Weierstrass points on a tropical curve

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What is a Weierstrass point?

Definition: $X$ a smooth algebraic curve, $D_N$ a divisor of degree $N$

$\leadsto$ projective embedding $\phi : X \to \mathbb{P}^r$. 
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Definition: \(X\) a smooth algebraic curve, \(D_N\) a divisor of degree \(N\)
\(\leadsto\) projective embedding \(\phi : X \to \mathbb{P}^r\).

\[W(D_N) = \{ x \in X : \exists H \subset \mathbb{P}^r \text{ s.t. } m_x(H \cap X) \geq r + 1 \}\]
\[= \left\{ x \in X : \text{“higher-than-expected” tangency with some hyperplane } H \text{ at } x \right\} \]
What is a Weierstrass point?

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Example: \( X = \{xyz + x^3 + y^3 + z^3 = 0\} \subset \mathbb{P}_\mathbb{C}^2 \)

\( N = 3 \)
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\[ N = 3 , \quad \# W(D) = 9 \]
What is a Weierstrass point?

Intuition (Mumford):

\[ N \text{-torsion points} \leftrightarrow \text{Weierstrass points of } D_N \]

on an elliptic curve on a higher-genus curve

Numerical "evidence": as \( N \) grows, \( \#(\text{Weierstrass points of } D_N) = gN^2 + O(N) \)

Problem

How are Weierstrass points distributed on an algebraic curve?
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Problem

How are Weierstrass points distributed on an algebraic curve?
Weierstrass points: genus 1, complex case

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How are Weierstrass points distributed on genus 1 curve $X/\mathbb{C}$?
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$\rightsquigarrow$ Weierstrass points distribute \textit{uniformly}, w.r.t. $\mathbb{C} \rightarrow \mathbb{C}/\Lambda$
Problem

How are Weierstrass points distributed on higher genus curve $X/\mathbb{C}$?
Weierstrass points: genus $\geq 2$, complex case

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How are Weierstrass points distributed on higher genus curve $X/\mathbb{C}$?

Theorem (Neeman, 1984)
Suppose $X$ is a complex algebraic curve of genus $g \geq 2$. Then $W(D_N)$ distributes according to the Bergman measure as $N \to \infty$. 
Weierstrass points: genus $\geq 2$, complex case

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Tropical Weierstrass points
20 November 2019
Weierstrass points: genus $\geq 2$, complex case

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Weierstrass points: non-Archimedean case

Problem

How are Weierstrass points distributed on $X/K$, $\text{val} : K^\times \to \mathbb{R}$?

Source: Matt Baker’s math blog
Weierstrass points: non-Archimedean case

Problem

How are Weierstrass points distributed on $X_K/K$ $X^\text{an}$?

Source: Matt Baker’s math blog
Weierstrass points: non-Archimedean case

Problem
How are Weierstrass points distributed on $\frac{X}{K}$ $X^{an}$?

Theorem (Amini, 2014)

Suppose $X^{an}$ is a Berkovich curve of genus $g \geq 2$. Then $W(D_N)$ distributes according to the Zhang measure as $N \rightarrow \infty$. 

Source: Matt Baker’s math blog
Weierstrass points: non-Archimedean case

Problem
How are Weierstrass points distributed on $X/K$? $X^{an}$?

Source: Matt Baker’s math blog

Problem (Amini, 2014)
Does the distribution follow from considering only the skeleton $\Gamma \subset X^{an}$?
Problem

How are Weierstrass points distributed on $X/K / \mathcal{X}^{an}$?

Problem (Amini, 2014)

Does the distribution follow from considering only the skeleton $\Gamma \subset \mathcal{X}^{an}$?
What is a tropical curve?

Tropical curve ( = a skeleton of $X^{\text{an}}$)
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Tropical curve ( = a skeleton of $X^{an}$) (combinatorics) = finite graph with edge lengths
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(combinatorics) = finite graph with edge lengths
(alg. geometry) = model for a degenerating algebraic curve
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Tropical curve ( = a skeleton of $X^\text{an}$)

(combinatorics) = finite graph with edge lengths
(alg. geometry) = model for a degenerating algebraic curve

Example: $X_t = \{xyz - tx^3 + t^2y^3 + t^5z^3 = 0\} \subset \mathbb{P}_\mathbb{C}^2$

$t=1$  $t=\varepsilon$  $t=0$
What is a tropical curve?

Tropical curve ( = a skeleton of $X^{an}$)

(combinatorics) = finite graph with edge lengths
(alg. geometry) = model for a degenerating algebraic curve

Example: $X_t = \{xyz - t^1 x^3 + t^2 y^3 + t^5 z^3 = 0\} \subset \mathbb{P}_\mathbb{C}^2$

$\rightsquigarrow$ dual graph of $X_0$
What is a tropical curve?

Tropical curve (= a skeleton of \(X^{an}\))

(combinatorics) = finite graph with edge lengths metric graph
(alg. geometry) = model for a degenerating algebraic curve

Example: \(X_t = \{xyz - t^1x^3 + t^2y^3 + t^5z^3 = 0\} \subset \mathbb{P}^2_\mathbb{C}\)
Tropical curves: divisor theory

Tropical curve $= \text{metric graph}$

<table>
<thead>
<tr>
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<tbody>
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Intuition: linear equivalence on $\Gamma = \text{"discrete current flow"}$

$q$-reduced divisor $\text{red } q[D] = \text{"energy-minimizing" divisor in } |D|$
Tropical curves: divisor theory

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$|D| = \{E \text{ lin. equiv. to } D, E \geq 0\}$
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$q$-reduced divisor $\text{red}_q[D] = \text{“energy-minimizing” divisor in } |D|$
Tropical curves: reduced divisors

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Example:

What happens as $q$ varies?
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Example:
Problem

How are Weierstrass points distributed on a tropical curve?

Definition: $\Gamma =$ metric graph, $D_N$ divisor of degree $N$
$\leadsto$ Baker–Norine rank $r = r(D_N)$

$$W(D_N) = \{ x \in X : \text{red}_x[D_N] \geq (r + 1)x \}$$
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Observation: as \( N \) grows,

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EXCEPT sometimes $\#(\text{Weierstrass points}) = \infty$
Example: Genus $g(\Gamma) = 1$:

degree $D = 6$, 
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degree $D = 6$, $\sim$ $\#(W(D)) = 6$
Example: Genus $g(\Gamma) = 3$:

degree $D = 4$, 
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degree $D = 4$, $\leadsto$ $\#(W(D)) = 8$
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\[ \text{degree } D = 4, \quad \leadsto \quad \#(W(D)) = \infty! \]
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Weierstrass points: tropical case

In general, this problem doesn’t happen!

Theorem (R)

For a generic divisor class \([D]\), the Weierstrass locus \(W(D)\) is finite.
Weierstrass points: tropical case

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**Theorem (R)**

*For a generic divisor class \([D]\), the Weierstrass locus \(W(D)\) is finite.*

So, we can still ask

**Problem**

How are Weierstrass points distributed *supposing* \(W(D)\) is finite?
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*For a generic divisor class \([D]\), the Weierstrass locus \(W(D)\) is finite.*

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How are Weierstrass points distributed supposing \(W(D)\) is finite for **generic** \([D]\)?
Weierstrass points: tropical case

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For a generic divisor class $[D]$, the Weierstrass locus $W(D)$ is finite.

So, we can still ask

Problem

How are Weierstrass points distributed supposing $W(D)$ is finite for generic $[D]$?

Theorem (R)

For a sequence of generic divisor classes $[D_N]$ on $\Gamma$, the Weierstrass locus $W(D_N)$ distributes according to Zhang’s canonical measure $\mu$. 
\( \Gamma = \text{electrical network by replacing each edge } \sim \text{ resistor} \)
Electrical networks

\[ \Gamma = \text{electrical network by replacing each edge } \rightsquigarrow \text{resistor} \]

Given \( y, z \in \Gamma \), let

\[ j^y_z = \begin{pmatrix} \text{voltage on } \Gamma \text{ when 1 unit of} \\ \text{current is sent from } y \text{ to } z \end{pmatrix} \]

By Ohm's law, \( \text{current} = \frac{\text{voltage}}{\text{resistance}} = \text{slope of } j^y_z \)
Electrical networks

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Example: current \( = (j^y_z)' \)
**Electrical networks**

**Example:** current $= (j^y_z)'$

![Diagram of electrical network with node labels and currents]

Satisfies Laplacian $\Delta(j^y_z) = z - y$

![Diagram of electrical network with node labels and currents]
Electrical networks: Canonical measure

\[ \Gamma = \text{metric graph} \]

**Definition ("electrical" version, Chinburg–Rumely–Baker–Faber)**

Zhang’s **canonical measure** \( \mu \) on an edge is the “current defect”

\[
\mu(e) = \text{current bypassing } e \text{ when 1 unit sent from } e^- \text{ to } e^+ \\
= 1 - \text{(current through } e \text{ when ... )}
\]
Electrical networks: Canonical measure

Γ = metric graph

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Example:

![Diagram](image)

$\mu(e) = \frac{7}{12}$

$\mu(e) = \frac{1}{3}$

$\mu(e) = 0$

$\mu(e) = 1$
Gamma = metric graph

**Definition ("electrical" version, Chinburg–Rumely–Baker–Faber)**

Zhang’s **canonical measure** $\mu$ on an edge is the “current defect”

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$$= 1 - (\text{current through } e \text{ when ... })$$

Generally:

- $0 \leq \mu(e) \leq 1$
- $\mu(e) = 0 \iff e \text{ a bridge}$
- $\mu(e) = 1 \iff e \text{ a loop}$

**Foster’s Theorem:** $\mu(\Gamma) = \sum_{e \in E} \mu(e) = g$
Theorem (R)

For a sequence of generic divisor classes \([D_N]\) on \(\Gamma\), the Weierstrass locus \(W(D_N)\) distributes according to Zhang’s canonical measure \(\mu\).

Namely, for any edge \(e\)

\[
\frac{\#(W(D_N) \cap e)}{N} \rightarrow \mu(e) \quad \text{as} \quad N \rightarrow \infty.
\]
Theorem (R)

For a sequence of generic divisor classes $[D_N]$ on $\Gamma$, the Weierstrass locus $W(D_N)$ distributes according to Zhang’s canonical measure $\mu$.

Namely, for any edge $e$

$$\frac{\#(W(D_N) \cap e)}{N} \to \mu(e) \quad \text{as} \quad N \to \infty.$$ 

Idea:

(discrete current flow) $\xrightarrow{N \to \infty}$ (continuous current flow)
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For a sequence of generic divisor classes $[D_N]$ on $\Gamma$, the Weierstrass locus $W(D_N)$ distributes according to Zhang's canonical measure $\mu$.

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canonical measure $\mu(e)$
Theorem (R)

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Namely, for any edge \( e \)

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\]

Idea:

(discrete current flow) \( \xrightarrow{N \rightarrow \infty} \) (continuous current flow)

\#(Weierstrass points on \( e \)) \[\uparrow\] canonical measure \( \mu(e) \)
References

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Weierstrass points on a tropical curve

Thank you!