Equidistribution of Weierstrass points on tropical curves

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Overview

1. Weierstrass points
   - complex case
   - non-Archimedean case

2. Tropical curves
   - what is it?
   - divisors
   - tropical Weierstrass points

3. Electrical networks
   - voltage
   - canonical measure

4. Proof
Problem

How are Weierstrass points distributed on a curve?

Definition: $X$ a curve, $D_N$ a divisor of degree $N$

$\rightsquigarrow$ projective embedding $\phi : X \to \mathbb{P}^r$. 
Weierstrass points

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$\leadsto$ projective embedding $\phi : X \to \mathbb{P}^r$.

$W(D_N) = \{x \in X : \exists H \subset \mathbb{P}^r \text{ s.t. } m_x(H \cap X) \geq r + 1\}$

$= \{x \in X : \text{“higher-than-expected” tangency with } H \text{ at } x\}$
Weierstrass points

Problem

How are Weierstrass points distributed on a curve?

Example: \( X = \{xyz + x^3 + y^3 + z^3 = 0\} \subset \mathbb{P}^2_C \)
Weierstrass points

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Example: \( X = \{ xyz + x^3 + y^3 + z^3 = 0 \} \subset \mathbb{P}^2_{\mathbb{C}} \)

\( D = 3 \cdot \infty, \quad \# W(D) = 9 \)
Weierstrass points

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Intuition (Mumford):

$N$-torsion points on an elliptic curve $\leftrightarrow$ Weierstrass points of $D_N$ on a higher-genus curve
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Intuition (Mumford):

$N$-torsion points $\leftrightarrow$ Weierstrass points of $D_N$ on a higher-genus curve

Numerical “evidence”: as $N$ grows,

$\#(\text{Weierstrass points of } D_N) = gN^2 + O(N)$
Weierstrass points: genus 1, complex case

Problem

How are Weierstrass points distributed on genus 1 curve $X/\mathbb{C}$?
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How are Weierstrass points distributed on genus 1 curve $X/\mathbb{C}$?

$\leadsto$ Weierstrass points distribute **uniformly**, w.r.t. $\mathbb{C} \rightarrow \mathbb{C}/\Lambda$
Weierstrass points: genus $\geq 2$, complex case

Problem

How are Weierstrass points distributed on higher genus curve $X/C$?

Theorem (Neeman, 1984)

Suppose $X$ is a complex algebraic curve of genus $g \geq 2$. Then $W(D_\mathcal{N})$ distributes according to the Bergman measure as $N \to \infty$. 

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Weierstrass points: non-Archimedean case

Problem

How are Weierstrass points distributed on $X/K$, $val : K^\times \to \mathbb{R}$?

Source: Matt Baker’s math blog

Theorem (Amini, 2014)

Suppose $X$ is an is a Berkovich curve of genus $g \geq 2$. Then $W(D_N)$ distributes according to the Arakelov-Zhang measure as $N \to \infty$. 

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Theorem (Amini, 2014)

Suppose $X^{an}$ is a Berkovich curve of genus $g \geq 2$. Then $W(D_N)$ distributes according to the Arakelov-Zhang measure as $N \to \infty$. 

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How are Weierstrass points distributed on $X/K$, $val : K^\times \to \mathbb{R}$?

Problem (Amini, 2014)

Does the distribution follow from considering only the skeleton $\Gamma \subset X^{an}$?
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What is ... a tropical curve?

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Example: $X_t = \{xyz - tx^3 + t^2y^3 + t^5z^3 = 0\} \subset \mathbb{P}^2_{\mathbb{C}}$
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(Combinatorics) = metric graph  
= finite graph with edge lengths  

(algebraic geometry) = model for a degenerating algebraic curve

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Tropical Weierstrass equidistribution  
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Tropical curves: divisor theory

Tropical curve = metric graph

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Intuition: linear equivalence on $\Gamma$ = "discrete current flow" $|D| = \{ E \text{ lin. equiv. to } D, E \geq 0 \}$

$q$-reduced divisor $\text{red}_q[D] = \text{"energy-minimizing" divisor in } |D|$
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Example:
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Definition: $\Gamma = \text{metric graph}$, $D_N$ divisor of degree $N$
$\leadsto$ Baker–Norine rank $r = r(D_N)$

$$W(D_N) = \{x \in X : \text{red}_x[D_N] \geq (r + 1)x\}$$
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EXCEPT sometimes $\#(\text{Weierstrass points}) = \infty$
Example

Genus $g(\Gamma) = 1$:
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degree $D = 6$, 

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Degree $D = 6$, \quad \#(W(D)) = 6
Example

Genus $g(\Gamma) = 3$: 

[Diagram of a 3-genus graph with marked points]
Weierstrass points: tropical case

Example

Genus $g(\Gamma) = 3$:

degree $D = 4$. 
Example

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degree $D = 4$,  \[\sim\] \[\#(W(D)) = 8\]
Example

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Weierstrass points: tropical case
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Example

Genus $g(\Gamma) = 3$:

degree $D = 4$, $\leadsto \#(W(D)) = \infty$!
Weierstrass points: tropical case

**Example**

Genus $g(\Gamma) = 3$:

![Diagram showing a graph with degree $D = 4$, and $\#(W(D)) = \infty$](image-url)
Example

Genus $g(\Gamma) = 3$:

Degree $D = 4$,
Weierstrass points: tropical case

Example

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$\text{degree } D = 4, \quad \sim \quad \#(W(D)) = \infty!$
In general, this problem doesn’t happen!

**Theorem (R)**

*For a generic divisor class \([D]\), the Weierstrass locus \(W(D)\) is finite.*
Weierstrass points: tropical case

In general, this problem doesn’t happen!

Theorem (R)

For a generic divisor class \([D]\), the Weierstrass locus \(W(D)\) is finite.

So, we can still ask

Problem

How are Weierstrass points distributed supposing \(W(D)\) is finite?
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*For a generic divisor class \([D]\), the Weierstrass locus \(W(D)\) is finite.*

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*How are Weierstrass points distributed supposing \(W(D)\) is finite for *generic* \([D]\)?
Weierstrass points: tropical case

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How are Weierstrass points distributed supposing \(W(D)\) is finite for **generic** \([D]\)?

**Theorem (R)**

For a sequence of generic divisor classes \([D_N]\) on \(\Gamma\), the Weierstrass locus \(W(D_N)\) distributes according to Zhang’s canonical measure \(\mu\).
\[ \Gamma = \text{electrical network by replacing each edge } \sim \text{ resistor} \]
Electrical networks

Γ = electrical network by replacing each edge \( \leadsto \) resistor

Given \( y, z \in \Gamma \), send 1 unit of current from \( y \) to \( z \). Let

\[
j^y_z = \begin{pmatrix} \text{voltage on } \Gamma \text{ when 1 unit of} \\ \text{current is sent from } y \text{ to } z \end{pmatrix}
\]

where \( \Gamma \) is “grounded” at \( z \).
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where \( \Gamma \) is “grounded” at \( z \). By Ohm’s law, \( \textbf{current} = \textbf{slope} \) of \( j^y_z \).
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Given $y, z \in \Gamma$, send 1 unit of current from $y$ to $z$. Let

$$j_y^z = \begin{pmatrix} \text{voltage on } \Gamma \text{ when 1 unit of current is sent from } y \text{ to } z \end{pmatrix}$$

where $\Gamma$ is “grounded” at $z$. By Ohm’s law, current $= \text{slope of } j_y^z$

Example:
Electrical networks: Canonical measure

\( \Gamma = \) electrical network by replacing each edge \( \sim \) resistor

**Definition ("electrical" version, Chinburg–Rumely–Baker–Faber)**

Zhang's **canonical measure** \( \mu \) on an edge is the "current defect"

\[
\mu(e) = \text{current bypassing } e \text{ when 1 unit sent from } e^- \text{ to } e^+
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= 1 - (\text{current through } e \text{ when } \ldots )
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**Example:**

\[
\mu(e) = \frac{1}{3} \quad \mu(e) = 0 \quad \mu(e) = \frac{7}{12} \quad \mu(e) = \frac{7}{12} \quad \mu(e) = \frac{7}{12} \quad \mu(e) = 1
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\]

Generally:

- \( 0 \leq \mu(e) \leq 1 \)
- \( \mu(e) = 0 \Leftrightarrow e \text{ a bridge} \)
- \( \mu(e) = 1 \Leftrightarrow e \text{ a loop} \)

**Foster's Theorem:** \( \mu(\Gamma) = \sum_{e \in E} \mu(e) = g \)
Theorem (R)

For a sequence of generic divisor classes $[D_N]$ on $\Gamma$, the Weierstrass locus $W(D_N)$ distributes according to Zhang’s canonical measure $\mu$.

Namely, for any edge $e$

$$\frac{\#(W(D_N) \cap e)}{N} \to \mu(e) \quad \text{as} \quad N \to \infty.$$
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Idea:

(discrete current flow) \quad \leftarrow \quad (\text{continuous current flow})

$$\#(\text{Weierstrass points on } e) \quad \overset{N \to \infty}{\longrightarrow} \quad \text{canonical measure } \mu(e)$$
References

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Thank you!