Distribution of tropical Weierstrass points

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University of Michigan

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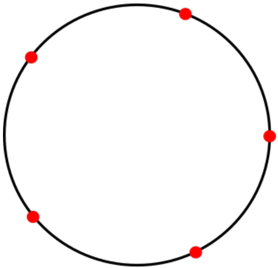
SIAM Chip-firing and tropical curves
13 July 2019

“Torsion points” on graphs

A circle has N-torsion points:



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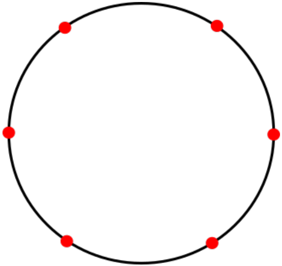

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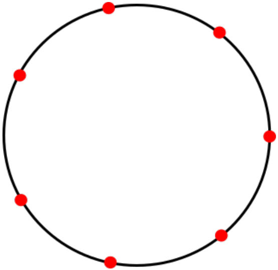

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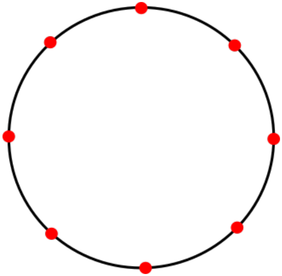

"Torsion points" on graphs

Vague Problem

What are the "N-torsion points" on an arbitrary metric graph Γ?

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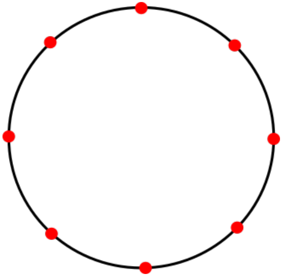

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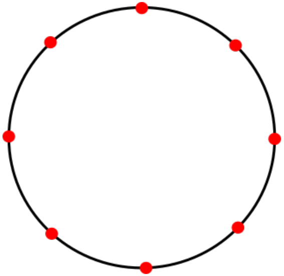

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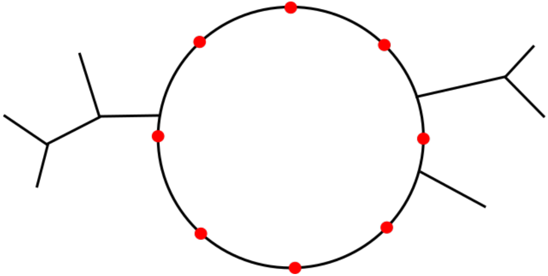

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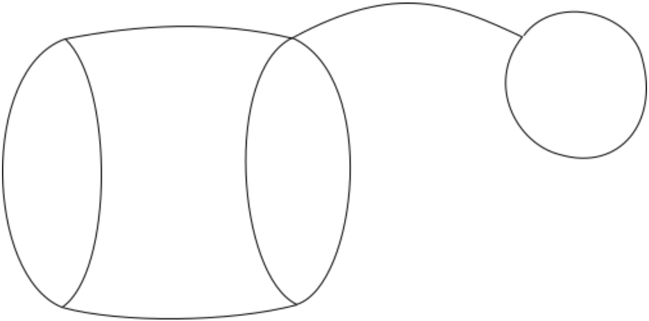

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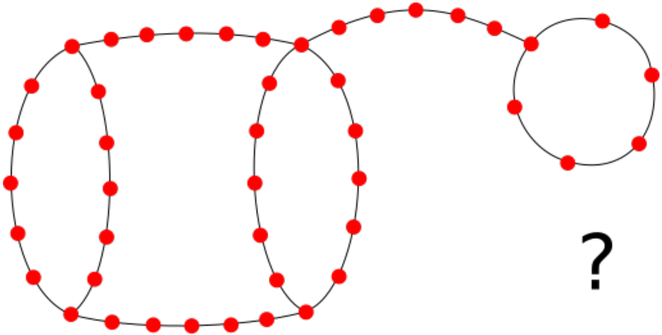

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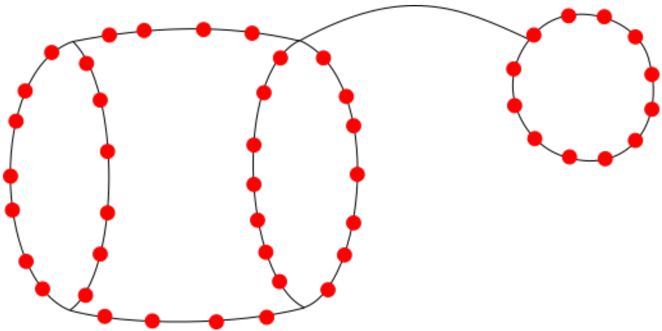

Weierstrass points

Problem

How are Weierstrass points distributed on an algebraic curve?

Definition: X a curve, D_N a divisor of degree N
$\rightsquigarrow$ projective embedding $\phi : X \to \mathbb{P}^r$.

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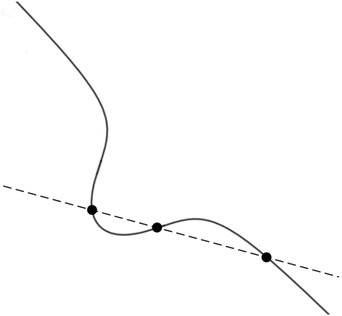
$$W(D_N) = \{x \in X : \exists H \subset \mathbb{P}^r \text{ s.t. } m_x(H \cap X) \geq r+1\}$$
$$= \{x \in X : \text{“higher-than-expected” tangency with } H \text{ at } x\}$$

Problem

How are Weierstrass points distributed on an algebraic curve?

Example: $X = \{xyz + x^3 + y^3 + z^3 = 0\} \subset \mathbb{P}^2_{\mathbb{C}}$



$N = 3$

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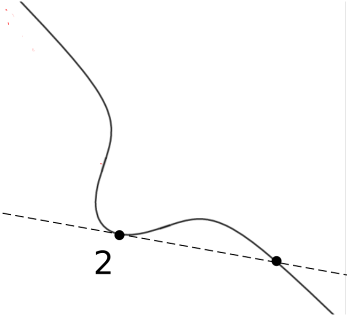

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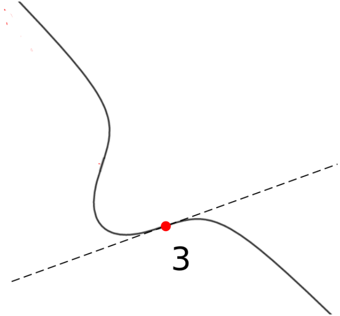

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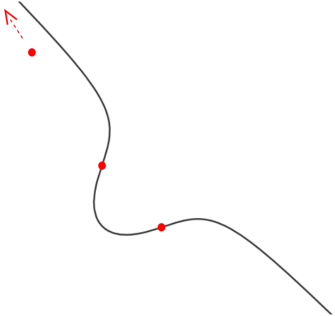

$N = 3$, $\#W(D) = 9$

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Intution (Mumford):

N-torsion points on an elliptic curve	$\leftrightarrow$	Weierstrass points of D_N on a higher-genus curve

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Numerical "evidence": as N grows,

$$\#(\text{Weierstrass points of } D_N) = gN^2 + O(N)$$

Weierstrass points: genus 1, complex case

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How are Weierstrass points distributed on genus 1 curve $X/\mathbb{C}$?

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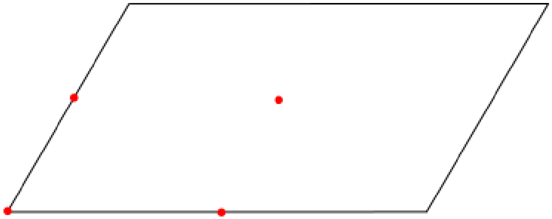

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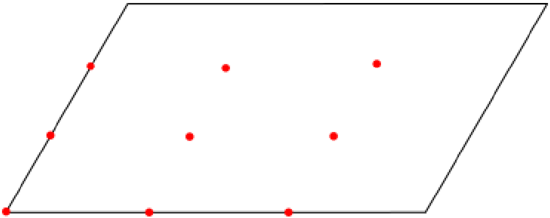

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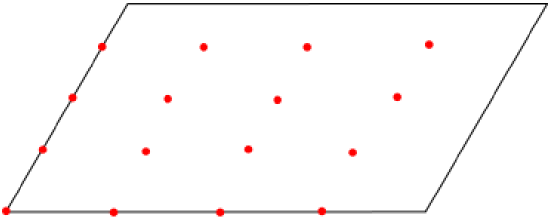

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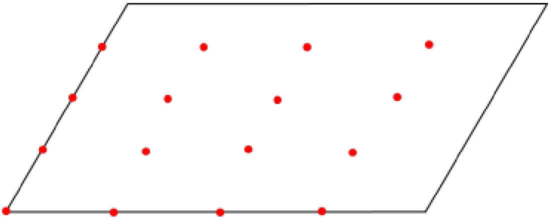

$\rightsquigarrow$ Weierstrass points distribute **uniformly**, w.r.t. $\mathbb{C} \to \mathbb{C}/\Lambda$

Weierstrass points: genus ≥ 2, complex case

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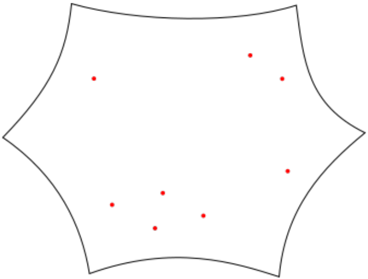

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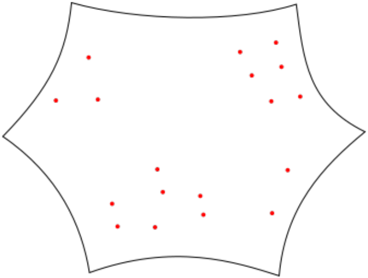

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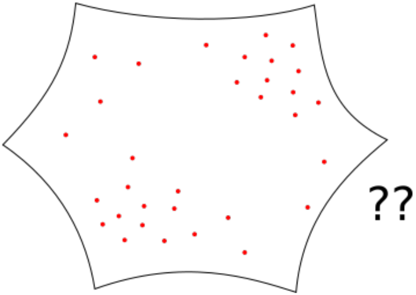

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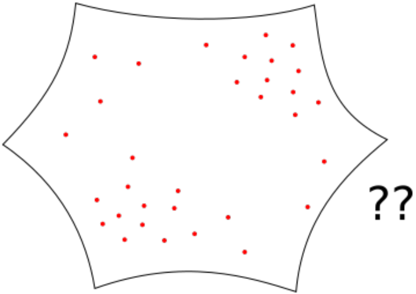


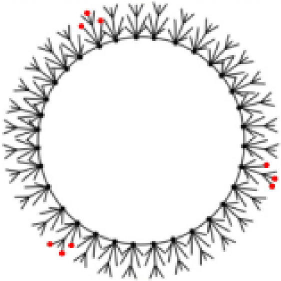
Theorem (Neeman, 1984)

Suppose X is a complex algebraic curve of genus $g \geq 2$. Then $W(D_N)$ distributes according to the Bergman measure as $N \to \infty$.

Weierstrass points: non-Archimedean case

Problem

How are Weierstrass points distributed on $X/\mathbb{K}$, $val : \mathbb{K}^\times \to \mathbb{R}$?



Source: Matt Baker's math blog

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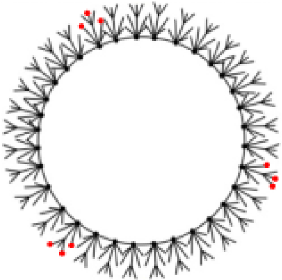
How are Weierstrass points distributed on ~~$X/\mathbb{K}$~~ X^{an}?



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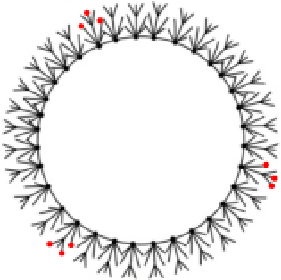

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Theorem (Amini, 2014)

Suppose X^{an} is a Berkovich curve of genus $g \geq 2$. Then $W(D_N)$ distributes according to the Zhang measure as $N \to \infty$.

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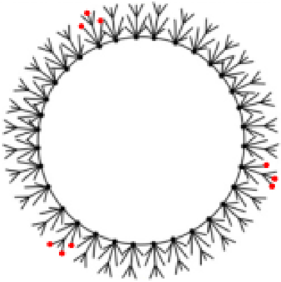

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Does the distribution follow from considering <u>only</u> the skeleton $\Gamma \subset X^{an}$?

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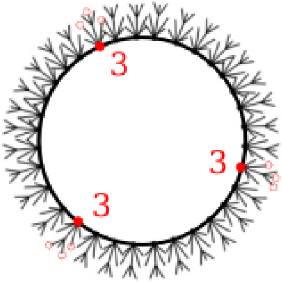

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Tropical curves: divisor theory

Tropical curve = metric graph

alg. curve X		tropical curve Γ
divisors $\mathrm{Div}(X)$	$\rightsquigarrow$	divisors $\mathrm{Div}(\Gamma)$
meromorphic functions	$\rightsquigarrow$	piecewise $\mathbb{Z}$-linear functions
linear system $\lvert D \rvert$	$\rightsquigarrow$	linear system $\lvert D \rvert$
$= \mathbb{P}^r$		$=$ polyhedral complex of dim $\geq r$
rank $r = \dim \lvert D \rvert$	$\rightsquigarrow$	rank $r =$ Baker-Norine rank

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Intuition: linear equivalence on Γ = "discrete current flow"

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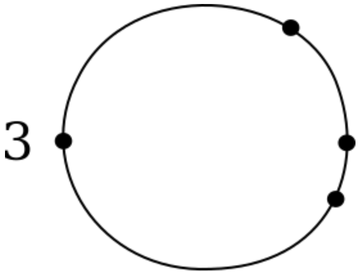

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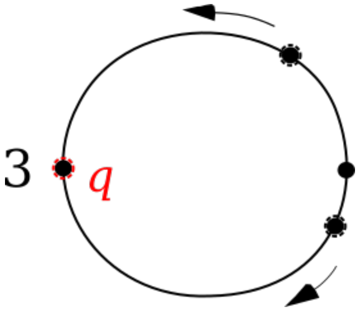

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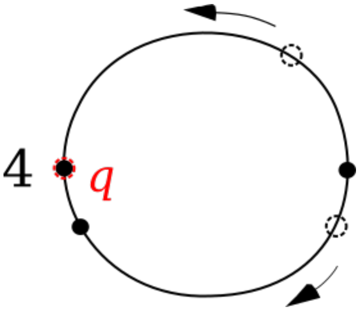

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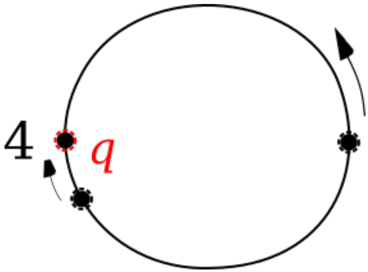

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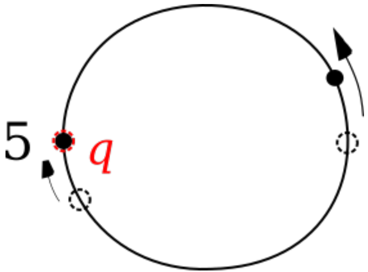

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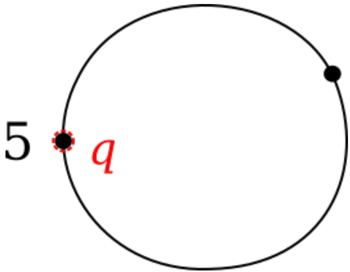

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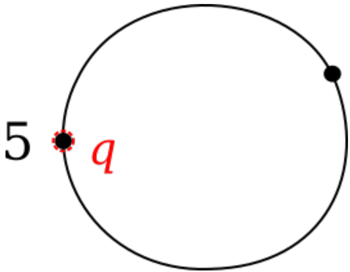

What happens as q varies?

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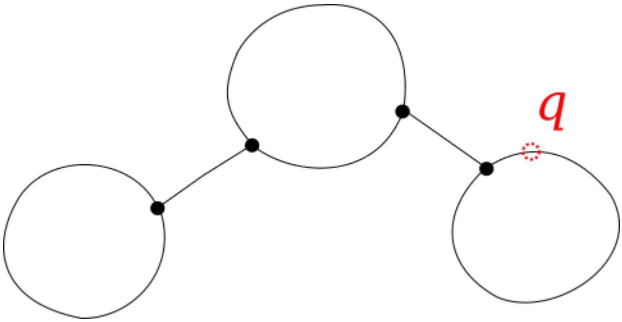

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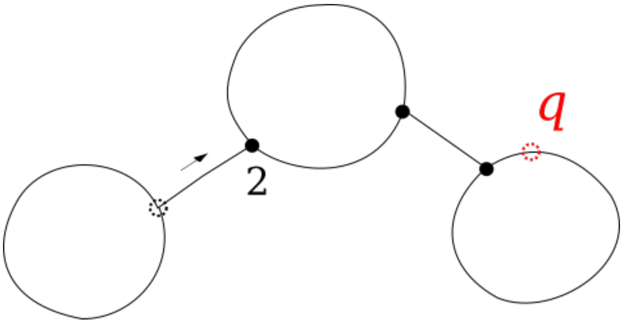

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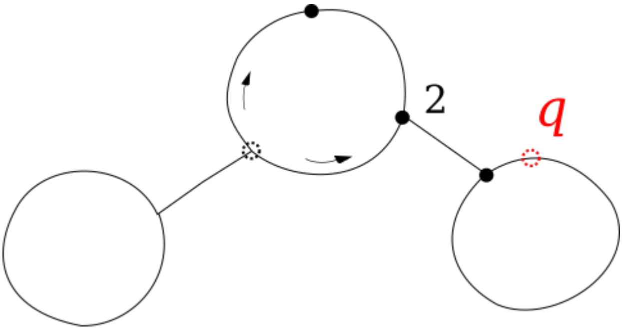

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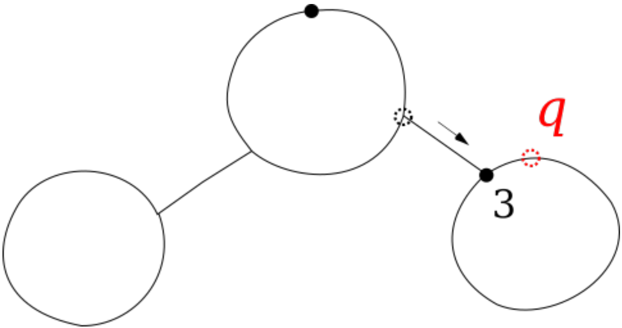

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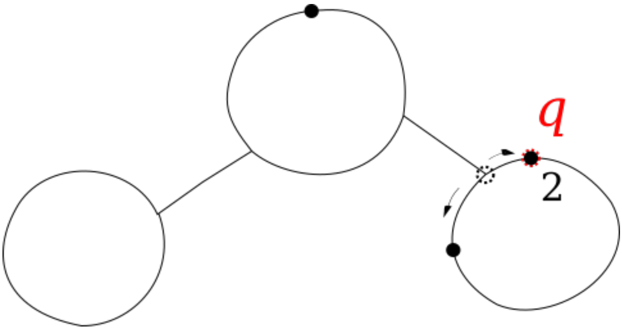

Tropical curves: Weierstrass points

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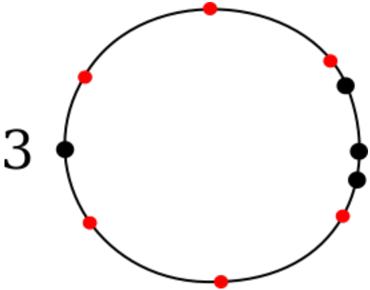
EXCEPT sometimes $\#(\text{Weierstrass points}) = \infty$

Weierstrass points: tropical case

Example: Genus $g(\Gamma) = 1$:



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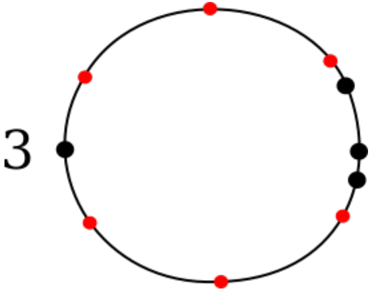
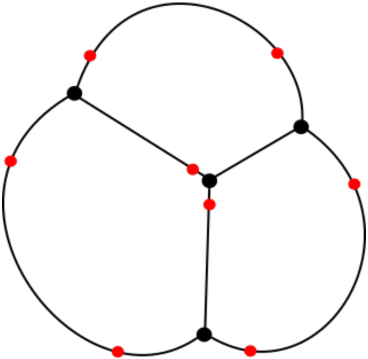


degree $D = 6$, $\rightsquigarrow$ $\#(W(D)) = 6$

Weierstrass points: tropical case

Example: Genus $g(\Gamma) = 3$:



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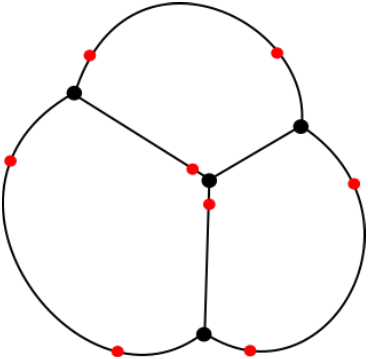

degree $D = 4$, $\rightsquigarrow$ $\#(W(D)) = 8$

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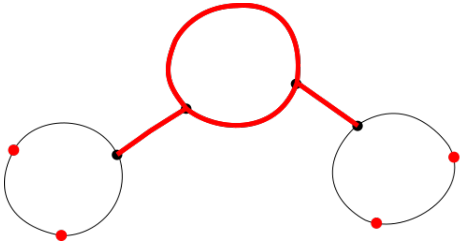

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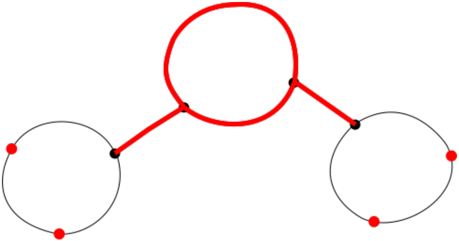

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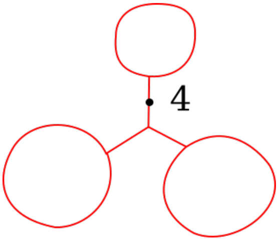

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In general, this problem doesn't happen!

Theorem (R)

For a generic divisor class $[D]$, the Weierstrass locus $W(D)$ is finite.

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For a sequence of generic divisor classes $[D_N]$ on Γ, the Weierstrass locus $W(D_N)$ distributes according to Zhang's canonical measure μ.

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$\Gamma =$ electrical network by replacing each edge $\rightsquigarrow$ resistor

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Given $y, z \in \Gamma$, let

$$j_z^y = \left(\begin{array}{l} \text{voltage on } \Gamma \text{ when 1 unit of} \\ \text{current is sent from } y \text{ to } z \end{array} \right)$$

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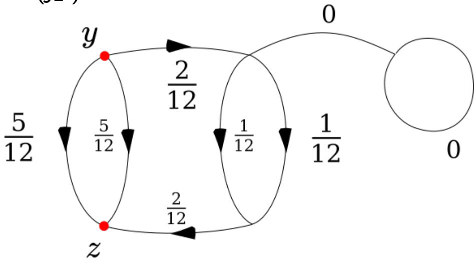

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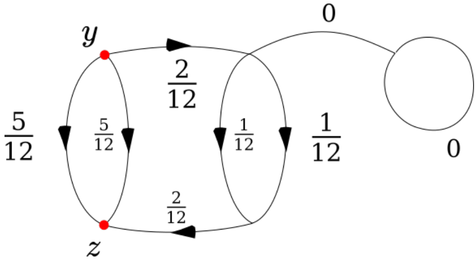
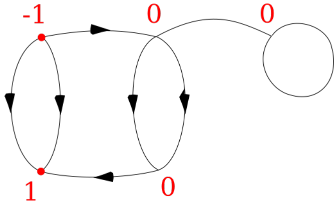


satisfies Laplacian $\Delta(j_z^y) = z - y$



Electrical networks: Canonical measure

Γ = metric graph

Definition ("electrical" version, Chinburg–Rumely–Baker–Faber)

Zhang's **canonical measure** μ on an edge is the "current defect"

$$\begin{aligned} \mu(e) &= \text{current bypassing } e \text{ when 1 unit sent from } e^- \text{ to } e^+ \\ &= 1 - (\text{current through } e \text{ when ... }) \end{aligned}$$

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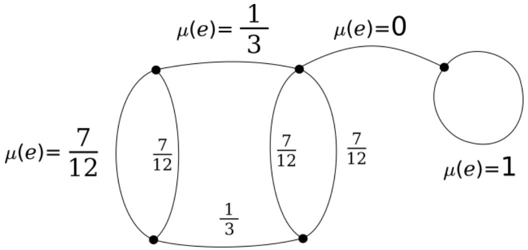

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Generally:

- $0 \leq \mu(e) \leq 1$
- $\mu(e) = 0 \Leftrightarrow e$ a bridge
- $\mu(e) = 1 \Leftrightarrow e$ a loop

Foster's Theorem: $\mu(\Gamma) = \sum_{e \in E} \mu(e) = g$

Electrical networks: Canonical measure

$\Gamma =$ metric graph

Definition ("tropical" version)

Zhang's **canonical measure** μ on an edge is the probability

$$\mu(e) = \mathbb{P}(\text{break divisor supported on } e)$$

Recall: a break divisor on Γ has degree g

Generally:

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Tropical Weierstrass distribution: proof idea

Theorem (R)

For a sequence of generic divisor classes $[D_N]$ on Γ, the Weierstrass locus $W(D_N)$ distributes according to Zhang's canonical measure μ.

Namely, for any edge e

$$\frac{\#(W(D_N) \cap e)}{N} \to \mu(e) \qquad \text{as} \qquad N \to \infty.$$

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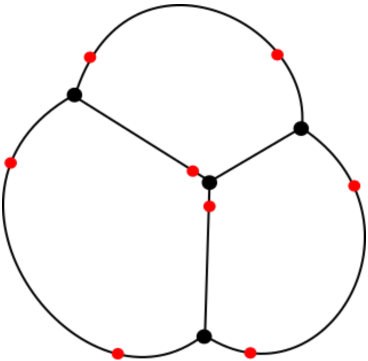

Thank you!