Equidistribution of Weierstrass points on tropical curves

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Overview

1. Weierstrass points
   - complex case
   - non-Archimedean case
2. Tropical curves
   - what is it?
   - divisors
   - tropical Weierstrass points
3. Electrical networks
   - voltage
   - effective resistance
   - canonical measure
4. Proof

Weierstrass points

Problem

How are Weierstrass points distributed?

Definition: X a curve, D_N a divisor of degree N
$\rightsquigarrow$ projective embedding $\phi : X \to \mathbb{P}^r$.

$$W(D_N) = \{x \in X : \exists H \subset \mathbb{P}^r \text{ s.t. } m(H \cap X) \geq r+1\}$$

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Example: $X = \{xyz + x^3 + y^3 + z^3 = 0\} \subset \mathbb{P}^2_{\mathbb{C}}$

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Intution (Mumford):

$$\begin{array}{ccc} N\text{-torsion points} & \leftrightarrow & \text{Weierstrass points of } D_N \\ \text{on an elliptic curve} & & \text{on a higher-genus curve} \end{array}$$

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Intution (Mumford):

N-torsion points on an elliptic curve	$\leftrightarrow$	Weierstrass points of D_N on a higher-genus curve

Numerical "evidence": as N grows,

$$\#(\text{Weierstrass points of } D_N) = gN^2 + O(N)$$

Weierstrass points: genus 1, complex case

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How are Weierstrass points distributed on genus 1 curve $X/\mathbb{C}$?

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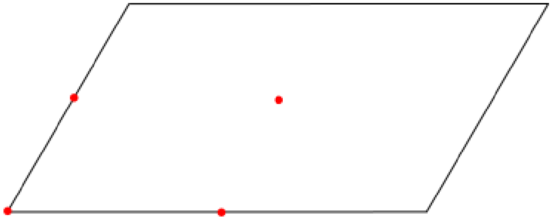

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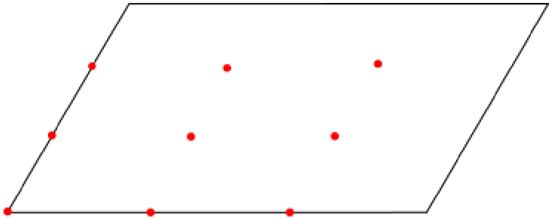

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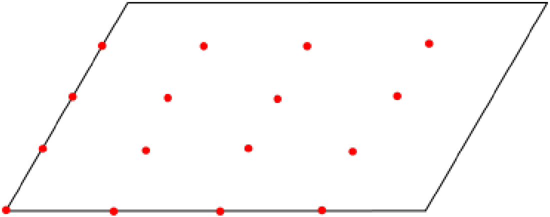

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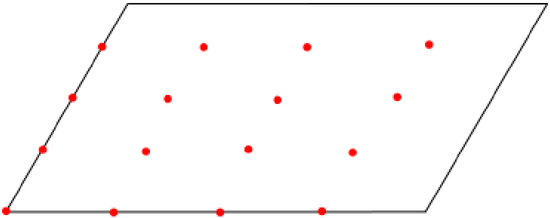

$\Rightarrow$ Weierstrass points distribute uniformly, w.r.t. $\mathbb{C} \to \mathbb{C}/\Lambda$

Weierstrass points: genus ≥ 2, complex case

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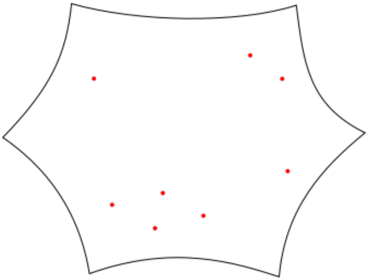

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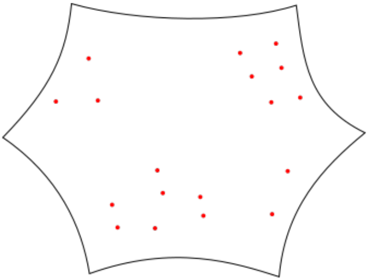

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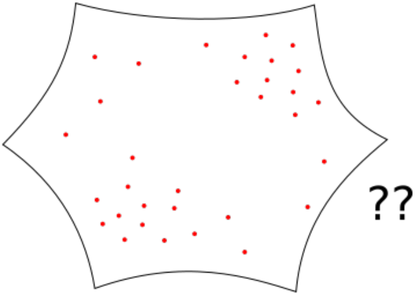

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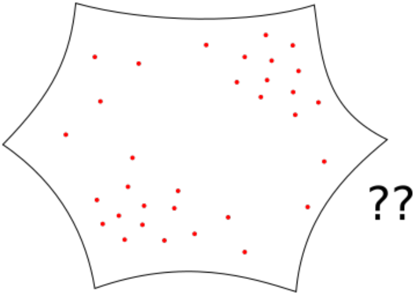


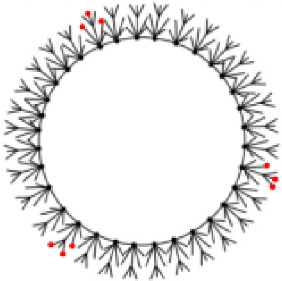
Theorem (Neeman, 1984)

Suppose X is a complex algebraic curve of genus $g \geq 2$. Then $W(D_N)$ distributes according to the Bergman measure as $N \to \infty$.

Weierstrass points: non-Archimedean case

Problem

How are Weierstrass points distributed on $X/\mathbb{K}$, $val : \mathbb{K}^\times \to \mathbb{R}$?



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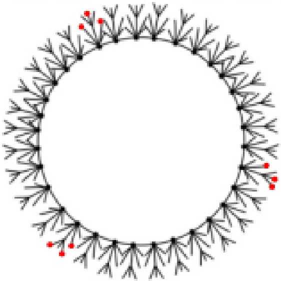

Theorem (Amini, 2014)

Suppose X^{an} is a Berkovich curve of genus $g \geq 2$. Then $W(D_N)$ distributes according to the Arakelov-Zhang measure as $N \to \infty$.

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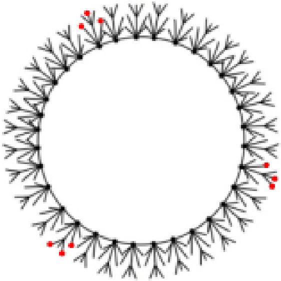

Problem (Amini, 2014)

Does the distribution depend <u>only</u> on the skeleton $\Gamma \subset X^{an}$?

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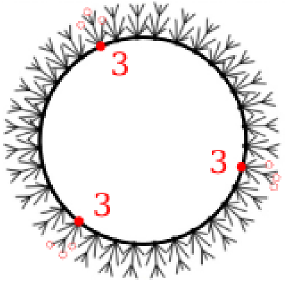

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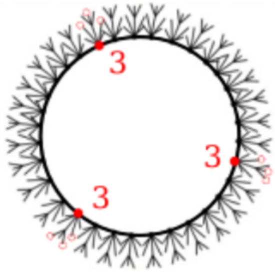
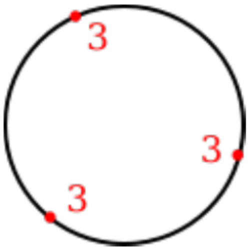
Tropical curves: what is it?

Tropical curve (= **abstract** tropical curve)



$\rightsquigarrow$



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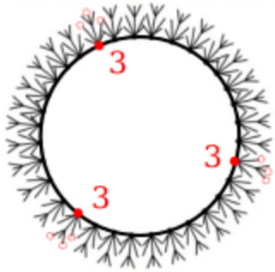

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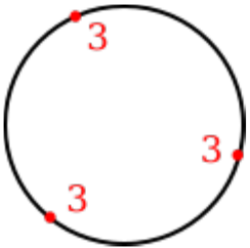

Tropical curves: what is it?

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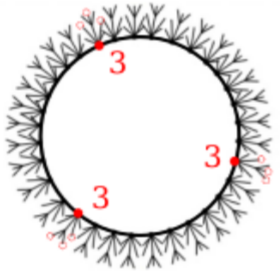

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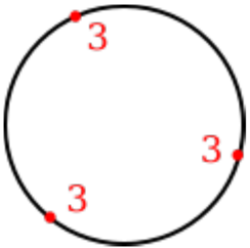

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Tropical curve ($=$ **abstract** tropical curve)
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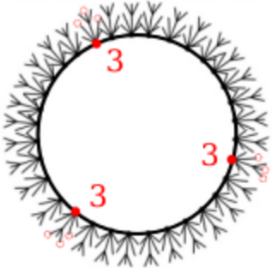
$=$ combinatorial model for a degenerating algebraic curve



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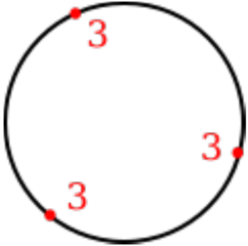

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Example: $X_t = \{xyz - t(x^3 + y^3 + z^3) = 0\} \subset \mathbb{P}^2$

Tropical curves: divisor theory

Tropical curve = finite graph with edge lengths

alg. curve X		tropical curve Γ
divisors $\mathrm{Div}(X)$	$\rightsquigarrow$	divisors $\mathrm{Div}(\Gamma)$
meromorphic functions	$\rightsquigarrow$	piecewise $\mathbb{Z}$-linear functions

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Intuition: linear equivalence = "discrete current flow"

Tropical curves: reduced divisors

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q-reduced divisor $\mathrm{red}_q[D]$ = "energy-minimizing" divisor in $[D]$

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$\rightsquigarrow$ Baker-Norine rank $r = r(D_N)$.

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EXCEPT sometimes $\#(\text{Weierstrass points of } D_N) = \infty$

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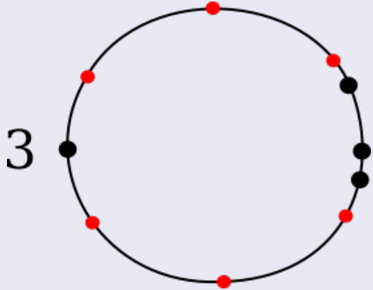
Example

Genus $g(\Gamma) = 1$:



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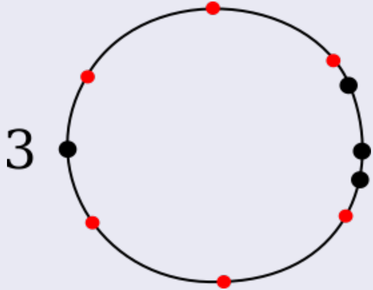

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Weierstrass points: tropical case

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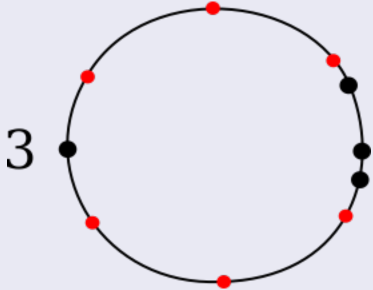

degree $D = 6$, $\quad \rightsquigarrow \quad$ $\#(W(D)) = 6$

Weierstrass points: tropical case

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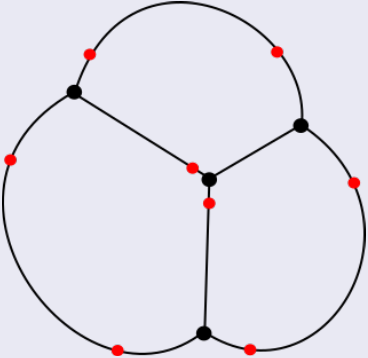

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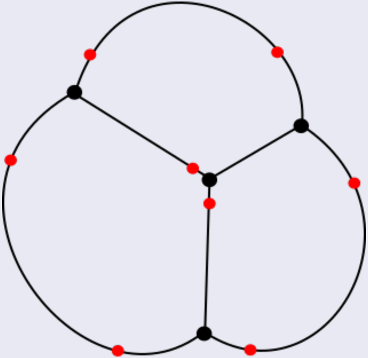

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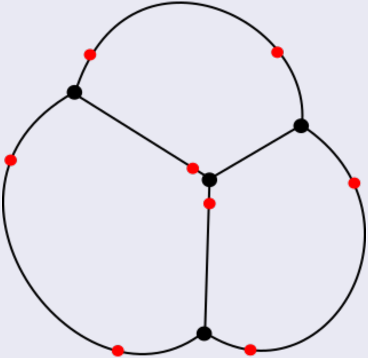

degree $D = 4$, $\rightsquigarrow$ $\#(W(D)) = 8$

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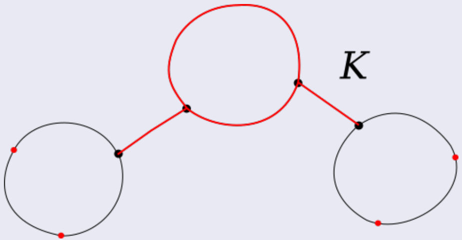

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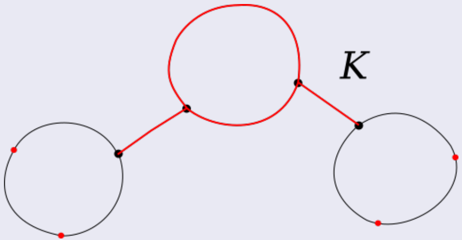

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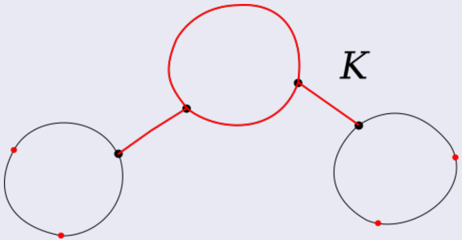

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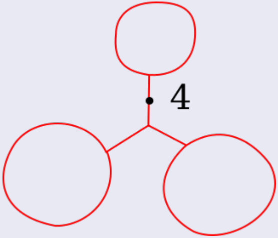

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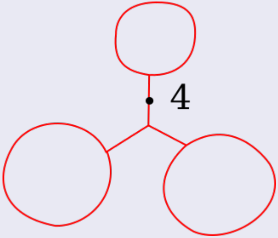

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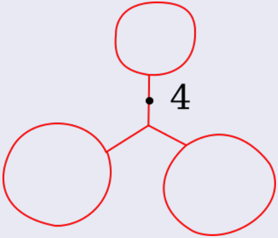

degree $D = 4$, $\rightsquigarrow$ $\#(W(D)) = \infty$!

In general, this problem doesn't happen!

Theorem (R)

For a generic divisor class $[D]$, the Weierstrass locus $W(D)$ is finite.

So, we can still ask

Problem

How are Weierstrass points distributed, **supposing** $W(D)$ is finite?

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Electrical networks

Γ = electrical network by replacing each edge $\rightsquigarrow$ resistor

Electrical networks: Canonical measure

$\Gamma =$ electrical network by replacing each edge $\rightsquigarrow$ resistor

Definition ("electrical" version)

The canonical measure μ on an edge is the "current defect"

$$\begin{aligned}\mu(e) &= \text{current bypassing } e \text{ when 1 unit sent from } e^- \text{ to } e^+\\ &= 1 - (\text{current through } e \text{ when ...}).\end{aligned}$$

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Namely:

- $0 \leq \mu(e) \leq 1$
- $\mu(e) = 0 \Leftrightarrow e$ a bridge
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Foster's Theorem: $\mu(\Gamma) = g$, since $\#(\text{break divisor}) = g$

Tropical Weierstrass distribution

Theorem (R)

For a sequence of divisor classes $[D_N]$, the Weierstrass locus $W(D_N)$ distributes according to the canonical measure μ.

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Namely, for any edge e

$$\frac{\#(W(D_N) \cap e)}{N} \to \mu(e) \qquad \textit{as} \qquad N \to \infty.$$

Tropical Weierstrass distribution: proof idea

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Idea:

$$\text{(discrete current flow)} \qquad \xrightarrow{N \to \infty} \qquad \text{(continuous current flow)}$$

Tropical Weierstrass distribution: proof idea

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Idea:

$$\begin{array}{ccc} \text{(discrete current flow)} & \xrightarrow{N\to\infty} & \text{(continuous current flow)} \\ \updownarrow & & \\ \#(\text{Weierstrass points on } e) & & \end{array}$$

Tropical Weierstrass distribution: proof idea

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\updownarrow & & \updownarrow \\
\#(\text{Weierstrass points on } e) & & \text{canonical measure } \mu(e)
\end{array}$$

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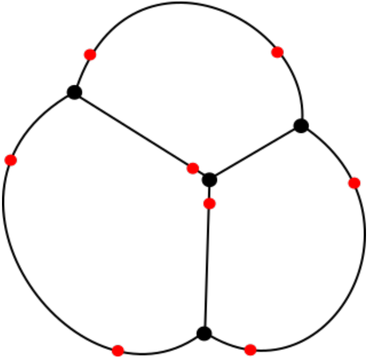

Thank you!