, x_m$.

3.1. **Proof of Theorem** 1.6**(1).** We denote by $[n]$ the finite set $\{1, \dots, n\}$, and by $[n] - i$ the set $\{j \in [n] : j \neq i\}$. For any finite set I of integers, we denote by $E_1(Y, I)$ a copy of $E_1(Y, |I|)$, but with lower indices of the generators taken from I instead of $\{1, \dots, |I|\}$. Note that $E_1(Y, n)$ and $E_1(Y, [n])$ are precisely the same.

Comparing the cases of Proposition 2.3 for $r - 1$ and r, we get a presentation of $E_1(Y - P, n)_{\mathbb{Q}}$ as an $E_1(Y, n)_{\mathbb{Q}}$-module:

$$E_1(Y - P, n)_{\mathbb{Q}} = \frac{E_1(Y, n)_{\mathbb{Q}}[g_i^r : i \in [n]]/(3.1)}{((3.2), (3.3))}$$

$$= \frac{E_1(Y, n)_{\mathbb{Q}}\langle 1, g_i^r : i \in [n]\rangle}{(g_i^s g_i^r, \alpha_i g_i^r : i \in [n], s \neq r, \alpha \in H^{\geq 1}(X))} \tag{3.4}$$

$$= E_1(Y, n)_{\mathbb{Q}} \oplus \bigoplus_{i=1}^{n} E_1(Y, n)_{\mathbb{Q}}\langle g_i^r\rangle/(g_i^s g_i^r, \alpha_i g_i^r : s \neq r) \tag{3.5}$$

Here, the second equality is due to the discussion above about quadratic algebras.

Next, we fix $i \in [n]$ and study $E_1(Y, n)_{\mathbb{Q}}$ as an $E_1(Y, [n] - i)_{\mathbb{Q}}$-module. We continue to work on cohomology in *rational* coefficients because we need

Künneth's formula. In the following computation, the lower indices j, k always range over $[n]-i$, the upper indices s, t always run over $\{1, \dots, r-1\}$, and α is taken from $H^{\geq 1}(X)$. The new generators and relations are precisely the ones that involve the lower index i, so as an algebra,

$$E_1(Y,n)_{\mathbb{Q}} = \frac{E_1(Y,[n]-i)_{\mathbb{Q}}[g_{ij}, g_i^s] \otimes_{\mathbb{Q}} H^*(X)}{\left(\begin{array}{rcll} g_{ij}g_{ik} & = & -g_{jk}g_{ij} + g_{jk}g_{ik}, & j \neq k \\ g_{ij}g_i^s & = & -g_j^s g_i^s - g_j^s g_{ij}, & j, s \\ g_i^s g_i^t & = & 0, & s \neq t \\ \alpha_i g_{ij} & = & \alpha_j g_{ij}, & j, \alpha \\ g_i^s \alpha_i & = & 0, & s, \alpha \end{array}\right)} \tag{3.6}$$

where $H^*(X)$ in the tensor factor contributes to the i-th coordinate, namely, $\{\alpha_i : \alpha \in H^*(X)\}$.

Applying the fact about quadratic algebras again, we get

$$E_1(Y,n)_{\mathbb{Q}} = \frac{E_1(Y,[n]-i)_{\mathbb{Q}} \otimes H^*(X)\langle 1, g_{ij}, g_i^s\rangle}{((\alpha_i - \alpha_j)g_{ij}, \alpha_i g_i^s)} \tag{3.7}$$

$$= E_1(Y,[n]-i)_{\mathbb{Q}} \otimes (H^*(X) \oplus \mathbb{Q}\langle g_{ij}, g_i^s\rangle) \tag{3.8}$$

$$= E_1(Y,[n]-i)_{\mathbb{Q}} \otimes (H^{\geq 1}(X) \oplus \mathbb{Q}\langle 1, g_{ij}, g_i^s\rangle) \tag{3.9}$$

as a module over $E_1(Y,[n]-i)_{\mathbb{Q}}$, where the last two equalities use the decomposition $H^*(X) = \mathbb{Q} \oplus H^{\geq 1}(X)$. We shall view $E_1(Y,n)_{\mathbb{Q}}$ as the module (3.9), but with a multiplication table given by the relations in (3.6).

Now we give a presentation of the direct summand of (3.5) indexed by i, as an $E_1(Y,[n]-i)_{\mathbb{Q}}$-module. In the computation below, the convention for α, j, s, t is as before, and β ranges over $H^{\geq 1}(X)$.

(3.10)
$$\frac{E_1(Y,n)_{\mathbb{Q}}\langle g_i^r\rangle}{E_1(Y,n)_{\mathbb{Q}}(g_i^s g_i^r, \alpha_i g_i^r : s, \alpha)} = \frac{E_1(Y,[n]-i)_{\mathbb{Q}} \otimes (H^{\geq 1}(X) \oplus \mathbb{Q}\langle 1, g_{ij}, g_i^s\rangle)}{(g_i^s, \alpha_i)E_1(Y,n)}\langle g_i^r\rangle$$

$$= \frac{E_1(Y,[n]-i)_{\mathbb{Q}} \otimes (H^{\geq 1}(X) \oplus \mathbb{Q}\langle 1, g_{ij}, g_i^s\rangle)}{E_1(Y,[n]-i)_{\mathbb{Q}}\left(\begin{array}{cccc} g_i^s\beta_i, & g_i^s, & g_i^s g_{ij}, & g_i^s g_i^t \\ \alpha_i\beta_i, & \alpha_i, & \alpha_i g_{ij}, & \alpha_i g_i^t \end{array}\right)}\langle g_i^r\rangle \tag{3.11}$$

$$\overset{(3.6)}{=} \frac{E_1(Y,[n]-i)_{\mathbb{Q}} \otimes (H^{\geq 1}(X) \oplus \mathbb{Q}\langle 1, g_{ij}, g_i^s\rangle)}{E_1(Y,[n]-i)_{\mathbb{Q}}\left(\begin{array}{rrrr} 0, & g_i^s, & g_j^s g_i^s + g_j^s g_{ij}, & 0 \\ (\alpha\beta)_i, & \alpha_i, & \alpha_j g_{ij}, & 0 \end{array}\right)}\langle g_i^r\rangle \tag{3.12}$$

$$= \frac{E_1(Y,[n]-i)_{\mathbb{Q}}\langle 1, g_{ij} : j \in [n]-i\rangle}{(g_j^s g_{ij}, \alpha_j g_{ij} : j \in [n]-i, s \neq r, \alpha \in H^{\geq 1}(X))}\langle g_i^r\rangle. \tag{3.13}$$

(In the last equality, note that the generators g_i^s and the summand $H^{\geq 1}(X)$ are eliminated by the relations.)

Applying (3.4) to the index set $[n]-i$, we have

$$E_1(Y-P,[n]-i)_{\mathbb{Q}} = \frac{E_1(Y,[n]-i)_{\mathbb{Q}}\langle 1, g_j^r : j\in[n]-i\rangle}{(g_j^s g_j^r, \alpha_j g_j^r : j\in[n]-i, s\neq r, \alpha\in H^{\geq 1}(X))}. \tag{3.14}$$

We get an isomorphism of the above two left $E_1(Y,[n]-i)_{\mathbb{Q}}$-modules since they have the same presentations. Combined with (3.5), we get the following isomorphism.

Lemma 3.1. *We have a linear isomorphism*

$$\Phi : E_1(Y,n)_{\mathbb{Q}} \oplus \bigoplus_{i=1}^{n} E_1(Y-P,[n]-i)_{\mathbb{Q}} \to E_1(Y-P,n)_{\mathbb{Q}}$$

such that $\Phi|_{E_1(Y,n)_{\mathbb{Q}}}$ is the natural map, and $\Phi|_{E_1(Y-P,[n]-i)_{\mathbb{Q}}}$ is the $E_1(Y,[n]-i)_{\mathbb{Q}}$-module map that sends 1 to g_i^r and sends g_j^r to $g_{ij}g_i^r$ for all $j\in[n]-i$.

We note that $\bigoplus_{i=1}^n E_1(Y-P,[n]-i)_{\mathbb{Q}}$ is isomorphic to $\mathrm{Ind}_{S_{n-1}}^{S_n} E_1(Y-P,n-1)_{\mathbb{Q}}$ as an S_n-module, by recalling that the symmetric groups act by permuting the lower indices in the generators. This finishes the proof of Theorem 1.6(1), where the degree shift is because the elements g_i^r and g_{ij} have bidegree $(2d,1)$ and Hodge type (d,d).

Remark 3.2. Proposition 2.3 and Theorem 1.6(1) hold in integer coefficients (i.e., without tensoring with $\mathbb{Q}$), if the Künneth map $H^*(X,\mathbb{Z})^{\otimes m} \to H^*(X^m,\mathbb{Z})$ is an isomorphism for all m. This always happens when $H^*(X,\mathbb{Z})$ has no torsion.

Question 3.3. Can we extend Φ to a morphism of spectral sequences? If this can be done, then we immediately verify Conjecture 1.7 for Y and rational coefficients. A morphism of spectral sequences is usually constructed by applying the functoriality of the relevant spectral sequences to a morphism of varieties. However, we point out that any such construction must make use of the noncompactness of X somehow, because the question has a negative answer if X is compact, in which case Theorem 1.6(2) fails.

3.2. **Proof of Theorem** 1.6**(2).** Now assume that X is a d-dimensional complex variety that is not compact. We need to show that Φ constructed above preserves the differential.

In $E_1(Y-P,n)_{\mathbb{Q}}$, we claim that $dg_j^r=0$ for all $j\in[n]$, and $d(g_{ij}g_i^r)=0$ for $i\neq j$.

Since X is not compact, the top cohomology $H^{2d}(X)$ vanishes. We have $dg_j^r = [\Delta_j^r] = p_i^*([P^r])$ in the notation of 2.3. But $[P^r]\in H^{2d}(X)$ and $H^{2d}(X)=0$, so $dg_j^r=0$.

To compute $d(g_{ij}g_i^r)$, use the graded Leibniz rule, noting that $dg_i^r=0$.

$$d(g_{ij}g_i^r) = d(g_{ij})g_i^r - g_{ij}dg_i^r = d(g_{ij})g_i^r = p_{ij}^*[\Delta]g_i^r.$$

Note that $[\Delta]\in H^{2d}(X\times X)$, but $H^{2d}(X)=0$, so Künneth's formula gives $[\Delta]\in\bigoplus_{p=1}^{2d-1} H^p(X)\otimes H^{2d-p}(X)$. In particular, $p_{ij}^*[\Delta]$ can be expressed as a

$\mathbb{Q}$-linear combination of terms of the form $\alpha_i\beta_j$, where $\alpha, \beta \in H^{\geq 1}(X)$. Since $\alpha_i g_i^r = 0$ in $E_1(Y-P,n)_{\mathbb{Q}}$ for every $\alpha \in H^{\geq 1}(X)$, we see that $d(g_{ij}g_i^r) = 0$ in $E_1(Y-P,n)_{\mathbb{Q}}$.

Finally, from the construction of Φ and the graded Leibniz rule, it follows that Φ commutes with the differential map on E_1.

3.3. **Proof of Theorem** 1.6**(3).** We shall prove that the spectral sequence $E^{i,j}(X_r,n)_{\mathbb{Q}}$ degenerates at E_2 for all r. Recall that the spectral sequence is described by Proposition 2.1 using the arrangement described in Section 2.3. We notice that every stratum F is isomorphic to X^m for some $m \geq 0$. By Künneth's formula, $H^i(X^m,\mathbb{Q})$ is pure of weight wi as well. As a result, $E_1^{i,j}(X_r,n)_{\mathbb{Q}}$ is pure of weight $w(i-2dj)+2dj$. Since all the subsequent pages are subquotients of E_1, the same purity holds for the page E_h ($h \geq 1$).

If $H^i(X,\mathbb{Q}) = 0$ for all $i > 0$, then the degeneracy holds trivially. We assume it is not the case, so w is uniquely determined. If $w = 1$, observe that $E_h^{i,j}(X_r,n)_{\mathbb{Q}}$ is pure of weight i. Since the differential is in the category of mixed Hodge structures, any differential between pieces pure of different weights must be zero. It follows that the only nonzero differential is on the first page. This is the classical argument of [20].

If $w > 1$, then we claim that the spectral sequence actually degenerates at E_1. We will prove it using the same argument but with slightly more calculations.

The differential d_h on the E_h page sends $E_h^{i,j}(X_r,n)_{\mathbb{Q}}$ to $E_h^{i-h+1,j-h}(X_r,n)_{\mathbb{Q}}$. In order for d_h to be nonzero, the source and the target must have the same weight, i.e.,

$$w(i-2dj)+2dj = w(i-2dj+1-h+2dh)+2dj-2dh,$$

which gives

$$w = \frac{2dh}{1+2dh-h}.$$

But since X is noncompact, the cohomology of X is concentrated in degrees $0 \leq i \leq 2d-1$, so the denominator of w is at most $2d-1$. It follows that $w \geq \frac{2d}{2d-1}$, a contradiction to $w = \frac{2dh}{1+2dh-h}$. This shows that d_h is zero for all $h \geq 1$, finishing the proof of the claim and thus of Theorem 1.6.

Remark 3.4. We wonder if any degeneracy property can be proved in the absence of the weight argument. See Example 4.1 for an example where the degeneracy assertion could be false.

3.4. **Finishing the proof of Theorem** 1.1**.** It remains to show that any Y in the three cases of Theorem 1.1 is of the form $X - \{\text{zero or more points}\}$ where the noncompact variety X satisfies the weight assumption of Theorem 1.6(3).

(1) Let X be a one-punctured smooth projective variety. Then $H^i(X)$ is pure of weight i, as stated in the proof of [8, Theorem 2.10]. The assumption of Theorem 1.6(3) holds with $w = 1$.

(2) Let X be the complement of a hyperplane arrangement or a toric arrangement. Then the assumption of Theorem 1.6(3) holds with $w = 2$ by [8, Theorems 3.7, 3.8].

(3) Let C be a smooth projective curve of genus g embedded in the projective plane $\mathbb{P}^2$, and let $X = \mathbb{P}^2 - C$. Then it is not hard to check that

$$H^i(X, \mathbb{Q}) = \begin{cases} \mathbb{Q}, \text{pure of weight } 0, & i = 0 \\ \mathbb{Q}^{2g}, \text{pure of weight } 3, & i = 2 \\ 0, & i \neq 0, 2, \end{cases}$$

so that X satisfies the assumption of Theorem 1.6(3) with $w = 3/2$.

This completes the proof of Theorem 1.1.

Remark 3.5. In certain cases, statements stronger than Theorem 1.1 can be said.

(1) Suppose X satisfies the assumption of Theorem 1.6(3) with $w = 1$, for example, when X is a one-punctured smooth projective variety or $\mathbb{C}^d$. Then as is discussed in [20, §4] (see also [2, Theorem 3.3]), the spectral sequence filtration is the weight filtration, so as an $\mathbb{Q}$-algebra (but not as mixed Hodge structures), $E_2(X_r, n)_{\mathbb{Q}}$ is isomorphic to the associated graded algebra of $H^*(F(X_r, n), \mathbb{Q})$ with respect to the weight filtration, which is isomorphic to $H^*(F(X_r, n), \mathbb{Q})$ itself by Deligne's observation [6, p.81]. The isomorphism Φ in Lemma 3.1 descends to an isomorphism

$$H^*(F(X_{r-1}, n), \mathbb{Q}) \oplus \operatorname{Ind}_{S_{n-1}}^{S_n} H^*(F(X_r, n-1), \mathbb{Q}) \to H^*(F(X_r, n), \mathbb{Q})$$

such that its restriction on $H^*(F(X_r, n-1), \mathbb{Q})$ is the $H^*(F(X_{r-1}, n-1), \mathbb{Q})$-module map that sends 1 to g_n^r and g_j^r to $g_{ij} g_n^r$ for $j \in [n-1]$. Note that these special elements are in the E_2 page because their first-page differentials vanish, and we view them in the cohomology ring via the ring isomorphism $E_2(X_r, n)_{\mathbb{Q}} \cong H^*(F(X_r, n), \mathbb{Q})$ above.

(2) If X satisfies the assumption of Theorem 1.6(3) with $w \neq \frac{2d}{2d-1}$, then $H^*(F(X_r, n), \mathbb{Q})$ is an iterated extension of pieces pure of distinct weights. As a result, the pure Hodge structure of a weight-graded piece of $H^*(F(X_r, n), \mathbb{Q})$ is isomorphic to the piece of $E_2(X_r, n)_{\mathbb{Q}}$ of the corresponding weight. This implies that the virtual equivalence in Theorem 1.1 can be upgraded to a stronger sense: two rational mixed Hodge structures H and H' are said to be equivalent in this stronger sense if $\operatorname{Gr}_m^W H \cong \operatorname{Gr}_m^W H'$ as pure Hodge structures for all m. This is strictly stronger that virtual equivalence because there are nontrivial extensions in the category of pure Hodge structures of weight m.

4. Further remarks

The main goal of the next step is to extend Theorem 1.1 to more examples of noncompact X. There are two independent possible approaches:

(1) Trying to control the pages after E_2, either by looking for a degeneracy statement without the weight argument (Remark 3.4), or by extending Φ in Lemma 3.1 to a morphism of spectral sequence (Question 3.3).
(2) Studying the spectral sequence for $F(X - \bigcup_{i=1}^r Z_i, n)$ where X is a one-punctured smooth projective variety and Z_i's are closed subvarieties of X (not necessarily points). Then we have degeneracy while still covering a wide class of noncompact varieties. However, the first page of the spectral sequence will be more complicated and no longer be described by the Orlik–Solomon algebra, because the poset associated to the arrangement needed is not isomorphic to the lattice of a hyperplane arrangement.

We wonder if Theorem 1.1 holds in $\mathbb{Z}$ coefficients, as in the case of a punctured algebraic curve (1.1). We summarize the places where we require rational coefficients in our method. First of all, the Künneth formula only holds in $\mathbb{Q}$ coefficients, though it holds in the setting of (1.1) because the cohomology of a one-punctured algebraic curve has no torsion. Second, we need degeneracy in the spectral sequence, which requires a weight argument, but the weight filtration is defined on the rational cohomology. Third, the cohomology in question is built up from the E_∞ page of the spectral sequence via an iterated extension, but the extension problem does not have a unique answer if some of the pieces have torsion. Finally, taking S_n invariants require rational coefficients because $H^*(\mathrm{Conf}^n(X), R) \cong (H^*(F(X,n), R)^{S_n}$ is only guaranteed to hold if $n! = |S_n|$ is invertible in the coefficient ring R. We remark that for $X = \mathbb{P}^2 - C$ where C is a smooth plane curve that is not a line, there is a nontrivial torsion in $H^*(X, \mathbb{Z})$, so the case (3) of Theorem 1.1 is a good place to look for a potential counterexample to the statement in $\mathbb{Z}$ coefficients.

Example 4.1. We give an example where the degeneracy is not guaranteed by the weight argument. We do not know whether the degeneracy holds in this case and whether it provides a counterexample to Conjecture 1.7.

Let E be an elliptic curve, and O be a point of E. Consider the smooth surface $X = \mathrm{Conf}^2(E - O)$. Then [4] gives

$$H^i(X, \mathbb{Q}) = \begin{cases} \mathbb{Q}, \text{pure of weight } 0, & i = 0 \\ \mathbb{Q}^2, \text{pure of weight } 1, & i = 1 \\ \mathbb{Q}^2, \text{pure of weight } 3, & i = 2 \\ 0, & i \geq 3 \end{cases}$$

We observe that when $n \geq 6$, the weight-8 part of $H^7(X^n, \mathbb{Q})$, namely, $\mathrm{Gr}_8^W H^7(X^n, \mathbb{Q})$, is nonzero ($7 = 1+1+1+1+1+2$, $8 = 1+1+1+1+1+3$).

If X_r is X with r punctures, then the second-page differential on a specific degree reads

$$E_2^{8,2}(X_r, n) \otimes \mathbb{Q} \to E_2^{7,0}(X_r, n) \otimes \mathbb{Q}$$

is not yet known to be zero, because the left hand side is nonzero and pure of weight 8 and the right hand side is $H^7(X^n, \mathbb{Q})$, which has a nonzero weight-8 part.

References

[1] V. I. Arnol'd. The cohomology ring of the group of dyed braids. *Mat. Zametki*, 5:227–231, 1969.

[2] C. Bibby. Cohomology of abelian arrangements. *Proc. Amer. Math. Soc.*, 144(7):3093–3104, 2016.

[3] C.-F. Bödigheimer, F. Cohen, and L. Taylor. On the cohomology of configuration spaces. *Topology*, 28:111–123, 1989.

[4] G. Cheong and Y. Huang. Rationality for the Betti numbers of the unordered configuration spaces of a punctured torus. arXiv: 2009.07976.

[5] P. Deligne. Théorie de Hodge. II. *Inst. Hautes Études Sci. Publ. Math.*, (40):5–57, 1971.

[6] P. Deligne. Poids dans la cohomologie des variétés algébriques. In *Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1*, pages 79–85, 1975.

[7] G. C. Drummond-Cole and B. Knudsen. Betti numbers of configuration spaces of surfaces. *J. Lond. Math. Soc. (2)*, 96(2):367–393, 2017.

[8] C. Dupont. Purity, formality, and arrangement complements. *Int. Math. Res. Not. IMRN*, (13):4132–4144, 2016.

[9] J. Folkman. The homology groups of a lattice. *J. Math. Mech.*, 15:631–636, 1966.

[10] M. Goresky and R. MacPherson. *Stratified Morse theory*, volume 14 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1988.

[11] T. Hausel and F. Rodriguez-Villegas. Mixed Hodge polynomials of character varieties. *Invent. Math.*, 174(3):555–624, 2008. With an appendix by Nicholas M. Katz.

[12] S. Kallel. Symmetric products, duality and homological dimension of configuration spaces. In *Groups, homotopy and configuration spaces*, volume 13 of *Geom. Topol. Monogr.*, pages 499–527. Geom. Topol. Publ., Coventry, 2008.

[13] E. Looijenga. Cohomology of $\mathcal{M}_3$ and $\mathcal{M}_3^1$. *Mapping class groups and moduli spaces of Riemann surfaces*, pages 205–228, 1993.

[14] L. Moci and R. Pagaria. On the cohomology of arrangements of subtori. 2020. arXiv: 2001.05180.

[15] F. Napolitano. On the cohomology of configuration spaces on surfaces. *J. London Math. Soc. (2)*, 68(2):477–492, 2003.

[16] P. Orlik and H. Terao. *Arrangements of hyperplanes*, volume 300 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.

[17] C. A. M. Peters and J. H. M. Steenbrink. Mixed Hodge structures. 52:xiv+470, 2008.

[18] D. Petersen. A spectral sequence for stratified spaces and configuration spaces of points. *Geom. Topol.*, 21(4):2527–2555, 2017.

[19] P. Tosteson. Lattice spectral sequences and cohomology of configuration spaces. 2016. arXiv: 1612.06034.

[20] B. Totaro. Configuration spaces of algebraic varieties. *Topology*, 35(4):1057–1067, 1996.

[21] R. Vakil and M. M. Wood. Discriminants in the Grothendieck ring. *Duke Math. J.*, 164(6):1139–1185, 2015.

[22] M. L. Wachs. Poset topology: tools and applications. In *Geometric combinatorics*, volume 13 of *IAS/Park City Math. Ser.*, pages 497–615. Amer. Math. Soc., Providence, RI, 2007.

[23] C. A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.