Weierstrass points on tropical curves

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A **metric graph** is a network made of vertices and edges, where each edge has a fixed length.
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The **genus** of a metric graph is the number of “independent cycles”.
Problem
How to place \( N \) points on a metric graph in an “evenly spaced” way?
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On a circle:
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On a more complicated metric graph:
Introduction

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Objectives:
- capture global topological structure
- ignore “dead ends”
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How to place $N$ points on a metric graph in an “evenly spaced” way?

Objectives:
- capture global topological structure
- ignore “dead ends”
Introduction: What is tropical geometry?

connection between algebraic geometry and combinatorics

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<td>Riemann surface</td>
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What is algebraic geometry?

Study polynomial equations and their solution sets.

**Example:** \[ x^3 + y^3 + xy + 1 = 0, \quad x, y \in \mathbb{R} \]

Source: desmos.com
What is algebraic geometry?

Study polynomial equations and their solution sets.

Example: \[ x^3 + y^3 + xy + 1 = 0, \quad x, y \in \mathbb{C} \]

The solution set in \( \mathbb{C} \) is a **Riemann surface**.
What is algebraic geometry?

Study polynomial equations and their solution sets.

**Example:** \(x^3 + y^3 + xy + 1 = 0, \quad x, y \in \mathbb{C}\)

(projectivized: \(x^3 + y^3 + xyz + z^3 = 0, \quad [x : y : z] \in \mathbb{P}^2_{\mathbb{C}}\))

The solution set in \(\mathbb{C}\) is a **Riemann surface**.
What is algebraic geometry?

Study polynomial equations and their solution sets.

Example: \[ x^4 + y^4 + xy + 1 = 0, \quad x, y \in \mathbb{C} \]

The solution set in \( \mathbb{C} \) is a **Riemann surface**.
The **genus** is the number of “holes”.

Harry Richman (U. Michigan)
What is tropical geometry?

Study piecewise-linear functions ("tropical polynomials") and their break loci ("solution sets").

Example: \( \min \{ a + 2x, b + x + y, c + 2y, d + y, \ldots \} \)

Source: Richter-Gebert, Sturmfels, Theobald
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Example: \( \min\{1 + 3x, 2 + 3y, x + y, 5\} \)
Turn a Riemann surface into a metric graph via a degenerating family.

Example: $X_t = \{ tx^3 + t^2y^3 + t^5z^3 + xyz = 0 \} \subset \mathbb{P}^2_{\mathbb{C}}$
Tropicalizing algebraic curves

Turn a Riemann surface into a metric graph via a degenerating family.

Example: \( X_t = \{ tx^3 + t^2y^3 + t^5z^3 + xyz = 0 \} \subseteq \mathbb{P}^2 \)

\[ \begin{array}{c}
\text{t=1} \\
\text{t=\epsilon} \\
\text{t=0}
\end{array} \]

\[ \begin{array}{c}
\text{dual graph} \\
\text{of } X_0
\end{array} \]

\[ \begin{array}{c}
L=5 \\
L=1 \\
L=2
\end{array} \]

\{y=0\} \quad \{z=0\} \quad \{x=0\} \]
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\[ X_t = \{ tx^3 + t^2 y^3 + t^5 z^3 + xyz = 0 \}, \quad t, t^2, t^5 \in \mathbb{C}[t] \]
Turn a Riemann surface into a metric graph via a degenerating family.

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\[ X_t = \{ a(t)x^3 + b(t)y^3 + c(t)z^3 + xyz = 0 \}, \quad a(t) \in \mathbb{C}((t)) \]

\( \mathbb{C}((t)) = \) field of Laurent series, e.g. \( a(t) = t + 2t^2 + 6t^3 + \cdots \)
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What is a Weierstrass point?

Idea: point whose tangent line has “higher-than-expected” tangency

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Generalize to high-dimensional embedding \( X \to \mathbb{P}^r \)

(projective embedding \( \phi : X \to \mathbb{P}^r \) \( \Leftrightarrow \) (linear equivalence class of \( D \)))
A **divisor** is a finite collection of points, $D = p_1 + \cdots + p_N$.

The **degree** is the number of points, $N$. 
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A rational function \( f(z) \) has **order of vanishing** \( m \) at \( z_i \) if

\[
 f(z) = c(z - z_i)^m + \text{(higher powers of } z - z_i) .
\]
A **divisor** is a finite collection of points, $D = p_1 + \cdots + p_N$.

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A rational function $f(z)$ has **order of vanishing** $m$ at $z_i$ if

$$f(z) = c(z - z_i)^m + \text{(higher powers of } z - z_i).$$

Divisors $D_1, D_2$ are **linearly equivalent** if there is a rational function $f(z)$ and constants $a_1, a_2$ such that

$$D_1 = \sum_{f(z_i) = a_1} \text{ord}_{z_i}(f(z) - a_1) \cdot z_i, \quad D_2 = \sum_{f(z_i) = a_2} \text{ord}_{z_i}(f(z) - a_2) \cdot z_i$$

Idea: $D_1 \sim D_2$ are different “level sets” of the same function
A divisor is a finite collection of points, $D = p_1 + \cdots + p_N$.

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X a smooth algebraic curve, $D = p_1 + \cdots + p_N$ a divisor

(linear equivalence class of $D$) $\leftrightarrow$ (projective embedding $\phi : X \to \mathbb{P}^r$)

The rank of $D$ is the dimension $r = r(D)$
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**Definition:**

$$W(D) = \{p \in X : D \sim (r + 1)p + E \text{ for some } E\}$$
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“higher-than-expected” tangency with some hyperplane at $p$
What is a Weierstrass point?

Example: $X =$ genus 1 curve over $\mathbb{C}$

$$\{x^3 + y^3 + xy + 1 = 0\}$$
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\( \leadsto \)

Weierstrass points are \textbf{evenly spaced} \,(w.r.t. addition law)
What is a Weierstrass point?

Intuition (Mumford):

\[ N \text{-torsion points } \leftrightarrow \text{ Weierstrass points of } D_N \]

on an elliptic curve on a higher-genus curve

Numerical “evidence”: as \( N \) grows,

\[
\#(\text{Weierstrass points of } D_N) = gN^2 + O(N) \\
= g(N - g + 1)^2
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How are Weierstrass points distributed on an algebraic curve?
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How are Weierstrass points distributed on higher genus curve $X/\mathbb{C}$?

Theorem (Neeman, 1984)

Suppose $X$ is a complex algebraic curve of genus $g \geq 2$. Then $W(D_N)$ distributes according to the Bergman measure as $N \to \infty$. 
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Tropical Weierstrass points
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Weierstrass points: complex case

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How are Weierstrass points distributed on $X^\text{an}/\mathbb{C}((t))$?

Source: Matt Baker
Problem

How are Weierstrass points distributed on $X^{\text{an}}/\mathbb{C}((t))$?

Theorem (Amini, 2014)

Suppose $X^{\text{an}}$ is a Berkovich curve of genus $g \geq 2$. Then $W(D_N)$ distributes according to the Zhang measure as $N \to \infty$. 

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Problem

How are Weierstrass points distributed on $\mathcal{X}^{an}/\mathcal{C}((t))$?

Problem (Amini, 2014)

Does the distribution follow from considering only the skeleton $\Gamma \subset \mathcal{X}^{an}$?

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Problem (Amini, 2014)

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Tropical linear equivalence

Study piecewise-$\mathbb{Z}$-linear function and its break locus ("solution set")

Idea: $D_1 \sim D_2$ for level sets of the same piecewise-linear function

Example:

$$f(x) = \min \{0, x - 4, 2x - 7, 4x - 9\} - \min \{0, 2x - 7, 3x - 9, 4x - 9.5\}$$
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Weierstrass points: tropical case

**Problem**

How are Weierstrass points distributed on a tropical curve?

Γ a metric graph, \( D = p_1 + \cdots + p_N \) a divisor

**Definition:**

\[
W(D) = \{ p \in \Gamma : D \sim (r + 1)p + E \text{ for some } E \}
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Here \( r \) is the Baker–Norine rank \( r = r(D) = N - g \) when \( N \gg 0 \)
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\]

EXCEPT sometimes \( \#(\text{Weierstrass points}) = \infty \)
How to compute Weierstrass locus?

\[ W(D) = \{ q \in \Gamma : D \sim (r + 1)q + E \text{ for some } E \} \]
How to compute Weierstrass locus?

\[ W(D) = \{ q \in \Gamma : D \sim (r + 1)q + E \text{ for some } E \} \]
\[ = \{ q \in \Gamma : \text{red}_q[D] \geq (r + 1)q \} \]

Intuition: linear equivalence on \( \Gamma \) = “discrete current flow”
\( q \)-reduced divisor \( \text{red}_q[D] \) = “energy-minimizing” divisor \( \sim D \)
Tropical curves: reduced divisors

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Example:

What happens as $q$ varies?
Example: Genus $g(\Gamma) = 1$, degree $D = 6$:

$\#(W(D)) = 6$
Example: Genus $g(\Gamma) = 3$, degree $D = 4$:

$\leadsto \#(W(D)) = 8$
Example: Genus $g(\Gamma) = 3$, degree $D = 4$:

$$\#(W(D)) = \infty!$$
Example: Genus $g(\Gamma) = 3$, degree $D = 4$:

$\#(W(D)) = \infty$!

In general, this problem doesn’t happen.

**Theorem (R)**

*For a generic divisor class $[D]$, the Weierstrass locus $W(D)$ is finite.*
Weierstrass points: tropical case

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**Example:**

So, we can still ask about distribution of $W(D)$ “generically”
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**Theorem (R)**

For a generic divisor class \([D]\), the Weierstrass locus \(W(D)\) is finite.

**Example:**

So, we can still ask about distribution of \(W(D)\) “generically”

**Theorem (R)**

For a sequence of generic divisor classes \([D_N]\) on \(\Gamma\), the Weierstrass locus \(W(D_N)\) distributes according to Zhang’s canonical measure \(\mu\).
Electrical networks

\( \Gamma = \text{electrical network by replacing each edge } \sim \text{ resistor} \)

Given \( y, z \in \Gamma \), let

\[ j_{yz} = \begin{pmatrix} \text{voltage on } \Gamma \text{ when 1 unit of } \\
\text{current is sent from } y \text{ to } z \end{pmatrix} \]

Observation: voltage function is piecewise-linear

By Ohm’s law, current = \( \frac{\text{voltage}}{\text{resistance}} = \text{slope of } j_{yz} \)
Electrical networks

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Example: current = \( (j^y_z)' \)
Electrical networks: canonical measure

\[ \Gamma = \text{metric graph} \]

**Definition ("electrical" version, Chinburg–Rumely–Baker–Faber)**

Zhang’s **canonical measure** \( \mu \) on an edge is the “current defect”

\[
\mu(e) = \text{current bypassing } e \text{ when 1 unit sent from } e^- \text{ to } e^+
\]

\[ = 1 - \text{(current through } e \text{ when ... )} \]

Example:
Electrical networks: canonical measure

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\[ = 1 - \left( \text{current through } e \text{ when } \ldots \right) \]

**Example:**

\[ \mu(e) = \frac{7}{12} \]
Electrical networks: canonical measure

Γ = metric graph

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\]

**Example:**

\[
\mu(e) = \frac{7}{12} \quad \mu(e) = \frac{1}{3} \quad \mu(e) = 0 \quad \mu(e) = 1
\]
Example:

\[
\begin{array}{c}
\frac{1}{3} \\
7\frac{1}{12} & 7\frac{1}{12}
\end{array}
\]

\[
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Electrical networks: canonical measure
Example:

Foster’s Theorem: \( \mu(\Gamma) = \sum_{e \in E} \mu(e) = g \)
Theorem (R)

For a sequence of generic divisor classes $[D_N]$ on $\Gamma$, the Weierstrass locus $W(D_N)$ distributes according to Zhang’s canonical measure $\mu$.

Namely, for any edge $e$

$$\frac{\#(W(D_N) \cap e)}{N} \to \mu(e) \quad \text{as} \quad N \to \infty.$$
Theorem (R)

For a sequence of generic divisor classes \([D_N]\) on \(\Gamma\), the Weierstrass locus \(W(D_N)\) distributes according to Zhang’s canonical measure \(\mu\).

Namely, for any edge \(e\)

\[
\frac{\#(W(D_N) \cap e)}{N} \rightarrow \mu(e) \quad \text{as} \quad N \rightarrow \infty.
\]

Idea:

(continuous current flow)

\[\uparrow\]

canonical measure \(\mu(e)\)
Theorem (R)

For a sequence of generic divisor classes \([D_N]\) on \(\Gamma\), the Weierstrass locus \(W(D_N)\) distributes according to Zhang’s canonical measure \(\mu\).

Namely, for any edge \(e\)

\[
\frac{\#(W(D_N) \cap e)}{N} \to \mu(e) \quad \text{as} \quad N \to \infty.
\]

Idea:

(discr. current flow) \(\xrightarrow{N \to \infty}\) (cont. current flow)

\[\uparrow\]

canonical measure \(\mu(e)\)
Theorem (R)

For a sequence of generic divisor classes \([D_N] \) on \(\Gamma\), the Weierstrass locus \(W(D_N)\) distributes according to Zhang's canonical measure \(\mu\).

Namely, for any edge \(e\)

\[
\frac{\#(W(D_N) \cap e)}{N} \rightarrow \mu(e) \quad \text{as} \quad N \rightarrow \infty.
\]

Idea:

(discrete current flow) \[\xrightarrow[N\to\infty]{}\] (continuous current flow)

\[
\#(\text{Weierstrass points on } e) \quad \uparrow \quad \text{canonical measure } \mu(e)
\]
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Thank you!