Chern-Weil Construction of Chern Classes

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Abstract

This is a study note under the guideline of an exercise for the course *MATH5230: Differential Topology* taught by Prof. Guowu Meng. In this article we are aimed at giving a definition of Chern classes of smooth complex vector bundles over smooth manifolds and proving the definition is a topological invariant.

1 Introduction

Let $\xi : E \to M$ denote a smooth complex vector bundle (simply called *bundle* throughout the article). Given a bundle $\xi : E \to M$ and a smooth map $f : M' \to M$, the **pullback** bundle $f^*\xi$ of ξ via f is defined as $\xi' : E' \to M'$, where E is the categorical pullback of the diagram $E, M' \to M$. It can be verified that ξ' has a local trivialization and transition functions naturally given by those of ξ via f, and is thus a smooth (complex) vector bundle.

A **characteristic class** with coefficient ring R is a pullback preserving way to assign to each isomorphic class of bundles $\xi : E \to M$ an element in $H^*(M; R)$, the cohomology ring of M with coefficients in R. In other words, if χ is a characteristic class, then $\chi(f^*\xi) = f^*(\chi(\xi))$.

Choosing $R = \mathbb{R}$, the cohomology ring $H^*(M; \mathbb{R})$ is isomorphic to the de Rham cohomology ring $H^*_{dR}(M)$. In the next section we will construct Chern classes for a given bundle ξ as cohomology classes of real differential forms. Then in Section 3 we will prove basic properties of Chern classes.

2 Construction

2.1 The curvature form

Suppose we are given a $\mathbb{C}$-linear connection $\nabla : \Gamma(\xi) \to \Gamma(\xi \otimes_{\mathbb{R}} T^*M)$ on ξ (Hereafter we simply write $\otimes$ for $\otimes_{\mathbb{R}}$). We are going to define the curvature form $F_\nabla \in \Gamma(\mathrm{End}_{\mathbb{C}}(\xi) \otimes \Lambda^2 T^*M)$ associated to the connection ∇.

Definition 2.1. Given a bundle $\xi : E \to M$, write $\Omega^k(\xi) = \Omega^k(M; E) := \Gamma(\xi \otimes \Lambda^k T^*M)$, whose elements are called E-**valued differential** k-**forms** on M. The wedge product of an ordinary differential form with an E-valued differential form is defined as

$$(s \otimes \omega) \wedge \rho = \omega \wedge (s \otimes \rho) := s \otimes (\omega \wedge \rho),\ \omega, \rho \in \Omega(M), s \in \Gamma(\xi).$$

We clearly have the graded commutative law inherited from that of ordinary differential forms. In particular, noting that $\Gamma(\xi)$ is just $\Omega^0(\xi)$, a section s of ξ is of even degree, so $s \otimes \omega = s \wedge \omega = \omega \wedge s$.

There is no canonical exterior derivative for E-valued differential forms. However if a connection on ξ is given, then we can define the **exterior derivative associated to the connection** ∇:

$$\begin{aligned} \mathrm{d}_\nabla : \Omega^k(\xi) &\to \Omega^{k+1}(\xi) \\ s \otimes \omega &\mapsto \nabla s \wedge \omega + s \wedge \mathrm{d}\omega \end{aligned}$$

When $k = 0$ above, we can set $\omega = 1$ and see that d_∇ is just ∇.

Proposition 2.2. *The exterior derivative associated to a connection is well-defined, $\mathbb{C}$-linear and it satisfies the graded Leibniz rule:*

$$\mathrm{d}_\nabla(\omega \wedge \rho) = \mathrm{d}_\nabla \omega \wedge \rho + (-1)^{\deg \omega} \omega \wedge \mathrm{d}_\nabla \rho$$

where ω is homogeneous, and exactly one of ω and ρ is an E-valued form while the other is an ordinary form. Here d_∇ of an ordinary form is just its exterior derivative.

Proof. Let us verify that d_∇ is well defined. By a lemma in manifold theory, $\Gamma(\xi \otimes_\mathbb{R} \Lambda^k T^*M) \cong \Gamma(\xi) \otimes_{C^\infty(M)} \Gamma(\Lambda^k T^*M)$. By the Leibniz rule of the ordinary exterior derivative and ∇, we have $\mathrm{d}_\nabla(fs \otimes \omega) - \mathrm{d}_\nabla(s \otimes f\omega) = (s \otimes df) \wedge \omega - s \wedge (df \wedge \omega) = 0$, where $f \in C^\infty(M)$. Hence d_∇ is $C^\infty(M)$-balanced, and thus well-defined by the universal properties of tensor products.

The lemma above also show that pure tensors $s \otimes \omega$ generates $\Omega^k(\xi)$ as an abelian group. The remaining properties of d_∇ can be therefore proved by direct computation on pure tensors. □

Unlike the exterior derivative of ordinary forms, the derivative of E-valued forms does not square to 0. This is exactly because there may not exist a local trivialization of E where ∇ is the usual differentiation of vector-valued function. In other words, the connection ∇ may not be **flat**. This motivates us to use d_∇^2 as a measure of *curvature*.

Proposition 2.3. d_∇^2 *is $C^\infty(M)$ linear on ξ-sections, and thus define a tensor field F_∇ in $\Gamma(\mathrm{End}_\mathbb{C}(\xi) \otimes \Lambda^2 T^*M)$, called the* ***curvature form*** *of ∇.*

Proof. Let $f \in C^\infty(M), s \in \Gamma(\xi)$. We have

$$\begin{aligned} \mathrm{d}_\nabla^2(fs) &= \mathrm{d}_\nabla(f\nabla s + s \otimes df) \\ &= df \wedge \nabla s + f\mathrm{d}_\nabla^2 s + \nabla s \wedge df + s \otimes d^2 f \\ &= f\mathrm{d}_\nabla^2 s \end{aligned}$$

□

2.2 Local computations

On a local trivialization, sections of ξ can be represented by a $\mathbb{C}$-vector-valued function $s = [f_i]$, and E-valued k-forms by $\omega = [\omega_i]$, where f_i are complex-valued functions and ω_i are complex k-forms. The componentwise derivative $\mathrm{d} : [f_i] \mapsto [df_i]$ is a connection. Recall that the difference of any two connections on ξ is tensorial, so $\nabla - \mathrm{d} \in \Gamma(\mathrm{End}_{\mathbb{C}}(\xi) \otimes T^*M)$ and we can locally write $\nabla = \mathrm{d} + A$ where $A = [A_{ij}]$ is a complex one-form valued matrix. We have $\nabla([f_i]) = [df_i + A_{ij} f_j]$. Writing $s_i = [\delta_{ij}]_j$, the standard frame of the local trivialization, we have $\nabla s_i = s_j \otimes A_{ji}$.

Lemma 2.4. *On a trivial bundle ξ with a connection $\nabla = \mathrm{d} + A$, we have*
(a) $\mathrm{d}_\nabla = \mathrm{d} + A\wedge$, i.e. $\mathrm{d}_\nabla([\omega_i]) = [\mathrm{d}\omega_i + A_{ij} \wedge \omega_j]_i$ for any E-valued k-form $[\omega_i]$.
(b) $F_\nabla = \mathrm{d}A + A \wedge A$, i.e. as a matrix of 2-forms, $F_\nabla = [\mathrm{d}A_{ij} + A_{ik} \wedge A_{kj}]_{ij}$.

Proof. (a) Given an E-valued k-form $[\omega_i]$, we have

$$\begin{aligned} \mathrm{d}_\nabla([\omega_i]) &= \mathrm{d}_\nabla(s_i \otimes \omega_i) \\ &= \nabla s_i \wedge \omega_i + s_i \otimes \mathrm{d}\omega_i = s_j \otimes A_{ji} \wedge \omega_i + s_i \otimes \mathrm{d}\omega_i \\ &= s_i \otimes (A_{ij} \wedge \omega_j + \mathrm{d}\omega_i) = [\mathrm{d}\omega_i + A_{ij} \wedge \omega_j]_i. \end{aligned}$$

(b) Given a section $s = [f_i]$ of ξ, we have

$$\begin{aligned} F_\nabla s :&= \mathrm{d}_\nabla^2 s \\ &= (\mathrm{d} + A\wedge)((\mathrm{d} + A)s) = \mathrm{d}^2 s + \mathrm{d}(As) + A \wedge As + A \wedge \mathrm{d}s \\ &= \mathrm{d}A \wedge s - A \wedge \mathrm{d}s + A \wedge As + A \wedge \mathrm{d}s \\ &= (\mathrm{d}A + A \wedge A)s \end{aligned}$$

□

By a standard facts about tensor fields, the connection ∇ on ξ extends naturally to connections (still written ∇) on all **tensor bundles** $\xi^{\otimes k} \otimes \xi^{*\otimes l}$ (whose sections are called **tensor fields**), which satisfy the Leibniz rule with respect to tensor products and commute with contractions. Applying Proposition 2.2, we get bundle-valued exterior derivatives d_∇ for all tensor bundles of ξ. The operator d_∇ satisfies the graded Leibniz rule and commutes with contraction, which can be proved by examining tensors of the form $A \otimes \omega$, where A is a tensor fields of ξ and ω is an ordinary differential form.

In particular, $\mathrm{End}(\xi) \cong \xi \otimes \xi^*$ is a tensor bundle, so we can talk about the exterior derivative of its bundle-valued forms. Under this identification, the matrix multiplication is exactly a tensor product followed by a contraction, so the Leibniz rule applies.

Lemma 2.5. *Let $F \in \Omega^k(\mathrm{End}(\xi))$. Using the local setting as before, F can be expressed by a k-form-valued matrix. Then*

$$\mathrm{d}_\nabla F = \mathrm{d}F + A \wedge F - (-1)^k F \wedge A,$$

where $\nabla = \mathrm{d} + A$.

Proof. Let $s \in \Gamma(\xi)$. By the graded Leibniz rule, we have

$$\begin{aligned}(\mathrm{d}_\nabla F)s &= \mathrm{d}_\nabla(Fs) - (-1)^k F \wedge \nabla s \\ &= (\mathrm{d} + A\wedge)(Fs) - (-1)^k F \wedge (\mathrm{d} + A)s \\ &= (\mathrm{d}F)s + (-1)^k F \wedge \mathrm{d}s + A \wedge Fs - (-1)^k (F \wedge \mathrm{d}s + F \wedge As) \\ &= (\mathrm{d}F + A \wedge F - (-1)^k F \wedge A)s\end{aligned}$$

□

We are ready to end this section with a global result as a consequence of the local computations above:

Proposition 2.6. *The curvature form satisfies* $\mathrm{d}_\nabla F_\nabla = 0$*.*

Proof. Noting that F_∇ has degree 2, we have

$$\begin{aligned}\mathrm{d}_\nabla F_\nabla &= \mathrm{d}F_\nabla + [A, F_\nabla] = \mathrm{d}(\mathrm{d}A + A \wedge A) + [A, \mathrm{d}A + A \wedge A] \\ &= \mathrm{d}(A \wedge A) + [A, \mathrm{d}A] + A \wedge A \wedge A - A \wedge A \wedge A \\ &= \mathrm{d}A \wedge A - A \wedge \mathrm{d}A + A \wedge \mathrm{d}A - \mathrm{d}A \wedge A = 0\end{aligned}$$

□

2.3 Definition of Chern classes

Note that the space of two-forms lies in the commutative ring $\Omega^{2*}(M)$ of even-degree differential forms. Pointwise, $F_\nabla(p) \in \mathrm{End}_{\mathbb{C}}(\xi|_p) \otimes_{\mathbb{R}} \Omega_p^{2*}(M) \cong \mathrm{End}_{\mathbb{C}\otimes\Omega_p^{2*}(M)}(\xi|_p \otimes \Omega_p^{2*}(M))$, so its trace, determinant and characteristic polynomial make sense and lie in the ring of complex-valued even-degree differential forms. The Chern classes are almost the coefficients of the characteristic polynomial of F_∇, up to some normalization.

Definition 2.7. Given a complex vector bundle ξ of rank k and a connection ∇ over ξ, write

$$\det(I + \frac{\mathrm{i}t}{2\pi} F_\nabla) = \sum_0^\infty (t^i c_i) = 1 + tc_1 + ... + t^k c_k.$$

We call c_i the **i-th Chern form** of ∇. Note that $c_0 = 1, c_{k+1} = c_{k+2} = ... = 0$.

Recall the linear algebra fact that given a $k \times k$ matrix A, $\det(I + A)$ is a rational polynomial in $\mathrm{tr}(A^m), m = 1, ..., k$. Hence c_i is an $2i$-form and is a real polynomial of $\mathrm{tr}((\mathrm{i}F_\nabla)^m), m = 1, ..., i$.

Lemma 2.8. *The $2m$-form $\mathrm{tr}(F_\nabla^m)$ is closed, and if ∇' is another connection on ξ, then $\mathrm{tr}(F_\nabla^m) - \mathrm{tr}(F_{\nabla'}^m)$ is an exact complex form.*

Proof. First we show closedness. Viewing $\mathrm{End}(\xi)$ as $\xi \otimes \xi^*$, the trace operator is just the contraction, so it commutes with d_∇. Therefore it suffices to show $\mathrm{d}_\nabla(F_\nabla^m) = 0$. As the commutative Leibniz rule holds for degree reason, we have the usual power rule

$$\mathrm{d}_\nabla(F_\nabla^m) = mF_\nabla^{m-1} \wedge \mathrm{d}_\nabla F_\nabla,$$

which vanishes as $\mathrm{d}_\nabla F_\nabla = 0$.

Now suppose $\nabla = \mathrm{d} + A, \nabla' = \mathrm{d} + A'$ on a local trivializtion. Write $B := \nabla' - \nabla \in \Omega^1(\mathrm{End}(\xi))$, locally represented by a matrix of one-forms, also denoted by B. Writing $\nabla_t = \nabla + tB$, represented by the matrix A_t, we have

$$\begin{aligned}\mathrm{tr}(F_{\nabla'}^m) - \mathrm{tr}(F_\nabla^m) &= \int_0^1 \frac{d}{dt}\mathrm{tr}(F_{\nabla_t}^m)\,dt \\ &= \int_0^1 \mathrm{tr}(mF_{\nabla_t}^{m-1}\frac{d}{dt}F_{\nabla_t})\,dt\end{aligned}$$

Using the local expression,

$$\begin{aligned}\frac{d}{dt}F_{\nabla_t} &= \frac{d}{dt}(\mathrm{d}A_t + A_t \wedge A_t) \\ &= \mathrm{d}(\frac{d}{dt}A_t) + \frac{d}{dt}(A_t \wedge A_t) \\ &= \mathrm{d}B + B \wedge A_t + A_t \wedge B \\ &= \mathrm{d}_{\nabla_t}B\end{aligned}$$

as B is of degree 1.

As $\mathrm{d}_{\nabla_t}F_{\nabla_t} = 0$, we have

$$\mathrm{tr}(mF_{\nabla_t}^{m-1}\mathrm{d}_{\nabla_t}B) = \mathrm{tr}(\mathrm{d}_{\nabla_t}(mF_{\nabla_t}^{m-1}B)) = \mathrm{d}\;\mathrm{tr}(mF_{\nabla_t}^{m-1}B).$$

Hence,

$$\mathrm{tr}(F_{\nabla'}^m) - \mathrm{tr}(F_\nabla^m) = \mathrm{d}\int_0^1 \mathrm{tr}(mF_{\nabla_t}^{m-1}B)\,dt,$$

which is globally exact. □

Corollary 2.9. *The i-th Chern form represents a cohomology class $[c_i] \in H^{2i}_{\mathrm{dR}}(M;\mathbb{C})$, which is independent of the connection ∇. We call $[c_i]$, often written as $c_i(\xi)$, the* ***i-th Chern class*** *of ξ.*

Proof. The lemma exactly shows that the differential form $\mathrm{tr}(F_\nabla^m)$ represents a connection-independent cohomology class, for $m = 0, 1, ..., k$. Since the wedge product of de Rham cohomology classes is well-defined, $[c_i]$ can be expressed as a (complex) polynomial of $[\mathrm{tr}(F_\nabla^m)]$ and is thus also a connection-independent cohomology class. □

Now it remains to identify $[c_i]$ as a *real* cohomology class. Note that $H^*_{\mathrm{dR}}(M;\mathbb{R})$ is a subset of $H^*_{\mathrm{dR}}(M;\mathbb{C})$, since a real form is closed (exact) as a real form if and only if it is closed (exact) as a complex form. (We just verify one of the implications. Suppose ω is a real form such that $\omega = \mathrm{d}\alpha$ where α is a complex form. Then taking α' to be the real part of α, we get $\omega = \mathrm{d}\alpha'$, so that ω is exact as a real form.)

We claim that $[c_i] \in H^*_{\mathrm{dR}}(M;\mathbb{R})$. As $[c_i]$ is a real polynomial in $[\mathrm{tr}((\mathrm{i}F_\nabla)^m)], m = 0, 1, ..., k$, all we need is to show $[\mathrm{tr}((\mathrm{i}F_\nabla)^m)]$ is real. In fact, there is ∇ where the form $\mathrm{tr}((\mathrm{i}F_\nabla)^m)$ is real.

Definition 2.10. Suppose we are given an Hermitian structure $\langle\cdot,\cdot\rangle$ on ξ, with the convention $\langle\lambda v, w\rangle = \bar{\lambda}\langle v, w\rangle$. Then a connection ∇ is called an **Hermitian connection** if $d\langle s, s'\rangle = \langle\nabla s, s'\rangle + \langle s, \nabla s'\rangle$ for all ξ-sections s and s'.

If we locally trivialize ξ by using an orthonormal local frame, then d is an Hermitian connection. Using the partition of unity, a global Hermitian connection always exists on ξ.

Proposition 2.11. *If ∇ is an Hermitian connection on an Hermitian bundle ξ, then* $\mathrm{tr}((\mathrm{i}F_\nabla)^m)$ *is real.*

Proof. The question is local. Using the orthonormal trivialization described above, we may write $\nabla = \mathrm{d} + A$. As both d and ∇ are Hermitian connections, we have

$$d\langle s, s'\rangle = \langle \mathrm{d}s, s'\rangle + \langle s, \mathrm{d}s'\rangle$$

and

$$d\langle s, s'\rangle = \langle(\mathrm{d} + A)s, s'\rangle + \langle s, (\mathrm{d} + A)s'\rangle.$$

By subtraction, we get $\langle As, s'\rangle + \langle s, As'\rangle = 0$. Hence A is a skew-Hermitian matrix, i.e. $A^* + A = 0$.

We claim F_∇ is also skew-Hermitian. Indeed,

$$\begin{aligned} F^*_\nabla &= (\mathrm{d}A + A \wedge A)^* = \mathrm{d}A^* - A^* \wedge A^* \\ &= -\mathrm{d}A - (-A) \wedge (-A) = -F_\nabla, \end{aligned}$$

where the minus sign after the second equality is because the formula $(AB)^* = B^*A^*$ reverses the order of multiplication.

As a consequence, $\mathrm{i}F_\nabla$ is Hermitian. Since $\mathrm{i}F_\nabla$ commutes with itself, its powers are also Hermitian. Hence $\mathrm{tr}((\mathrm{i}F_\nabla)^m)$ is real.

□

Corollary 2.12. *Any Hermitian connection with respect to any Hermitian structure on ξ gives by its i-th Chern form a real representative of $c_i(\xi)$. In particular, $c_i(\xi) \in H^{2i}_{\mathrm{dR}}(M)$.*

3 Properties

Proposition 3.1. *(Whitney sum formula) Define the* ***total Chern class*** *of a bundle to be* $c(\xi) := c_0 + c_1 + c_2 + ... = \det(I + \frac{\mathrm{i}}{2\pi} F_\nabla)$. *Given two bundles* ξ_1 *and* ξ_2 *on a manifold* M*, we have*

$$c(\xi_1 \oplus \xi_2) = c(\xi_1)c(\xi_2)$$

Proof. Suppose we are given connections ∇_i on ξ_i. Define a connection ∇ on $\xi := \xi_1 + \xi_2$ by

$$\nabla(s_1 \oplus s_2) := \nabla_1 s_1 \oplus \nabla_2 s_2.$$

It is easy to check that ∇ is a connection. Choosing a local trivialization and writing $\nabla_i = \mathrm{d} + A_i$, we can also verify that $\nabla = \mathrm{d} + A_1 \oplus A_2$, where $A_1 \oplus A_2$ is the block diagonal matrix given by

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}.$$

As the matrix multiplication can be computed block by block, $F_\nabla = \mathrm{d}A + A \wedge A = F_{\nabla_1} \oplus F_{\nabla_2}$. Evaluating determinants in blocks, we get $c(\xi) = c(\xi_1)c(\xi_2)$. □

Remark. Comparing terms of degree $2k$, we get a more explicit though less elegant form of the Whitney sum formula:

$$c_k(\xi) = \sum_{i=1}^{k} c_i(\xi_1)c_{k-i}(\xi_2)$$

Proposition 3.2. *Let* ℓ_1 *and* ℓ_2 *be two complex line bundles (i.e. complex vector bundles of rank* 1*), and* $\ell := \ell_1 \otimes \ell_2$*. Then*

$$c_1(\ell) = c_1(\ell_1) + c_1(\ell_2)$$

Proof. Let ∇_i be a connection on ℓ_i. Define a connection ∇ on ℓ by

$$\nabla(s_1 \otimes s_2) = \nabla_1 s_1 \otimes s_2 + s_1 \otimes \nabla_2 s_2$$

To see ∇ is indeed a well-defined connection, note that the formal map $\nabla(\cdot \otimes \cdot)$ satisfies

$$\nabla(fs_1 \otimes s_2) = df \otimes s_1 \otimes s_2 + f\nabla_1 s_1 \otimes s_2 + fs_1 \otimes \nabla_2 s_2 = \nabla(s_1 \otimes fs_2)$$

for any $f \in C^\infty(M)$. Being $C^\infty(M)$-balanced, ∇ is well defined. The computation also proves the Leibniz rule.

Now under a local trivialization, ∇_i is written $\mathrm{d} + A_i$, where A_i is just a 1-form. Given a section s, locally represented by a function f, we have

$$\nabla(f) = \nabla(1 \cdot f) = (d(1) + A_1)f + 1 \cdot (df + A_2 f) = df + (A_1 + A_2)f$$

Hence ∇ is represented by $A = A_1 + A_2$. As $F_\nabla = \mathrm{d}A + A \wedge A = \mathrm{d}A$, linear in A, we get $F_\nabla = F_{\nabla_1} + F_{\nabla_2}$. Notice that $c_1(\nabla)$ is simply $\frac{\mathrm{i}}{2\pi} F_\nabla$, so it is linear in F_∇. Thus $c_1(\xi) = c_1(\xi_1) + c_1(\xi_2)$. □

Proposition 3.3. *Chern classes are characteristic classes, i.e. if $\xi : E \to M$ is a bundle, $\phi : M' \to M$ is a smooth map and $\xi' : E' \to M'$ is the pullback of ξ via ϕ, then $c_i(\xi') = \phi^*(\xi)$.*

Proof. Let ∇ be a connection on ξ. Let $\nabla' := \phi^*\nabla$, the pullback of ∇ via ϕ, which is defined as follows:

On an open set $U \subseteq M$ where ξ is trivial, write $\nabla = \mathrm{d} + A$, where $A = [A_{ij}]$ is a matrix of one-forms on U. Let $U' = \phi^{-1}(U)$. Then there is a trivialization on U' given by the pullback. We define $\nabla' = \mathrm{d} + [\phi^* A_{ij}]$ on U'.

To check it is globally defined, we shall give it a coordinate independent characterization. Define the pullback of E-valued k-forms by $\phi^* : \Omega^k(U;\xi) \to \Omega^k(U';\xi'), s \otimes \omega \mapsto \phi^* s \otimes \phi^*\omega$. We claim that on any trivial bundle U, the connection ∇' we have defined is the unique connection ∇' on U' such that for any section s of $\xi|_U$, we have

$$\nabla'(\phi^* s) = \phi^*(\nabla s). \quad (*)$$

Write $\nabla' = \mathrm{d} + A'$ on U'. We have

$$\nabla'(\phi^* s) = \phi^*(\nabla s) = \phi^* \mathrm{d}s + \phi^*(As) = (\mathrm{d} + \phi^* A)(\phi^* s).$$

On the other hand,

$$\nabla'(\phi^* s) = (\mathrm{d} + A')(\phi^* s).$$

Hence $\nabla' = \mathrm{d} + A'$ satisfies (*). To show the uniqueness, let s_i be represented by the constant vector $e_i = [\delta_{ij}]_j$, then $s_i' = \phi^* s_i$ is also represented by e_i. Thus $(\phi^* A)e_i = A' e_i$, i.e. the i-th column of A' and $\phi^* A$ is the same.

Now $F_\nabla = \mathrm{d}A + A \wedge A$ and $F_{\nabla'} = \mathrm{d}A' + A' \wedge A' = \phi^* F_\nabla$. Taking characteristic polynomials, we get $c_i(\xi') = \phi^*(c_i(\xi))$. ☐