Weierstrass Points and Torsion Points on Tropical Curves

by

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This thesis is dedicated to my father, David Ross Richman, and to my grandfather, Alexander Richman.
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# TABLE OF CONTENTS

Dedication ................................................................. ii
Acknowledgements ...................................................... iii
List of Figures .......................................................... vi
List of Symbols ........................................................ vii
Abstract ............................................................... viii

Chapter

1. Introduction ......................................................... 1
   1.1 Tropical geometry ................................................. 2
   1.2 Weierstrass points ............................................... 4
   1.3 Torsion points of the Jacobian ................................. 8
   1.4 Outline ......................................................... 10

2. Tropical Curves .................................................... 12
   2.1 Metric graphs and divisors ................................... 12
   2.2 Principal divisors and linear equivalence ................... 13
   2.3 Picard group and Jacobian .................................... 15
   2.4 Reduced divisors .............................................. 17
   2.5 Break divisors and ABKS decomposition ................. 18
   2.6 Rank and Riemann–Roch ...................................... 20
   2.7 Matroids ................................................... 21

3. Resistor Networks ................................................ 24
   3.1 Voltage function ................................................. 24
   3.2 Kirchhoff formulas ............................................. 27
   3.3 Energy and reduced divisors .................................. 28
   3.4 Resistance function ........................................... 29
   3.5 Canonical measure ............................................. 30

4. Weierstrass Points ................................................ 33
   4.1 Classical Weierstrass points ................................ 33
   4.2 Tropical Weierstrass points ................................... 34
   4.2.1 Stable tropical Weierstrass points ....................... 35
   4.3 Finiteness of Weierstrass points ............................ 36
   4.3.1 Setup .................................................. 37
4.3.2 Point-set topology ............................................ 39
4.3.3 Proofs .......................................................... 40
4.4 Distribution of Weierstrass points ................................................. 47
  4.4.1 Examples .......................................................... 47
  4.4.2 Proofs .......................................................... 47
4.5 Tropicalizing Weierstrass points ................................................. 53

5. Torsion Points of the Jacobian .................................................... 55
  5.1 The classical Manin–Mumford conjecture .......................................... 55
  5.2 The Manin–Mumford conjecture for tropical curves ...................................... 55
    5.2.1 Higher-degree Manin–Mumford ................................................. 57
  5.3 Tropical Manin–Mumford results .................................................... 58
  5.4 Setup .............................................................. 58
    5.4.1 Stabilization of metric graphs .................................................... 58
    5.4.2 Girth and independent girth ...................................................... 59
  5.5 Proofs .............................................................. 59
    5.5.1 Higher-degree Manin–Mumford ................................................. 62

Appendices ................................................................................. 67
Bibliography ................................................................................. 73
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>Torsion points on a circle.</td>
</tr>
<tr>
<td>1.2</td>
<td>A metric graph, with 5 nodes and 8 edges.</td>
</tr>
<tr>
<td>1.3</td>
<td>Torsion points on a complex elliptic curve.</td>
</tr>
<tr>
<td>1.4</td>
<td>Flex point (right) on an embedded curve in $\mathbb{P}^2$.</td>
</tr>
<tr>
<td>1.5</td>
<td>A genus three Riemann surface.</td>
</tr>
<tr>
<td>1.6</td>
<td>Amoeba and logarithmic limit set of $x^3 + y^3 + 4xy + 1 = 0$.</td>
</tr>
<tr>
<td>1.7</td>
<td>Break locus of trop$(x^3 + y^3 + t^C xy + 1)$.</td>
</tr>
<tr>
<td>1.8</td>
<td>Tropicalizing a Riemann surface (left) to a graph (right).</td>
</tr>
<tr>
<td>2.1</td>
<td>Metric graph of genus 0.</td>
</tr>
<tr>
<td>2.2</td>
<td>Metric graph of genus 2.</td>
</tr>
<tr>
<td>2.3</td>
<td>Chip firing across an elementary cut.</td>
</tr>
<tr>
<td>2.4</td>
<td>Linear interpolation showing the divisor $f^{-1}_\Delta(\lambda)$.</td>
</tr>
<tr>
<td>2.5</td>
<td>Break divisors and non-break divisors.</td>
</tr>
<tr>
<td>2.6</td>
<td>ABKS decomposition of Pic$_2^0(\Gamma)$.</td>
</tr>
<tr>
<td>2.7</td>
<td>Wheatstone graph.</td>
</tr>
<tr>
<td>3.1</td>
<td>A divisor and its reduced divisor representative.</td>
</tr>
<tr>
<td>3.2</td>
<td>Genus 2 metric graph with edge lengths $a, b, c$.</td>
</tr>
<tr>
<td>4.1</td>
<td>Weierstrass points, in red, on a genus 1 metric graph.</td>
</tr>
<tr>
<td>4.2</td>
<td>Metric graph with finite Weierstrass locus.</td>
</tr>
<tr>
<td>4.3</td>
<td>Divisor on metric graph with $W(D) = \Gamma$.</td>
</tr>
<tr>
<td>4.4</td>
<td>Metric graph with Weierstrass locus $W(K)$ not finite.</td>
</tr>
<tr>
<td>4.5</td>
<td>Divisor with Weierstrass locus and stable Weierstrass locus.</td>
</tr>
<tr>
<td>4.6</td>
<td>ABKS decomposition of $\tilde{\text{Br}}^2(\Gamma)$.</td>
</tr>
<tr>
<td>4.7</td>
<td>Function $g_n^{(1)}$ having zeros red$<em>{w</em>{i+1}}[D_n]$ and poles red$_{w_i}[D_n]$, with slopes are indicated above each affine part.</td>
</tr>
<tr>
<td>4.8</td>
<td>Function $g_n^{(0)}$ having zeros red$_{w_i}[D_n]$ and poles red$_s[D_n]$.</td>
</tr>
<tr>
<td>5.1</td>
<td>Critical group of order 3.</td>
</tr>
<tr>
<td>5.2</td>
<td>Critical group of order 11.</td>
</tr>
<tr>
<td>5.3</td>
<td>Metric graph and its stabilization.</td>
</tr>
<tr>
<td>5.4</td>
<td>Slopes on edge.</td>
</tr>
</tbody>
</table>
LIST OF SYMBOLS

Symbol

Here we collect notation which will be used in the thesis.

A blah
B blah
C blah
Γ a compact, connected metric graph
PL(Γ) continuous, piecewise linear functions on Γ
PLZ(Γ) continuous, piecewise ℤ-linear functions on Γ
Δ(f) the principal divisor associated to a piecewise (ℤ-)linear function f
D a divisor on a metric graph or algebraic curve
Dn a divisor of degree n
r(D) the Baker–Norine rank of D
Div(Γ) divisors on Γ (with ℤ-coefficients)
DivZ(Γ) divisors on Γ with ℤ-coefficients, i.e. Div(Γ) ⊗ ℤ ℤ
Divd(Γ) divisors of degree d on Γ
Picd(Γ) divisor classes of degree d on Γ
Symd(Γ) effective divisors of degree d on Γ
Effd(Γ) effective divisor classes of degree d on Γ
[D] a divisor class; the set of divisors linearly equivalent to D
[D] the space of effective divisors linearly equivalent to D
redx[D] the x-reduced divisor equivalent to D, where x ∈ Γ
br[D] the break divisor equivalent to D, where D has degree g
Brd(Γ) the space of break divisors on Γ
µ = µΓ the Zhang canonical measure on Γ
G a finite, connected graph with vertex set V(G) and edge set E(G)
(G, ℓ) a combinatorial model for a metric graph, where ℓ : E(G) → ℝ≥0
T(G) the set of spanning trees of a graph G
ABSTRACT

We investigate two constructions on metric graphs, using the framework of tropical geometry. On a metric circle, i.e. a genus 1 tropical curve, each of these constructions produces a set of \( n \) points which are evenly spaced around the circle.

In the first part, we define a set of Weierstrass points for a divisor on a tropical curve. This is analogous to the Weierstrass points of an algebraic curve, and for a curve over a non-Archimedean field the definitions are compatible under tropicalization via Baker’s specialization lemma. In contrast to that of an algebraic curve, the Weierstrass locus on a tropical curve is not always a finite set. We fix this failure of finiteness in two ways: we show that on an arbitrary tropical curve, the Weierstrass locus of a generic divisor class is a finite set; further, we define a stable Weierstrass locus which is finite for an arbitrary divisor class. We then investigate the distribution of Weierstrass points for a high-degree divisor. We show that for a high-degree divisor with finite Weierstrass locus, the distribution of Weierstrass points is well-approximated by Zhang’s canonical measure. This measure can be described by probabilities of weighted spanning trees, or alternatively by current flows in an electrical resistor network. The same distribution property holds for the stable Weierstrass locus of an arbitrary divisor of high degree. This distribution result is a tropical analogue of a theorem of Neeman concerning Weierstrass points on a complex algebraic curve.

In the second part, given a tropical curve we consider the torsion points of its Jacobian. The Manin–Mumford Conjecture states that for an algebraic curve of genus \( g \geq 2 \), these intersect in finitely many points. For complex curves this conjecture was proved by Raynaud. For a tropical curve, this conjecture fails for any graph whose edge lengths are rational numbers. However, we show that for a tropical curve of genus \( g \geq 2 \) the Manin–Mumford conjecture does hold for a very general choice of edge lengths, if we additionally assume that the graph is biconnected. Under these assumptions we prove a bound on the size of the intersection which depends only on the genus, namely \( \#(AJ(\Gamma) \cap \text{Jac}(\Gamma)_{\text{tors}}) \leq 3g - 3 \). This gives a tropical answer to a question raised by Baker and Poonen concerning the size of “torsion packets” on an algebraic curve. Next we consider how torsion points in the Jacobian intersect with the image of higher-dimensional analogues of the Abel–Jacobi map, which send a \( d- \)
tuple of points to the divisor class of their sum (up to a translation). This motivates
the definition of the “independent girth” of a graph. For a metric graph with large
genus $g$, the independent girth is bounded above by $O(\log g)$. 
CHAPTER 1

Introduction

In this thesis, we study two generalizations of a simple construction: dividing a circle in $N$ equal parts. These division points are called torsion points of the circle.

![Figure 1.1: Torsion points on a circle.](image)

The word “torsion” comes from algebraic terminology—if we equip the circle with the additive structure $\mathbb{R}/\mathbb{Z}$, i.e. how we usually think of adding angles together, then the $N$-torsion points are points $x$ which satisfy $N \cdot x = x + \cdots + x = 0$. (There are $N$ such points.)

A circle is a simple example of a metric graph. A metric graph captures the structure of a network, meaning something made up of nodes and edges, where additionally each edge is assigned a positive real length. If we take just one node and one edge, with the edge joined to the node at both ends, then we get a circle. If we use more nodes and edges, we can get a more complicated metric graph.

![Figure 1.2: A metric graph, with 5 nodes and 8 edges.](image)

For an arbitrary metric graph, we can ask: How does one divide this object into $n$ “equal parts”? There is probably no single good answer to such a question, but we consider two constructions which generalize $N$-torsion points of a circle to arbitrary metric graphs. Both constructions are taken from the study of complex algebraic curves, via the framework of tropical geometry.
In algebraic geometry, the analogue of a circle is an elliptic curve. An elliptic curve over the complex numbers is topologically equivalent to a parallelogram with opposite sides glued together. The elliptic curve also has an additive structure of $\mathbb{R}^2/\Lambda$, coming from addition of vectors in $\mathbb{R}^2$ modulo integer combinations of vectors forming the sides of the parallelogram. The $N$-torsion points are the points which satisfy the equation $Nx = x + \cdots + x = 0$ with respect to this addition law. In this case there are $N^2$ such points.

![Figure 1.3: Torsion points on a complex elliptic curve.](image)

The torsion points may also be constructed without reference to an addition law, as follows. Given a curve $X$ in projective space $\mathbb{P}^r$, a flex point is a point $p$ on $X$ such that some hyperplane intersects $X$ at $p$ with multiplicity at least $r + 1$. If we embed an elliptic curve into projective space $\mathbb{P}^r$ using a complete linear system of degree $N$ divisors, then the set of flex points is in fact a set of $N$-torsion points (for some choice of $0$ on the elliptic curve).

![Figure 1.4: Flex point (right) on an embedded curve in $\mathbb{P}^2$.](image)

1.1 Tropical geometry

Tropical geometry is a relatively new area of mathematics which allows one to translate statements about algebraic curves to graph theory, and vice versa.

Algebraic geometry is the study of solutions to polynomial equations such as $x^4 + y^4 = 1$. Over the complex numbers, the set of solutions is known as a Riemann surface. Tropical geometry allows us to turn a Riemann surface into a graph. This

![Figure 1.5: A genus three Riemann surface.](image)
may be achieved from either an “embedded” or “non-embedded” perspective.

In the embedded perspective, given a complex algebraic variety in $\mathbb{C}^n$ we may consider the image in $\mathbb{R}^n$ under the logarithm map $(z_1, \ldots, z_n) \mapsto (\log |z_1|, \ldots, \log |z_n|)$. This image is called the amoeba of the algebraic variety. Bergman [11] observed that the limit of this amoeba, when “zooming out to infinity”, forms a polyhedral complex.

**Example 1.1 (Logarithmic limit set).** Consider the solutions to $x^3 + y^3 + 4xy + 1 = 0$ where $x, y \in \mathbb{C}^2$. The amoeba of this complex curve is shown on the left side of Figure 1.1, shaded in gray, and the logarithmic limit set is on the right.

![Figure 1.6: Amoeba and logarithmic limit set of $x^3 + y^3 + 4xy + 1 = 0$.](image)

Under the process of sending $z_i \mapsto \log |z_i|$ and “zooming out to infinity”, the effect is that

$$\lim_{|z| \to \infty} \frac{\log |a_n z^n + (\text{lower-order terms})|}{\log |z|} = \lim_{|z| \to \infty} \frac{\log |a_n| + n \log |z|}{\log |z|} = n.$$

A way to algebraically formalize / mimic this process is to work with coefficients in the field of Laurent series $K = \mathbb{C}((t^{-1}))$ (or Puiseux series $\cup_{n \geq 1} \mathbb{C}((t^{-1/n}))$), equipped with the non-Archimedean valuation $\text{val}(a_n t^n + (\text{lower-order terms in } t)) = n$. Then given a variety $X \subset K^n$ cut out by polynomials in $K[x_1, \ldots, x_n]$, its tropicalization (or non-Archimedean amoeba) is the image of $X$ under $(z_1, \ldots, z_n) \mapsto (\text{val } z_1, \ldots, \text{val } z_n)$.

A fundamental theorem of tropical geometry is that the tropicalization, as defined above, may be computed via the following process on the polynomials cutting out the given variety. (For simplicity, we describe the case of polynomials in two variables.) Given a polynomial $f = \sum_{i,j \geq 0} a_{i,j}(t)x^iy^j \in K[x, y]$, its tropicalization is defined as

$$\text{trop}(f) = \max_{i,j \geq 0} \{\text{val}(a_{i,j}(t)) + ix + jy\}.$$

This expression $\text{trop}(f)$ defines a piecewise-linear function on $\mathbb{R}^2$. Its break locus is the subset of $\mathbb{R}^2$ where the function is not linear.

**Theorem 1.2 (Fundamental theorem of tropical geometry).** Suppose $(z_1, \ldots, z_n) \in K^n$ lies on the variety cut out by $f = 0$ for some polynomial $f \in K[x_1, \ldots, x_n]$. Then the point $(\text{val } z_1, \ldots, \text{val } z_n) \in \mathbb{R}^n$ lies in the break locus of $\text{trop}(f)$. 

3
The converse of Theorem 1.2 is not true, but there is some sense in which the converse holds for a “sufficiently general” \( f \in K[x_1, \ldots, x_n] \).

**Example 1.3.** The tropicalization of the polynomial \( f = x^3 + y^3 + t^C xy + 1 \) is the piecewise-linear function

\[
trop(f) = \max\{3x, 3y, C + x + y, 0\}.
\]

If \( C > 0 \), this tropicalized function has four domains of linearity. Its break locus consists of three bounded segments and three unbounded segments. The bounded segments have endpoints \((C, C), (-C, 0), \) and \((0, -C)\).

![Figure 1.7: Break locus of trop(x^3 + y^3 + t^C xy + 1).](image)

In the abstract (non-embedded) perspective, tropicalization is achieved via *degenerating* a smooth algebraic curve to a curve with nodal singularities, along a one-parameter family, then taking the dual graph of the nodal curve. This degeneration process turns meromorphic (i.e. rational) functions on the Riemann surface (i.e. complex algebraic curve) to piecewise linear functions on the dual graph. These tools were developed by Baker–Norine \[8\] and others \[24, 17\].

\[15\] \[30\]

### 1.2 Weierstrass points

(Recall the above discussion of torsion points on an elliptic curve, as flex points of a projective embedding. For background on complex algebraic curves, see \[25\].)

Suppose \( X \) is a smooth, proper complex algebraic curve. The *Weierstrass points* of a divisor \( D \) on \( X \) are the flex points of the projective embedding \( X \rightarrow \mathbb{P}^r \) corresponding to the complete linear system of \( D \). This defines a finite subset of \( X \).
Historically, mathematicians were first interested in studying the Weierstrass points of the canonical divisor on a curve of genus \( g \geq 2 \). The fact that an algebraic curve of genus \( g \geq 2 \) has finite automorphism group can be shown by using the Weierstrass points of the canonical divisor. A generic curve has \( g^3 - g \) such points. In the literature, the Weierstrass points of a divisor \( D \), which is not the canonical divisor, are sometimes referred to as “higher Weierstrass points”. (Sometimes, “higher Weierstrass points” refers to Weierstrass points of \( nK \), where \( K \) is the canonical divisor and \( n \geq 2 \) is an integer.) See [13] for an extensive history of the study of Weierstrass points.

In [26], Mumford notes that the Weierstrass points associated to a divisor of degree \( n \) should be viewed as a higher-genus analogue of the \( n \)-torsion points on an elliptic curve. The fact that \( n \)-torsion points on a complex elliptic curve become “evenly distributed” as \( n \) grows large leads one to ask whether the same phenomenon holds for Weierstrass points on other algebraic curves.

An answer was given by Neeman [27], who showed that for a complex curve (i.e. Riemann surface) of genus \( g \geq 2 \), when \( n \to \infty \) the Weierstrass points of degree \( n \) divisors become distributed according to the Bergman measure.

**Theorem 1.4** (Neeman [27]). Let \( X \) be a compact Riemann surface of genus \( g \geq 2 \), and let \( \{D_n : n \geq 1\} \) be a sequence of divisors on \( X \) with \( \deg D_n = n \). Let \( W_n \) denote the Weierstrass locus of the divisor \( D_n \), and let \( \delta_n = \frac{1}{gn^2} \sum_{x \in W_n} \delta_x \) denote the normalized discrete measure on \( X \) associated to \( W_n \) (where \( \delta_x \) is the Dirac measure at \( x \)). Then as \( n \to \infty \), the measures \( \delta_n \) converge weakly to the Bergman measure on \( X \).

Before Neeman’s result, Olsen [28] showed that given a positive-degree divisor \( D \) on a complex algebraic curve \( X \), the union of the Weierstrass points of the multiples \( nD \), over all \( n \geq 1 \), is dense in \( X \) in the complex topology.

If one replaces the ground field \( \mathbb{C} \) with a non-Archimedean field, one may consider the same question of how Weierstrass points are distributed inside the Berkovich analytification \( X^\text{an} \) of an algebraic curve, say after retracting to a compact skeleton \( \Gamma \). This was addressed by Amini in [2]. Here the Weierstrass points are distributed according to the **Zhang canonical admissable measure**, constructed by Zhang in [33].

**Theorem 1.5** (Amini [2]). Let \( X \) be a smooth proper curve of genus \( g \geq 1 \) over a complete, algebraically closed, non-Archimedean field \( K \) with non-trivial valuation and residue characteristic 0. Let \( \Gamma \) be a skeleton of the Berkovich analytification \( X^\text{an} \) with retraction map \( \rho : X^\text{an} \to \Gamma \). Let \( D \) be a positive-degree divisor on \( X(K) \). Let \( W_n \) denote the Weierstrass locus of the divisor \( nD \), and let \( \delta_n = \frac{1}{\#W_n} \sum_{x \in W_n} \delta_{\rho(x)} \) denote the normalized discrete measure on \( \Gamma \) associated to \( W_n \) (where \( \delta_x \) is the Dirac
measure at $x$). Then as $n \to \infty$, the measures $\delta_n$ converge weakly to the Zhang canonical measure on $\Gamma$, up to a factor of $g$.

Zhang’s canonical measure does not have support on bridge edges, so it is independent of the choice of skeleton. Zhang’s construction was motivated by Arakelov’s pairing for divisors on a Riemann surface [4], for the purpose of answering questions in arithmetic geometry. Here we use a definition of $\mu$ along more elementary lines from Chinburg–Rumely [14] and Baker–Faber [6], using the notions of current flow and electric potential in a network of resistors.

In [2] Amini raises the question of whether the distribution of Weierstrass points is possibly intrinsic to the metric graph $\Gamma$, without needing to identify $\Gamma$ with the skeleton of some Berkovich curve $X^{an}$. One major obstacle to this idea is that on a metric graph, the Weierstrass locus for a divisor may fail to be a finite set of points. Our approach is to sidestep this issue by showing that finiteness does hold for a generic choice of divisor class. With this assumption of genericity, we are able to show that distribution of Weierstrass points is intrinsic to $\Gamma$.

In [5], Baker studies ordinary Weierstrass points on graphs and on metric graphs, and mentions several applications of number theoretic significance. These results are stated only for Weierstrass points associated to the canonical divisor; higher Weierstrass points for general divisors are not considered.

In this thesis we prove analogous results for Weierstrass points on a tropical curve. We show that, for a tropical curve $\Gamma$, the Weierstrass locus is finite for a generic divisor class. Generically, the number of Weierstrass points depends only on the degree of the divisor and the genus of the underlying curve. We further prove that, for any degree-increasing sequence of such generic divisors, the Weierstrass points become distributed according to the Zhang canonical measure on $\Gamma$. This measure can be described via interpreting $\Gamma$ as an electrical network of resistors.

We also define a stable Weierstrass locus which is finite for an arbitrary divisor class, and compute its cardinality for a generic divisor class, which depends only on the degree and genus.

Given a compact, connected metric graph $\Gamma$ and a divisor $D$ of rank $r = r(D)$, we define the Weierstrass locus $W(D)$ as

$$W(D) = \{ x \in \Gamma : D \sim (r+1)x + E \text{ for some } E \geq 0 \},$$

where $\sim$ denotes linear equivalence and $r(D)$ is the Baker–Norine rank (see Chapter 2 for definitions). The set $W(D)$ may fail to be finite; in some cases it contains all of $\Gamma$ (see Example 4.6).

For a divisor of degree $n \geq g$, we define the stable Weierstrass locus of $D$ as

$$W^{st}(D) = \{ x \in \Gamma : br[D - (n-g)x] = x + E \text{ for some } E \geq 0 \}$$
where $\text{br}[D]$ denotes the unique break divisor representative of a degree $g$ divisor $D$. The stable Weierstrass locus is finite for any divisor. If $D$ has rank $r(D) = n - g$, i.e. $D$ is nonspecial, then the stable Weierstrass locus is contained in $W(D)$. In particular, this containment holds when the degree $n \geq 2g - 1$. See Section 2 for definitions of linear equivalence, rank, and break divisor.

Our first result addresses the question of counting the number of Weierstrass points. Here “generic” means on a dense open subset of the space of divisor classes.

**Theorem 1.6.** Let $\Gamma$ be a compact, connected metric graph of genus $g$.

(a) For a generic divisor class of degree $n \geq g$, the Weierstrass locus $W(D)$ is finite with cardinality

$$\#W(D) = g(n - g + 1).$$

For a generic divisor class of degree $n < g$, $W(D)$ is empty.

(b) For an arbitrary divisor class of degree $n \geq g$, the stable Weierstrass locus $W_{\text{st}}(D)$ is a finite set with cardinality

$$\#W_{\text{st}}(D) \leq g(n - g + 1),$$

and equality holds for a generic divisor class.

Parts (a) and (b) of Theorem 1.6 are connected by showing that $W(D) = W_{\text{st}}(D)$ for a generic divisor class.

The next main theorem describes the distribution of tropical Weierstrass points. Here, note that the condition “$W_n = W(D_n)$ is a finite set” is satisfied for generic $[D_n] \in \text{Pic}^n(\Gamma)$ by Theorem 1.6.

**Theorem 1.7.** Let $\Gamma$ be a metric graph of genus $g$, and let $\{D_n : n \geq 1\}$ be a sequence of divisors on $\Gamma$ with $\text{deg} D_n = n$. Let $W_n$ be the Weierstrass locus of $D_n$. Suppose each $W_n$ is a finite set, and let

$$\delta_n = \frac{1}{n} \sum_{x \in W_n} \delta_x$$

denote the normalized discrete measure on $\Gamma$ associated to $W_n$ (where $\delta_x$ is the Dirac measure at $x$). Then as $n \to \infty$, the measures $\delta_n$ converge weakly to the Zhang canonical measure $\mu$ on $\Gamma$.

The Zhang canonical measure is defined in Section 3.5. (Warning: we use a different normalization for $\mu$ than previous authors; namely we have total measure $\mu(\Gamma) = g$ rather than $\mu(\Gamma) = 1$.) We also obtain a quantitative version of this distribution result which specifies a bound on the rate of convergence. (See Theorem 4.26.)

**Technical note:** our tropical curves $\Gamma$ have no “hidden genus” at vertices and no infinite legs, i.e. we restrict our attention to $X^{an}$ with totally degenerate reduction and no punctures.
1.3 Torsion points of the Jacobian

As discussed above, an algebraic curve \( X \) of genus one with a chosen basepoint \( x_0 \in X \) is equipped with a natural additive structure on points of \( X \).

Given an algebraic curve with fixed basepoint \( x_0 \), we say that \( x \) is a torsion point if the divisor \( n(x-x_0) \) is linearly equivalent to 0 for some positive \( n \). Equivalently, \( x \) is a torsion point if the Abel–Jacobi embedding (with respect to \( x_0 \)) sends \( x \) to the torsion subgroup \( \text{Jac}(X)_{\text{tors}} \) of the Jacobian. The Jacobian of a genus \( g \) algebraic curve (over \( \mathbb{C} \)) is a compact abelian group, isomorphic to \( \mathbb{C}^g/\mathbb{Z}^g \cong H^1(X,\mathbb{C})/H^1(X,\mathbb{Z})^\vee \).

Motivated by analogy with Mordell’s conjecture on finiteness of rational points, Manin and Mumford conjectured that an algebraic curve of genus 2 or more has finitely many torsion points. This conjecture was proved by Raynaud [31].

**Theorem 1.8** (Raynaud; formerly the Manin–Mumford Conjecture). *For any smooth curve \( X \) of genus \( g \geq 2 \), the set of torsion points \( \varnothing \cap \text{Jac}(X)_{\text{tors}} \) is finite.*

The following stronger result remains open, though it is suspected to be true [9]. (Baker and Poonen use the equivalent language of torsion packets on curves.)

Is there a constant \( N(g) \) such that any algebraic curve \( X \) of genus \( g \geq 2 \) has \( \#(\varnothing \cap \text{Jac}(X)_{\text{tors}}) \leq N(g) \)?

Suppose \( \Gamma = (G, \ell) \) is a metric graph of genus \( g \geq 2 \) whose edge lengths are all rational, i.e. \( \ell(e) \in \mathbb{Q}_{>0} \) for all \( e \in E(G) \). Then \( \Gamma \) does not satisfy the Manin–Mumford condition.

This observation is a consequence of the fact that on a graph with unit edge lengths the degree-0 divisor classes supported on vertices form a finite abelian group, known as the critical group of the graph. In other words, vertex-supported divisor classes are always torsion.

We say that a property holds for a very general point of some real parameter space if it holds outside of a countable collection of proper Zariski-closed subsets. A Zariski-closed subset is the set of zeros of a polynomial function. Recall that a graph \( G \) is biconnected (or two-connected) if \( G \) is connected after deleting any vertex.

**Theorem 1.9.** *Let \( \Gamma \) be a connected metric graph of genus \( g \geq 2 \). If the set of torsion points \( AJ_q(\Gamma) \cap \text{Jac}(\Gamma)_{\text{tors}} \) is finite, then we have the uniform bound* \( \#(AJ_q(\Gamma) \cap \text{Jac}(\Gamma)_{\text{tors}}) \leq 3g - 3 \).*

**Theorem 1.10** (Uniform Manin–Mumford bound). *Let \( G \) be a finite connected graph of genus \( g \geq 2 \). If \( G \) is biconnected, then for a very general choice of edge lengths
\( \ell : E(G) \to \mathbb{R}_{>0} \), the metric graph \( \Gamma = (G, \ell) \) satisfies the Manin–Mumford condition. Moreover, under these conditions \( \Gamma \) satisfies the bound
\[
\#(AJ_q(\Gamma) \cap \text{Jac}(\Gamma)_{\text{tors}}) \leq g + 1
\]
where \( AJ_q : \Gamma \to \text{Jac}(\Gamma) \) denotes the Abel–Jacobi map with basepoint \( q \in \Gamma \).

Say a graph \( G \) is \textit{Manin–Mumford finite} if the metric graph \( (G, \ell) \) satisfies the Manin–Mumford condition for a very general choice of edge lengths \( \ell \).

Note that if a metric graph \( \Gamma \) is the wedge sum of two subgraphs \( \Gamma = \Gamma_1 \vee \Gamma_2 \), then the Jacobian decomposes as \( \text{Jac}(\Gamma) = \text{Jac}(\Gamma_1) \oplus \text{Jac}(\Gamma_2) \). This implies the following corollary.

**Theorem 1.11.** Let \( G \) be a finite connected graph whose biconnected components are \( G_1, \ldots, G_k \). Then \( G \) is Manin–Mumford finite if and only if each component \( G_i \) has genus \( g_i \neq 1 \) for \( i = 1, \ldots, k \).

We say a metric graph \( \Gamma \) satisfies the \textit{generalized Manin–Mumford condition} in degree \( d \) if the image of the “higher Abel–Jacobi map”
\[
AJ_D^{(d)} : \Gamma^d \to \text{Jac}(\Gamma)
\]
\[
(p_i)_i \mapsto \left\lceil \sum_i p_i - D \right\rceil
\]
intersects only finitely many torsion points of \( \text{Jac}(\Gamma) \), for any choice of effective base-divisor \( D \in \text{Sym}^d(\Gamma) \). We say a graph \( G \) is \textit{Manin–Mumford finite in degree} \( d \) if for very general edge lengths \( \ell \), the metric graph \( (G, \ell) \) satisfies the degree \( d \) Manin–Mumford condition.

When \( d = 1 \), this is the usual Manin–Mumford condition on \( \Gamma \). When \( d \geq g \) and \( g(\Gamma) \geq 1 \), the generalized Manin–Mumford condition cannot hold, since the higher Abel–Jacobi map will be surjective and \( \text{Jac}(\Gamma)_{\text{tors}} \) is infinite. If the generalized Manin–Mumford condition holds in degree \( d \), then it also holds in degree \( d' \) for any \( 1 \leq d' \leq d \).

Recall that the \textit{girth} of a graph is the minimal length of a simple cycle.

Let \( G \) be a finite connected graph with girth \( \gamma \). Then for any choice of edge lengths the metric graph \( \Gamma = (G, \ell) \) does not satisfy the generalized Manin–Mumford condition in degree \( d \geq \gamma \).

Note that if \( G \) has girth 1, i.e. \( G \) has a loop edge, then the loop is a biconnected component of genus 1.

**Theorem 1.12.** Let \( \Gamma \) be a connected metric graph of genus \( g \geq 1 \). If \( G \) is Manin–Mumford finite in degree \( d \), then
\[
\#(AJ_D^{(d)}(\Gamma) \cap \text{Jac}(\Gamma)_{\text{tors}}) \leq \binom{3g - 3}{d}.
\]
We define the independent girth $\gamma_{\text{ind}}$ of a graph as

$$\gamma_{\text{ind}}(G) = \min_C(\#E(C) + 1 - h_0(G\setminus E(C)))$$

where the minimum is taken over all closed cycles $C$ in $G$, and $h_0$ denotes the number of connected components. Since the (usual) girth satisfies $\gamma(G) = \min_C(\#E(C))$ and $h_0 \geq 1$, we have the inequality $\gamma_{\text{ind}} \leq \gamma$. The independent girth is invariant under subdivision of edges, so it is well-defined for a metric graph.

**Theorem 1.13.** Let $G$ be a finite connected graph of genus $g \geq 1$ with independent girth $\gamma_{\text{ind}}$. Then $G$ is Manin–Mumford finite in degree $d$ if and only if $1 \leq d < \gamma_{\text{ind}}$.

1.4 Outline

In Chapter 2 we review background material on metric graphs and their divisor theory. In Chapter 3 we review the interpretation of a metric graph as an electrical resistor network, and define Zhang’s canonical measure. In Chapter 4 we define the Weierstrass locus and stable Weierstrass locus for a divisor on a metric graph, and give examples, and we prove that $W(D)$ is generically finite and compute its cardinality. In Chapter 4, we prove results on the distribution of Weierstrass points on a metric graph. In Chapter 5 we prove results on the Jacobian torsion points of a metric graph.

Here we collect some notation which will be used throughout the paper.
Γ a compact, connected metric graph
PL_\mathbb{R}(\Gamma) continuous, piecewise linear functions on \Gamma
PL_\mathbb{Z}(\Gamma) continuous, piecewise \mathbb{Z}-linear functions on \Gamma
S(\Gamma) “well-behaved” piecewise smooth functions on \Gamma
\Delta(f) the principal divisor associated to a piecewise (\mathbb{Z}-)linear function f
D a divisor on a metric graph or algebraic curve
D_n a divisor of degree n
K = K_\Gamma the canonical divisor on \Gamma
r(D) the Baker–Norine rank of D
\text{Div}(\Gamma) divisors on \Gamma (with \mathbb{Z}\text{-coefficients})
\text{Div}_\mathbb{R}(\Gamma) divisors on \Gamma with \mathbb{R}\text{-coefficients}, i.e. \text{Div}(\Gamma) \otimes \mathbb{Z} \mathbb{R}
\text{Div}^d(\Gamma) divisors of degree d on \Gamma
\text{Pic}^d(\Gamma) divisor classes of degree d on \Gamma
\text{Sym}^d(\Gamma) effective divisors of degree d on \Gamma
\text{Eff}^d(\Gamma) effective divisor classes of degree d on \Gamma
[D] a divisor class; the set of divisors linearly equivalent to D
|D| the space of effective divisors linearly equivalent to D
\text{red}_x[D] the x-reduced divisor equivalent to D, where x \in \Gamma
\text{br}[D] the break divisor equivalent to D, where D has degree g
\text{Br}^d(\Gamma) the space of break divisors on \Gamma
\mu = \mu_\Gamma the Zhang canonical measure on \Gamma
G a finite, connected graph with vertex set V(G) and edge set E(G)
(G, \ell) a combinatorial model for a metric graph, where \ell : E(G) \to \mathbb{R}_{\geq 0}
\mathcal{T}(G) the set of spanning trees of a graph G
CHAPTER 2

Tropical Curves

In this section we define metric graphs and linear equivalence of divisors on metric graphs. We use the terms “metric graph” and “abstract tropical curve” interchangeably. We recall the Baker–Norine rank of a divisor, and state the Riemann–Roch theorem which is satisfied by this rank function.

2.1 Metric graphs and divisors

A metric graph is a compact, connected metric space which comes from assigning positive real edge lengths to a finite connected combinatorial graph. Namely, we construct a metric graph \( \Gamma \) by taking a finite set of edges \( E = \{e_i\} \), each isometric to a real interval \( e_i = [0, L_i] \) of length \( L_i > 0 \), gluing their endpoints to a finite set of vertices \( V \), and imposing the path metric. The underlying combinatorial graph \( G = (E, V) \) is called a combinatorial model for \( \Gamma \). We allow loops and parallel edges in a combinatorial graph \( G \). We say \( e \) is a segment of \( \Gamma \) if it is an edge in some combinatorial model.

The valence \( \text{val}(x) \) of a point \( x \) on a metric graph \( \Gamma \) is defined to be the number on connected components of a sufficiently small punctured neighborhood of \( x \). Points in the interior of a segment of \( \Gamma \) always have valence 2. All points \( x \) with \( \text{val}(x) \neq 2 \) are contained in the vertex set of any combinatorial model.

The genus of a metric graph \( \Gamma \) is its first Betti number as a topological space,

\[
g(\Gamma) = b_1(\Gamma) = \dim_\mathbb{R} H_1(\Gamma, \mathbb{R}).
\]

If \( G \) is a combinatorial model for \( \Gamma \), the genus is equal to \( g(\Gamma) = \#E(G) - \#V(G) + 1 \).

**Example 2.1.** The metric graph in Figure 2.1 has genus 0. A minimal combinatorial model has 8 vertices and 7 edges.

**Example 2.2.** The metric graph in Figure 2.2 has genus 2. A minimal combinatorial model has 2 vertices and 3 edges.
A divisor on a metric graph Γ is a finite formal sum of points of Γ with integer coefficients. The degree of a divisor is the sum of its coefficients; i.e. for the divisor $D = \sum_{x \in \Gamma} a_x x$, we have $\deg(D) = \sum_{x \in \Gamma} a_x$. We let $\text{Div}(\Gamma)$ denote the set of all divisors on Γ, and let $\text{Div}^d(\Gamma)$ denote the divisors of degree $d$. We say a divisor is effective if all of its coefficients are non-negative; we write $D \geq 0$ to indicate that $D$ is effective. More generally, we write $D \geq E$ to indicate that $D - E$ is an effective divisor. We let $\text{Sym}^d(\Gamma)$ denote the set of effective divisors of degree $d$ on Γ. $\text{Sym}^d(\Gamma)$ inherits from Γ the structure of a polyhedral cell complex of dimension $d$.

We let $\text{Div}_\mathbb{R}(\Gamma)$ denote the set of divisors on Γ with coefficients in $\mathbb{R}$. In other words, $\text{Div}_\mathbb{R}(\Gamma) = \text{Div}(\Gamma) \otimes \mathbb{Z} \mathbb{R}$.

### 2.2 Principal divisors and linear equivalence

We define linear equivalence for divisors on metric graphs, following Gathmann–Kerber [17] and Mikhalkin–Zharkov [24]. This notion is analogous to linear equivalence of divisors on an algebraic curve, where rational functions are replaced with piecewise $\mathbb{Z}$-linear functions.

A piecewise linear function on Γ is a continuous function $f : \Gamma \to \mathbb{R}$ such that there is some combinatorial model for Γ such that $f$ restricted to each edge is a linear function, i.e. a function of the form

$$f(x) = ax + b, \quad a, b \in \mathbb{R},$$

where $x$ is a length-preserving parameter on the edge. We let $\text{PL}_\mathbb{R}(\Gamma)$ denote the set of all piecewise linear functions on Γ.

A piecewise $\mathbb{Z}$-linear function on Γ is a piecewise linear function such that all its slopes are integers, i.e. $f$ restricted to each edge has the form

$$f(x) = ax + b, \quad a \in \mathbb{Z}, \ b \in \mathbb{R}$$
(for some combinatorial model). We let \( \text{PL}_\mathbb{Z}(\Gamma) \) denote the set of all piecewise \( \mathbb{Z} \)-linear functions on \( \Gamma \). The functions \( \text{PL}_\mathbb{Z}(\Gamma) \) are closed under the operations of addition, multiplication by \( \mathbb{Z} \), and taking pairwise max and min.

We let \( UT_x \Gamma \) denote the unit tangent fan of \( \Gamma \) at \( x \), which is the set of “directions going away from \( x \)” on \( \Gamma \). For \( v \in UT_x \Gamma \), the symbol \( \epsilon v \) for sufficiently small \( \epsilon \geq 0 \) means the point in \( \Gamma \) that is distance \( \epsilon \) away from \( x \) in the direction \( v \). For \( v \in UT_x \Gamma \) and a function \( f : \Gamma \to \mathbb{R} \) we let

\[
D_v f(x) = \lim_{\epsilon \to 0^+} \frac{f(x + \epsilon v) - f(x)}{\epsilon}
\]

denote the slope of \( f \) while travelling away from \( x \) in the direction \( v \) (if it exists).

Given \( f \in \text{PL}_\mathbb{Z}(\Gamma) \), we define the **principal divisor** \( \Delta(f) \in \text{Div}^0(\Gamma) \) by

\[
\Delta(f) = \sum_{x \in \Gamma} a_x x \quad \text{where} \quad a_x = \sum_{v \in UT_x \Gamma} D_v f(x).
\]

In words, the coefficient in \( \Delta(f) \) of a point \( x \) is equal to the sum of the outgoing slopes of \( f \) at \( x \). On a given segment, this divisor is supported on the finite set of points at which \( f \) is not linear, sometimes called the “break locus” of \( f \). If \( \Delta(f) = D - E \) where \( D, E \) are effective divisors with disjoint support, then we call \( D = \Delta^+(f) \) the **divisor of zeros** of \( f \) and \( E = \Delta^-(f) \) the **divisor of poles** of \( f \).

We say two divisors \( D, E \) are **linearly equivalent**, denoted \( D \sim E \), if there exists a piecewise \( \mathbb{Z} \)-linear function \( f \) such that

\[
\Delta(f) = D - E.
\]

Note that linearly equivalent divisors must have the same degree. We let \([D] \) denote the linear equivalence class of divisor \( D \), i.e.

\[
[D] = \{ E \in \text{Div}(\Gamma) : E \sim D \} = \{ D + \Delta(f) : f \in \text{PL}_\mathbb{Z}(\Gamma) \}.
\]

We say a divisor class \([D] \) is **effective**, or write \([D] \geq 0 \), if there is an effective representative \( E \sim D, E \geq 0 \) in the equivalence class.

We let \(|D| \) denote the (complete) **linear system** of \( D \), which is the set of effective divisors linearly equivalent to \( D \). We have

\[
|D| = \{ E \in \text{Div}(\Gamma) : E \sim D, E \geq 0 \}
\]

\[
= \{ D + \Delta(f) : f \in \text{PL}_\mathbb{Z}(\Gamma), \Delta(f) \geq -D \}.
\]

Unlike \([D] \), the linear system \(|D| \) is naturally a compact polyhedral complex, with topology induced by the inclusion \(|D| \subset \text{Sym}^d(\Gamma) \).
Remark 2.3 (Linear equivalence as chip firing). We sometimes speak of a degree \( n \) effective divisor on \( \Gamma \) as a collection of \( n \) “chips” placed on \( \Gamma \). Changing the divisor \( D \) to a linearly equivalent divisor \( D' \) can be achieved through a sequence of “chip firing moves” where we choose an elementary cut\(^1\) of \( \Gamma \) consisting of \( m \) segments of length \( \epsilon \), and on each edge move a chip from one end to the other. The piecewise-linear function associated to such a chip firing move has slope 0 outside the cut segments, and slope 1 on the cut segments. For more discussion see [3, Remark 2.2], [8, Section 1.5] and the references therein.

Remark 2.4 (Linear interpolation along \( f \)). Given a function \( f \in \text{PL}_\mathbb{Z}(\Gamma) \), we may associate to \( f \) a 1-parameter family of effective divisors which “linearly interpolate” between the zeros \( \Delta^+(f) \) and poles \( \Delta^-(f) \). We can think of this construction as specifying a unique “geodesic path” between any two points in the complete linear system \( |D| \). This notion previously appeared in [23] under the name \( t \)-path.

Namely, for \( \lambda \in \mathbb{R} \) we let \( \lambda \in \text{PL}_\mathbb{Z}(\Gamma) \) also denote the constant function on \( \Gamma \) by abuse of notation, and we define the effective divisor \( \Delta^{-1}(\lambda) \) by

\[
\Delta^{-1}(\lambda) = \Delta^-(f) + \Delta(\max\{f, \lambda\}).
\]

See Figure 2.4 for an illustration. Note that according to this definition, \( \Delta^{-1}(\lambda) = \Delta^-(f) \) for \( \lambda \) sufficiently large and \( \Delta^{-1}(\lambda) = \Delta^+(f) \) for \( \lambda \) sufficiently small. It is clear from definition that for any \( \lambda \), \( \Delta^{-1}(\lambda) \) is linearly equivalent to \( \Delta^+(f) \) and to \( \Delta^-(f) \).

2.3 Picard group and Jacobian

We let \( \text{Pic}(\Gamma) \) denote the Picard group of \( \Gamma \), which is the abelian group of all linear equivalence classes of divisors on \( \Gamma \). The addition operation on \( \text{Pic}(\Gamma) \) is induced from

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\(^1\) An elementary cut is a collection of segments of \( \Gamma \) such that removing the interiors of these segments disconnects \( \Gamma \) into exactly two components.
addition of divisors in \( \text{Div}(\Gamma) \). In other words, \( \text{Pic}(\Gamma) \) is the cokernel of the map \( \Delta \) sending a piecewise \( \mathbb{Z} \)-linear function to its associated principal divisor:

\[
\text{PL}_\mathbb{Z}(\Gamma) \xrightarrow{\Delta} \text{Div}(\Gamma) \to \text{Pic}(\Gamma) \to 0.
\]

The kernel of \( \Delta \) is the set of constant functions on \( \Gamma \).

Since the degree of a divisor class is well-defined, we have a disjoint union decomposition

\[
\text{Pic}(\Gamma) = \bigsqcup_{d \in \mathbb{Z}} \text{Pic}^d(\Gamma),
\]

where \( \text{Pic}^d(\Gamma) \) consists of divisor classes of degree \( d \). The degree-0 component \( \text{Pic}^0(\Gamma) \) is known as the \textit{Jacobian} of \( \Gamma \), denoted \( \text{Jac}(\Gamma) := \text{Pic}^0(\Gamma) \). The Jacobian \( \text{Jac}(\Gamma) \) is a compact abelian group.

**Theorem 2.5** (Abel–Jacobi for metric graphs). Let \( \Gamma \) be a metric graph of genus \( g \). Then there is an isomorphism of compact abelian topological groups.

\[
\text{Jac}(\Gamma) \cong (S^1)^{\times g} = S^1 \times \cdots \times S^1.
\]

\textit{Proof.} See Mikhalkin–Zharkov [24]. The proof follows the same idea as the classical Abel-Jacobi theorem, to show that \( \text{Pic}^0(\Gamma) = H^1(\Gamma, \mathbb{R})/H^1(\Gamma, \mathbb{Z}) \cong \mathbb{R}^g/\mathbb{Z}^g \). \qed

Addition of divisor classes induces an action of \( \text{Jac}(\Gamma) \) on \( \text{Pic}^d(\Gamma) \), for any degree \( d \in \mathbb{Z} \). Since \( \text{Pic}^d(\Gamma) \) is a torsor (or principal homogeneous space) for \( \text{Jac}(\Gamma) \) under this action, the Abel–Jacobi theorem also implies there are homeomorphisms \( \text{Pic}^d(\Gamma) \cong (S^1)^{\times g} \).

We let \( \text{Eff}^d(\Gamma) \) denote the set of divisor classes on \( \Gamma \) of degree \( d \) which have an effective representative. In other words, \( \text{Eff}^d(\Gamma) \) is the image of \( \text{Sym}^d(\Gamma) \) under the (degree-\( d \) restriction of the) cokernel map \( \text{Div}(\Gamma) \to \text{Pic}(\Gamma) \):

\[
\begin{array}{ccc}
\text{Sym}^d(\Gamma) & \longrightarrow & \text{Div}^d(\Gamma) \\
\downarrow & & \downarrow \text{coker} \Delta \\
\text{Eff}^d(\Gamma) & \longrightarrow & \text{Pic}^d(\Gamma).
\end{array}
\]

The space \( \text{Eff}^d(\Gamma) \) is naturally a polyhedral complex of pure dimension \( d \) when \( 0 \leq d \leq g \) (see Gross et. al. [18]). When \( d \geq g \), we have \( \text{Eff}^d(\Gamma) = \text{Pic}^d(\Gamma) \), i.e. every divisor class has an effective representative. This fact follows from the theory of break divisors; see Section 2.5 below.

As a particularly important case, the \textit{theta divisor} \( \Theta = \Theta(\Gamma) \) is \( \Theta = \text{Eff}^{g-1}(\Gamma) \), which lives inside \( \text{Pic}^{g-1}(\Gamma) \) as a codimension 1 polyhedral complex.
**Remark 2.6.** The map $\Delta : \text{PL}_\mathbb{Z}(\Gamma) \to \text{Div}(\Gamma)$ is also known as the *metric graph Laplacian* on $\Gamma$. This comes from identifying $\text{Div}(\Gamma)$ with the space of integer-valued discrete measures on $\Gamma$, via
\[ D = \sum_{i=1}^{n} a_i x_i \iff \delta = \sum_{i=1}^{n} a_i \delta_{x_i} \]
so that $\Delta(f)$ coincides with the (distributional) second derivative $-\frac{d^2}{dx^2} f(x)$, at least for $x$ in the interior of an edge. The definition of metric graph Laplacian naturally extends to piecewise linear functions on $\Gamma$ with arbitrary real slopes, if we also allow real-valued coefficients in the divisor $\Delta(f)$. This yields a map
\[ \text{PL}_\mathbb{R}(\Gamma) \xrightarrow{\Delta} \text{Div}_\mathbb{R}(\Gamma). \]
The cokernel of this map is less interesting (e.g. it does not tell us the genus of $\Gamma$); it is simply the degree function $\text{Div}_\mathbb{R}(\Gamma) \xrightarrow{\deg} \mathbb{R}$. We will see why this is the cokernel in Section 3.1 on voltage functions. This fits in the short exact sequence
\[ 0 \to \mathbb{R} \xrightarrow{\text{const}} \text{PL}_\mathbb{R}(\Gamma) \xrightarrow{\Delta} \text{Div}_\mathbb{R}(\Gamma) \xrightarrow{\deg} \mathbb{R} \to 0. \]
(Compare to the integral case
\[ 0 \to \mathbb{R} \xrightarrow{\text{const}} \text{PL}_\mathbb{Z}(\Gamma) \xrightarrow{\Delta} \text{Div}(\Gamma) \xrightarrow{\deg} \text{Pic}(\Gamma) \to 0 \]
where $\text{Pic}(\Gamma) \cong \mathbb{Z} \times (S^1)^g$.)

### 2.4 Reduced divisors

A divisor class $[D]$ is typically very large, so it is convenient to have a method of choosing a (somewhat-)canonical representative divisor inside $[D]$. When $D$ has arbitrary degree, we can do so after fixing a basepoint $q$ on our metric graph $\Gamma$, using the $q$-reduced divisor construction.

Given a point $q \in \Gamma$, the *$q$-reduced divisor* $\text{red}_q[D]$ is the unique divisor in $[D]$ which is effective away from $q$, and which minimizes a certain energy function among such representatives. Intuitively, $\text{red}_q[D]$ is the divisor in $[D]$ whose chips are “as close as possible” to the basepoint $q$. We defer giving the full definition until Section 3.3, following [10, Appendix A]. For now, we state these important properties of the reduced divisor:

- **(RD1)** $[D] \geq 0$ if and only if $\text{red}_q[D] \geq 0$
- **(RD2)** for any integer $m$, $\text{red}_q[mq + D] = mq + \text{red}_q[D]$
- **(RD3)** the degree of $\text{red}_q[D]$ away from $q$ is at most $g$, the genus of $\Gamma$ (follows from Riemann’s inequality, Corollary 2.14)
- **(RD4)** for a fixed effective divisor $D$, the map $\Gamma \to |D|$ sending $q \mapsto \text{red}_q[D]$ is continuous (due to Amini [1, Theorem 3]).
2.5 Break divisors and ABKS decomposition

When a divisor $D$ has degree $g$, there is a canonical representative of $[D]$ without any choice of basepoint, using the concept of break divisor. This notion was introduced by Mikhalkin–Zharkov [24] and studied extensively by An–Baker–Kuperberg–Shokrieh [3]. We review some of their results in this section.

A break divisor is an effective divisor of degree $g$ (the genus) which can be constructed in the following manner: choose a combinatorial model $G = (V, E)$ for $\Gamma$ and choose a spanning tree $T$ of $G$, then place one chip on each edge in the complement $E \setminus E(T)$. (Note that $E \setminus E(T)$ contains exactly $g$ edges.) Placing a chip on the endpoint of an edge is allowed.

The set of break divisors does not depend on the choice of combinatorial model. We use $\text{Br}^g(\Gamma)$ to denote the set of all break divisors on $\Gamma$. We may view $\text{Br}^g(\Gamma)$ as a topological space, using the topology induced from the inclusion in $\text{Sym}^g(\Gamma)$.

**Example 2.7.** In Figure 2.5 we show three examples of break divisors, on the left, and three examples of non-break divisors, on the right, on a genus 3 metric graph.

For a divisor class $[D]$ whose degree is $g$, the genus of the underlying curve, there is a unique representative of $[D]$ which is a break divisor.

**Theorem 2.8** (see [3, Theorem 1.1], [24, Corollary 6.6]). Let $\Gamma$ be a metric graph of genus $g$.

(a) Every divisor class $[D] \in \text{Pic}^g(\Gamma)$ contains a unique break divisor, which we denote $\text{br}[D]$.

(b) The map $\text{br} : \text{Pic}^g(\Gamma) \to \text{Sym}^g(\Gamma)$ sending a divisor class to its break divisor representative is continuous and injective. Its image is the space of all break divisors $\text{Br}^g(\Gamma)$.

(c) The map $\text{br} : \text{Pic}^g(\Gamma) \to \text{Sym}^g(\Gamma)$ is the unique continuous section of the map $[-] : \text{Sym}^g(\Gamma) \to \text{Pic}^g(\Gamma)$ taking an effective divisor to its linear equivalence class. Namely, $\text{br}$ is the unique continuous map such that the composition

$$\text{Pic}^g(\Gamma) \xrightarrow{\text{br}} \text{Sym}^g(\Gamma) \xrightarrow{[-]} \text{Pic}^g(\Gamma)$$
is the identity homeomorphism.

If we choose a combinatorial model \((G, \ell)\) for the metric graph \(\Gamma\), An–Baker–Kuperberg–Shokrieh [3] showed that the theory of break divisors implies a nice combinatorial decomposition of \(\text{Pic}^g(\Gamma)\). (\(\text{Pic}^g(\Gamma)\) is defined in Section 2.3.)

**Theorem 2.9** (ABKS decomposition, see [3, Section 3.2]). Suppose \(\Gamma = (G, \ell)\) is a metric graph with a combinatorial model. Let \(\mathcal{T}(G)\) denote the set of spanning trees of \(G\). Then

\[
\text{Pic}^g(\Gamma) = \bigcup_{T \in \mathcal{T}(G)} C_T
\]

where

\[
C_T = \{[x_1 + \cdots + x_g] : E(G) \setminus E(T) = \{e_1, \ldots, e_g\}, x_i \in e_i\}
\]

denotes the set of divisor classes represented by summing a point from each edge of \(G\) not in \(T\). The cells \(C_T\) have disjoint interiors, as \(T \in \mathcal{T}(G)\) varies.

For fixed \(T\), if we parametrize each edge \(e_i \not\in E(T)\) as the closed real interval \([0, \ell(e_i)]\), there is a natural surjective map \(\prod_{i=1}^g [0, \ell(e_i)] \to C_T\). This map always restricts to a homeomorphism on the respective interiors \(\prod_{i=1}^g (0, \ell(e_i)) \to C_T^\circ\), but may be non-injective on the boundary.

The proof is to combine Theorem 2.8 with the definition of break divisor, using the auxiliary data of the spanning tree. Since \(\text{Pic}^g(\Gamma)\) is canonically homeomorphic to \(\text{Br}^g(\Gamma)\), we may view Theorem 2.9 as a decomposition of \(\text{Br}^g(\Gamma)\).

**Remark 2.10.** If we take the combinatorial model for \(\Gamma\) to be sufficiently subdivided, then for each \(T = G \setminus \{e_1, \ldots, e_g\}\), the surjection \(\prod_{i=1}^g [0, \ell(e_i)] \to C_T\) is a (global) homeomorphism. In particular, for this to hold it suffices that \(G\) has girth \(> g\) (i.e. every cycle contains more than \(g\) edges). A necessary condition is that \(G\) has no loops or parallel edges (if \(g \geq 2\)).

**Example 2.11.** Consider the metric graph shown on the left side of Figure 2.6. Its minimal combinatorial model \(\Gamma = (G, \ell)\) contains two vertices and three edges. The associated ABKS decomposition of \(\text{Pic}^2(\Gamma)\) is shown on the right side of Figure 2.6; segments on the boundary are glued to the parallel boundary segment. There are three cells, corresponding to the three spanning trees in \(G\).

Here \(\text{Pic}^2(\Gamma)\) is homeomorphic to a torus (cf. Theorem 2.5). Each cell \(C_T\) is homeomorphic to a rectangle with a pair of opposite vertices glued together.

**Proposition 2.12.** Let \(q \in \Gamma\) be an arbitrary basepoint on a genus \(g\) metric graph. (a) For a generic divisor class \([D]\) of degree \(g\), the reduced divisor \(\text{red}_q[D]\) is equal to the break divisor \(\text{br}[D]\).
(b) For a generic divisor class \([D]\) of degree \(n\), the reduced divisor \(\text{red}_q[D]\) is equal to
\[
\text{red}_q[D] = (n - g)q + E
\]
where \(E\) is a break divisor.

2.6 Rank and Riemann–Roch

We recall the definition of the rank of a divisor on a metric graph, which is due to Baker and Norine [8] (originally for divisors on a combinatorial graph) and extended to metric graphs by Gathmann–Kerber [17] and Mikhalkin–Zharkov [24]. The rank function is a natural way to extend the important distinction between effective and non-effective divisor classes on a metric graph. Divisor classes with larger rank are in a sense “further away” from the set of non-effective divisor classes, where distance between divisors is given by adding or subtracting single points.

The rank \(r(D)\) of a divisor \(D\) on \(\Gamma\) is defined as
\[
r(D) = \max\{r \geq 0 : [D - E] \geq 0 \text{ for all } E \in \text{Sym}^r(\Gamma)\}
\]
if \([D]\) is effective, and \(r(D) = -1\) otherwise. Equivalently,
\[
r(D) = \begin{cases} 
-1 & \text{if } [D] \text{ is not effective,} \\
1 + \min_{x \in \Gamma} \{r(D - x)\} & \text{if } [D] \text{ is effective.}
\end{cases}
\]

This second definition inductively gives the rank of a divisor in terms of divisors of smaller degree; the base case is the set of non-effective divisor classes.\(^2\) Note that the rank of a divisor \(D\) depends only on its linear equivalence class.

The canonical divisor on a metric graph \(\Gamma\) is defined as
\[
K = \sum_{x \in \Gamma} (\text{val}(x) - 2) \cdot x.
\]
The degree of the canonical divisor is \(\deg K = 2g - 2\), which agrees with the canonical divisor on an algebraic curve.

\(^2\) By Riemann’s inequality, Corollary 2.14, a non-effective divisor class has degree at most \(g - 1\).
Theorem 2.13 (Riemann-Roch for metric graphs). Let $\Gamma$ be a metric graph of genus $g$, and let $K$ be the canonical divisor on $\Gamma$. For any divisor $D$ on $\Gamma$,

$$r(D) - r(K - D) = \deg(D) + 1 - g.$$ 


Corollary 2.14 (Riemann’s inequality for metric graphs). For a divisor $D$ on a metric graph of genus $g$,

$$r(D) \geq \deg(D) - g.$$ 

Proof. This follows from Riemann–Roch since $r(K - D) \geq -1$.

By Riemann’s inequality, any divisor $D$ satisfies $r(D) \geq \max\{\deg(D) - g, -1\}$. We say $D$ is nonspecial if this bound on $r(D)$ is achieved.

2.7 Matroids

In this section we review the definition of a matroid. In particular, we define the graphic and cographic matroid associated to a connected graph. Cographic matroids will be useful for understanding the structure of the Jacobian of a graph. For a complete reference on matroids, see [29] or [22].

A matroid $M = (E, B)$ is a finite set $E$ equipped with a family $B \subset 2^E$ of subsets of $E$, called the bases of the matroid, satisfying the basis exchange axiom: for distinct $B_1, B_2 \in B$, there exists some $x \in B_1 \setminus B_2$ and $y \in B_2 \setminus B_1$ such that $(B_1 \setminus x) \cup y \in B$. In other words, we can produce a new basis by exchanging an element of $B_1$ with an element of $B_2$.

An independent set of a matroid $M = (E, B)$ is a subset of $E$ which is a subset of some basis. A cycle is a subset of $E$ which is minimal among non-independent sets, under the inclusion relation. The rank of a subset $A \subset E$ is the cardinality of a maximal independent set contained in $A$.

Given a graph $G = (V, E)$, the graphic matroid is the matroid $M_G$ on the ground set $E = E(G)$ and bases $B =$ spanning trees of $G$. An independent set in $M_G$ is a set of edges which does not contain a cycle. (i.e. $h^1(G|A) = 0$.)

Example 2.15. Suppose $G$ is the theta graph with edges $\{e_1, e_2, e_3\}$. The bases of $M_G$ are the singleton sets $\{e_1\}, \{e_2\}, \{e_3\}$. The cycles are $\{\{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}\}$.

Example 2.16. Suppose $G$ is the “Wheatstone” graph in Figure 2.7. The bases of $M_G$ are $\{abd, abe, acd, ace, ade, bcd, bee, bde\}$. The cycles are $\{abc, abde, cde\}$. 

21
Given a graph $G = (V, E)$, the **cographic matroid** is the matroid $M_G^\perp$ on the ground set $E = E(G)$ whose bases are complements of spanning trees of $G$. An independent set in $M_G^\perp$ is a set of edges whose removal does not disconnect $G$. (i.e. a set $A \subseteq E$ such that $G \setminus A$ is connected, equivalently $h^0(G \setminus A) = 1$.) (For more on cographic matroids, see [29, Chapter 2.3].) The cographic matroid is also known as the *bond matroid* or the *cocycle matroid* of $G$.

**Example 2.17.** Suppose $G$ is the “Wheatstone” graph in Figure 2.7. The bases of the cographic matroid $M_G^\perp$ are $\{ac, ad, ae, bc, bd, be, cd, ce\}$. The cycles of $M_G^\perp$ are $\{ab, acd, ace, bcd, bce, de\}$.

A consequence on Mikhalkin–Zharkov’s proof [24] of the tropical Abel–Jacobi theorem (Theorem 2.5) is that the Abel–Jacobi map $\Gamma \to \text{Jac}(\Gamma)$ is linear on each edge of $\Gamma$. The universal cover of $\text{Jac}(\Gamma)$ is naturally identified with $H^1(\Gamma, \mathbb{R})$. The Abel–Jacobi map, restricted to a single edge $e \subset \Gamma$, lifts locally to $e \to H^1(\Gamma, \mathbb{R})$.

The structure of the edge-vectors in the image $\Gamma \to \text{Jac}(\Gamma)$ is exactly recorded by the cographic matroid $M_G^\perp$, for any combinatorial model $\Gamma = (G, \ell)$.

**Definition 2.18.** Let $\Gamma = (G, \ell)$ be a metric graph. Given a set of edges $e_1, \ldots, e_k \in E(G)$, let $\text{Div}(e_1, \ldots, e_k)$ denote the set of all divisors formed by adding together one point from each edge $e_i$, and let $\text{Eff}(e_1, \ldots, e_k)$ denote the set of corresponding divisor classes.

**Theorem 2.19.** Let $\Gamma = (G, \ell)$ be a metric graph. The

(a) For each edge $e_i \in E(G)$, let $v_i \in H^1(\Gamma, \mathbb{R})$ denote a vector parallel to the Abel–Jacobi image of $e_i$ in $\text{Jac}(\Gamma)$. Then the set of vectors $\{v_i : e_i \in E(G)\}$ form a realization of the cographic matroid $M_G^\perp$.

(b) The dimension of $\text{Eff}(e_1, \ldots, e_k)$ is equal to the rank of $\{e_1, \ldots, e_k\}$ in the cographic matroid $M_G^\perp$.

**Proof.** Part (a) is stated as Definition 5.1.3 of [12, p. 156].

Part (b) is a straightforward consequence of (a), since the subset $\text{Eff}(e_1, \ldots, e_k) \subset \text{Pic}^k(\Gamma)$ is naturally identified with the Minkowski sum of the corresponding vectors $v_1, \ldots, v_k \in H^1(\Gamma, \mathbb{R})$. \qed
Corollary 2.20. Let $\Gamma = (G, \ell)$ be a metric graph of genus $g$. For any integer $d$ in the range $0 \leq d \leq g$, the space $\text{Eff}^d(\Gamma)$ of degree $d$ effective divisors is naturally a cellular complex whose maximal cells are indexed by independent sets of $M_G^1$ of size $d$. 
In this section we view a metric graph as a resistor network, where each edge is a resistor whose resistance is equal to the length of the edge. This allows us to derive useful properties of the local and global structure of the metric graph.

We define the Zhang canonical measure on a metric graph (due to Zhang [33]) via the perspective of resistor networks following Baker–Faber [6].

3.1 Voltage function

We view a metric graph Γ as a resistor network by interpreting an edge of length $L$ as a resistor of resistance $L$. Note that this is well-defined on a metric graph due to the series rule for combining resistances, so we have compatibility with subdividing an edge into edges of shorter length. This interpretation is not only mathematically convenient, but physically honest—the electrical resistance of a wire is directly proportional to its length, a fact known as Pouillet’s law.

On a resistor network we may send current from one point to another. On a given segment, the voltage drop across the segment is equal to the resistance (i.e. length) of the segment multiplied by the amount of current passing through the segment—this is Ohm’s law. Under an externally-applied current, the flow of current within the network is determined by Kirchoff’s circuit laws: the current law says that the sum of directed currents out of any point is equal to zero (accounting for external currents), and the voltage law says that the sum of directed voltage differences around any closed loop is equal to zero. Our convention is that current flows from higher voltage to lower voltage.

It is a well-known empirical fact that Kirchoff’s circuit laws can be solved uniquely for any externally-applied current flow which satisfies conservation of current (i.e. internal current flows are unique). To some, it is also a well-known mathematical result. This is expressed in the following two definitions.
Definition 3.1 (physics version). Given points $y, z \in \Gamma$, the voltage function (or electric potential function) $j^y_z : \Gamma \to \mathbb{R}$ is defined by

$$j^y_z(x) = \text{voltage at } x \text{ when sending one unit of current from } y \text{ to } z,$$

such that $j^y_z(z) = 0$, i.e. the network is “grounded” at $z$.

Definition 3.2 (math version; definition–theorem). Given points $y, z \in \Gamma$, the voltage function $j^y_z$ is the unique function in $\text{PL}_\mathbb{R}(\Gamma)$ satisfying the conditions

$$\Delta(j^y_z) = z - y \in \text{Div}^0_\mathbb{R}(\Gamma) \quad \text{and} \quad j^y_z(z) = 0.$$

Proof. For the existence and uniqueness of $j^y_z$, see Theorem 6 and Corollary 3 of Baker–Faber [6]. Note that they use the notation $j_z(y, -)$ for $j^y_z(-)$.

Note that $j^y_z$ satisfies the following properties:

(V1) for any $x \in \Gamma$, $0 = j^y_z(z) \leq j^y_z(x) \leq j^y_z(y)$,

(V2) $j^y_z(x)$ is piecewise linear in $x$,

(V3) $j^y_z(x)$ is continuous in $x, y, \text{and } z$.

Proposition 3.3. The voltage function $j^y_z$ obeys the following symmetries.

(a) For any three points $x, y, z \in \Gamma$,

$$j^y_z(x) = j^x_z(y)$$

(b) For any four points $x, y, z, w \in \Gamma$,

$$j^y_z(x) - j^y_z(w) = j^x_w(y) - j^x_w(z).$$

Proof. See Baker–Faber [6, Theorem 8]; they refer to (b) as the “Magical Identity”. Note that (a) follows from (b) by setting $z = w$.

Remark 3.4. We may interpret any function $f \in \text{PL}_\mathbb{R}(\Gamma)$ as a voltage function on $\Gamma$, which results from the externally applied current $\Delta(f) \in \text{Div}_\mathbb{R}(\Gamma)$. In other words, the voltage $f$ results from sending current from $\Delta^-(f)$ to $\Delta^+(f)$ in $\Gamma$.

The existence of $j^y_z \in \text{PL}_\mathbb{R}(\Gamma)$ for any $y, z \in \Gamma$ implies that the principal divisor map $\Delta : \text{PL}_\mathbb{R}(\Gamma) \to \text{Div}^0_\mathbb{R}(\Gamma)$ is surjective. This verifies the claim made in Remark 2.6 concerning the exactness of the sequence

$$0 \to \mathbb{R} \xrightarrow{\text{const}} \text{PL}_\mathbb{R}(\Gamma) \xrightarrow{\Delta} \text{Div}_\mathbb{R}(\Gamma) \xrightarrow{\text{deg}} \mathbb{R} \to 0.$$
Proposition 3.5 (Slope-current principle). Suppose $f \in \text{PL}_R(\Gamma)$ has zeros $\Delta^+(f)$ and poles $\Delta^-(f)$ of degree $d \in \mathbb{R}$. Then the slope of $f$ is bounded by $d$, i.e.

$$|f'(x)| \leq d \quad \text{for any } x \text{ where } f \text{ is linear}.$$  

(This bound is sharp; it is attained only on bridge edges, and only when all zeros are on one side of the bridge and all poles are on the other side.)

Proof. Let $\lambda = f(x)$. Then the “tropical preimage”

$$f^{-1}_\Delta(\lambda) := \Delta^-(f) + \Delta(\max\{f,\lambda\})$$

has multiplicity $|f'(x)|$ at $x$, since the outgoing slopes of $\max\{f,\lambda\}$ at $x$ are $|f'(x)|$ and 0. (Note $x$ cannot be in $\Delta^-\left(f\right)$ since $f$ is linear at $x$.) Since the divisor $f^{-1}_\Delta(\lambda)$ is effective of degree $d$, this implies $|f'(x)| \leq d$ as desired. \qed

Remark 3.6. The above proposition is obvious from its “physical interpretation”: $f$ gives the voltage in the resistor network $\Gamma$ when subjected to an external current described by $\Delta^-\left(f\right)$ units flowing into the network and $\Delta^+\left(f\right)$ units flowing out. The slope $|f'(x)|$ is equal to the current flowing through the wire containing $x$, which must be no more than the total in-flowing (or out-flowing) current.

Next we address how the voltage function $j_{yz} \in \text{PL}_R(\Gamma)$ may be approximated by a sequence of functions in $\text{PL}_Z(\Gamma)$ (up to rescaling), which depend on reduced divisors. We only use property (RD3) of reduced divisors.

Proposition 3.7 (Discrete approximation of voltage function). Let $\{D_n : n \geq 1\}$ be a sequence of divisors on $\Gamma$ with $\deg D_n = n$. Fix two points $y, z \in \Gamma$. Let $\text{red}_y[D_n]$ and $\text{red}_z[D_n]$ denote the $y$- and $z$-reduced representatives in the divisor class $[D_n]$, and let $f_n$ be the unique function in $\text{PL}_Z(\Gamma)$ satisfying

$$\Delta(f_n) = \text{red}_z[D_n] - \text{red}_y[D_n]$$

and $f_n(z) = 0$. Then the functions $\frac{1}{n}f_n$ converge uniformly to $j_{yz}$ as $n \to \infty$.

Proof. If the sequence $\frac{1}{n}h_n$ converges to a limit, then the sequence $\frac{1}{n+c}h_n$ must also converge to the same limit as $n \to \infty$, for any constant $c$. Thus it suffices to show that the functions $\frac{1}{n}f_n$ converge uniformly to $j_{yz}$.

Let $\phi_n = \frac{1}{n}f_{n+g} - j_{yz}$. We claim that the sequence of functions $\{\phi_n \in \text{PL}_R(\Gamma) : n \geq 1\}$ converges uniformly to 0. Note that each $\phi_n$ is a continuous, piecewise-differentiable function with $\phi_n(z) = 0$, so for an arbitrary $x \in \Gamma$ we may calculate the value of $\phi_n(x)$ by integrating the derivative of $\phi_n$ along some path in $\Gamma$ from $z$ to $x$. The length of such a path is bounded uniformly in $x$, since $\Gamma$ is compact, so to
show that $\phi_n \to 0$ uniformly it suffices to show that the magnitude of the derivative $|\phi'_n|$ approaches 0 uniformly.

Claim: For any $x \in \Gamma$, $|\phi'_n(x)| \leq \frac{g}{n}$.

This follows from the slope-current principle (Proposition 3.5). By Riemann’s inequality, the $y$-reduced representative in $[D_{(n+g)}]$ may be expressed as

$$\text{red}_y[D_{n+g}] = ny + E_n$$

for some effective divisor $E_n$ of degree $g$. Similarly, $\text{red}_z[D_{n+g}] = nz + F_n$ for some effective $F_n$ of degree $g$. Thus the principal divisor associated to $\frac{1}{n}f_{n+g}$ is

$$\Delta\left(\frac{1}{n}f_{n+g}\right) = z + \frac{1}{n}F_n - y - \frac{1}{n}E_n.$$  

Recall that $\Delta(j^y_z) = z - y$; it follows that the principal $\mathbb{R}$-divisor associated to $\phi_n$ is

$$\Delta(\phi_n) = \Delta\left(\frac{1}{n}f_{n+g} - j^y_z\right) = \frac{1}{n}F_n - \frac{1}{n}E_n.$$  

In particular, $\Delta(\phi_n)$ is a difference of effective $\mathbb{R}$-divisors of degree $\frac{g}{n}$, so the zeros $\Delta^+(\phi_n)$ and poles $\Delta^-(\phi_n)$ each have degree at most $\frac{g}{n}$. By Proposition 3.5, this implies $|\phi'_n(x)| \leq \frac{g}{n}$ as claimed. 

We separate the central claim in the above proof to a named proposition, for future reference.

**Proposition 3.8** (Quantitative version of voltage approximation). Let $\Gamma$ be a metric graph of genus $g$, and let $D_n$ be a degree $n$ divisor on $\Gamma$. Fix two points $y$ and $z$ on $\Gamma$, and let $f_n$ be the unique function in $\text{PL}_\mathbb{Z}(\Gamma)$ satisfying

$$\Delta(f_n) = \text{red}_z[D_n] - \text{red}_y[D_n]$$

and $f_n(z) = 0$. Then for $n > g$ and any $x \in \Gamma$, $|(\frac{1}{n-g}f_n - j^y_z)'(x)| \leq \frac{g}{n-g}$.

**Remark 3.9.** We can interpret Proposition 3.7 as follows: the existence of the voltage function $j^y_z : \Gamma \to \mathbb{R}$ follows from Riemann’s inequality for divisors on $\Gamma$.

### 3.2 Kirchhoff formulas

In this section we review Kirchhoff’s formula for the resistance function.

**Theorem 3.10.** Suppose $\Gamma = (G, \ell)$ is a metric graph. For vertices $y, z \in V(G)$, let $j^y_z : G \to \mathbb{R}$ denote the voltage function which sends one unit of current from $y$ to $z$.
1. The total voltage drop between $y$ and $z$ is

$$j^y_z(y) - j^y_z(z) = \frac{\sum_{T \in \mathcal{T}(G_0)} w(T)}{\sum_{T \in \mathcal{T}(G)} w(T)}$$

where $\mathcal{T}(G)$ denotes the spanning trees of $G$, the graph $G_0$ (in the numerator) is the graph obtained from $G$ by identifying vertices $y$ and $z$, and the weight $w(T)$ of a spanning tree is defined as

$$w(T) = \prod_{e_i \notin E(T)} \ell(e_i).$$

2. The voltage drop across a directed edge $e = (e_+, e_-)$ is

$$j^y_z(e_+) - j^y_z(e_-) = \frac{\ell(e) \sum_{T \in \mathcal{T}(G)} sgn(T, y, z, e) w(T)}{\sum_{T \in \mathcal{T}(G)} w(T)}$$

where

$$sgn(T, y, z, e) = \begin{cases} +1 & \text{if the path in } T \text{ from } y \text{ to } z \text{ passes through } e \\ -1 & \text{if the path in } T \text{ from } y \text{ to } z \text{ passes through } -e \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the formulas (above) expresses the total voltage drop as a ratio of homogeneous polynomials in the variables $\{\ell(e_i) : e_i \in E(G)\}$. The denominator is homogeneous of degree $g$, and the numerator is homogeneous of degree $g + 1$.

3.3 Energy and reduced divisors

Here we give a definition of $q$-reduced divisors on a metric graph. We will only need to use $q$-reduced divisors for effective divisor classes, so we restrict our discussion here to the effective case.

**Definition 3.11.** Given a basepoint $q$ on $\Gamma$, we define the $q$-energy $\mathcal{E}_q : \Gamma \to \mathbb{R}$ by

$$\mathcal{E}_q(y) = j^y_q(y) = r(y, q).$$

Given an effective divisor $D = \sum_i y_i$, we define the $q$-energy $\mathcal{E}_q(D)$ by

$$\mathcal{E}_q(D) = \sum_i \sum_j j^y_q(y_j).$$

Note that

- $\mathcal{E}_q(D) \geq 0,$
• \( E_q(D) \) is strictly positive if \( D \) has support outside of \( q \),
• \( E_q(D) \geq \sum_i E_q(y_i) \), and in general this inequality is strict.

**Theorem 3.12** (Baker–Shokrieh [10, Theorem A.7]). Fix a basepoint \( q \in \Gamma \), and let \( D \) be an effective divisor on \( \Gamma \). There is a unique divisor \( D_0 \in |D| \) which minimizes the \( q \)-energy, i.e. such that

\[
E_q(D_0) < E_q(E) \quad \text{for all} \quad E \in |D|, \ E \neq D_0.
\]

**Definition 3.13.** The \( q \)-reduced divisor \( \text{red}_q[D] \) is the unique divisor in \( |D| \) which minimizes the \( q \)-energy \( E_q \).

Note that this definition is non-standard; the standard definition for reduced divisor is a combinatorial condition which can be phrased in the language of chip-firing, see [1, p. 4854], [3, Definition 2.3].

**Example 3.14.** In Figure 3.1 we show a degree 4 divisor, on the left, and its reduced representative with respect to basepoint \( q \), on the right.

![Figure 3.1: A divisor and its reduced divisor representative](image)

### 3.4 Resistance function

In this section we recall the definition of the (Arakelov–Zhang–Baker–Faber) canonical measure \( \mu \) on a metric graph.

**Definition 3.15.** Let \( r : \Gamma \times \Gamma \to \mathbb{R} \) denote the effective resistance function on the metric graph \( \Gamma \). Namely, viewing \( \Gamma \) as a resistor network

\[
r(x, y) = \begin{cases} 
\text{effective resistance between } x \text{ and } y \\
\text{total voltage drop when sending 1 unit of current from } x \text{ to } y
\end{cases}
\]

If we wish to emphasize the underlying graph, we write \( r(x, y; \Gamma) \). In terms of the voltage function from Section 3.1, \( r(x, y) = j_y^x(x) \).

It is straightforward to verify that the resistance function satisfies the following properties:

1. \( r(x, x) = 0 \),
2. \( r(x, y) > 0 \) if \( x \neq y \),

3. \( r(x, y) \) is continuous with respect to \( x \) and \( y \)

4. \( r(x, y) = r(y, x) \)

In contrast with the voltage function \( j^y_x \), the function \( x \mapsto r(x, y) \) is not piecewise linear; we will see that it is instead piecewise quadratic.

There is a special case of effective resistance which will be particularly useful in the following sections.

**Definition 3.16.** Given a segment \( e \) in a metric graph \( \Gamma \), the deleted effective resistance \( \ell_{\text{eff}}(\Gamma \setminus e) \) is the effective resistance between endpoints of \( e \) in the \( e \)-deleted subgraph; that is, if \( s, t \) are the endpoints of \( e \)

\[
\ell_{\text{eff}}(\Gamma \setminus e) = r(s, t; \Gamma \setminus e).
\]

Note that \( \ell_{\text{eff}}(\Gamma \setminus e) = 0 \) when \( e \) is a loop, and \( \ell_{\text{eff}}(\Gamma \setminus e) = +\infty \) when \( e \) is a bridge.

The rule for combining resistances in parallel implies that for a segment \( e \) with endpoints \( s \) and \( t \),

\[
r(s, t; \Gamma) = \left( \frac{1}{\ell(e)} + \frac{1}{\ell_{\text{eff}}(\Gamma \setminus e)} \right)^{-1} = \frac{\ell(e)\ell_{\text{eff}}(\Gamma \setminus e)}{\ell(e) + \ell_{\text{eff}}(\Gamma \setminus e)}.
\]

**Example 3.17.** Let \( \Gamma \) be a circle of circumference \( L \). By choosing a basepoint which we denote as 0, we may parametrize \( \Gamma \) with the interval \([0, L] \). Identifying points in this way, we have

\[
r(x, 0) = \text{parallel combination of resistances } x \text{ and } L - x
\]

\[
= \frac{x(L - x)}{x + (L - x)} = x - \frac{1}{L}x^2.
\]

The effective resistance is maximized when \( x = \frac{1}{2}L \), with maximum value \( \frac{1}{4}L \). The effective resistance is minimized when \( x = 0 \) or \( x = L \), with effective resistance 0.

### 3.5 Canonical measure

**Definition 3.18.** The canonical measure \( \mu = \mu_{\Gamma} \) on a metric graph \( \Gamma \) is the continuous measure defined by

\[
\mu = \mu(dx) = -\frac{1}{2} \frac{d^2}{dx^2} r(x, y_0) \, dx.
\]

where \( x \) is a length-preserving parameter on a segment, \( dx \) is the Lebesgue measure, and \( y_0 \) is a fixed point in \( \Gamma \). This defines \( \mu \) on the open dense subset of \( \Gamma \) where the second derivative exists; at the finite set of points where \( r(\cdot, y_0) \) is not differentiable, or where the valence of \( x \) differs from 2, we let \( \mu_{\Gamma} = 0 \).
Remark 3.19. The first derivative of a smooth function on \( \Gamma \) is only well-defined up to a choice of sign, since there are two directions in which we could parametrize any segment. The second derivative, however, is well-defined on each segment (without choosing an orientation) because \((\pm 1)^2 = 1\) so either choice of direction yields the same second derivative.

Remark 3.20. The definition of canonical measure is independent of the choice of basepoint \( y_0 \) because of the “Magical Identity” in Proposition 3.3 (b). Namely, for two basepoints \( y_0, z_0 \) we have
\[
j_{y_0}^x(x) - j_{y_0}^x(z_0) = j_{z_0}^x(x) - j_{z_0}^x(y_0)
\]
which implies
\[
r(x, y_0) - r(x, z_0) = j_{y_0}^x(x) - j_{z_0}^x(x)
= j_{y_0}^x(z_0) - j_{z_0}^x(y_0)
= j_{y_0}^x(x) - j_{z_0}^y(y_0).
\]
Since the voltage functions \( j_{y_0}^x, j_{z_0}^x \) are piecewise linear, we have
\[
\frac{d^2}{dx^2} (r(x, y_0) - r(x, z_0)) = \frac{d^2}{dx^2} (j_{y_0}^x(x) - j_{z_0}^y(x)) = 0.
\]

Remark 3.21. The definition of canonical measure given here differs from that used by Baker–Faber [6], in that our \( \mu \) does not have a discrete part supported at the points of \( \Gamma \) with valence different from 2.

Remark 3.22. The definition of canonical measure given here is equal to Zhang’s canonical measure [33, Section 3, Theorem 3.2 c.f. Lemma 3.7] associated to the canonical divisor \( D = K \), up to a multiplicative factor. Our canonical measure is normalized to satisfy \( \mu(\Gamma) = g \) rather than \( \mu(\Gamma) = 1 \).

The canonical measure of Baker–Faber is equal to Zhang’s canonical measure associated to \( D = 0 \).

Example 3.23 (Canonical measure on cirlce). If \( \Gamma \) is a circle of circumference \( L \), by Example 3.17 we have \( r(x, 0) = x - \frac{1}{L}x^2 \) so the canonical measure is \( \mu = \frac{1}{L}dx \). The total measure on the metric graph is \( \mu(\Gamma) = 1 \).

Example 3.24 (Canonical measure on theta graph). Consider the metric graph \( \Gamma \) of genus 2 shown in Figure 3.2, with edge lengths \( a, b, c \).

![Figure 3.2: Genus 2 metric graph with edge lengths a, b, c.](image-url)
On the edge of length $a$, we have $\ell(e) = a$ and $\ell_{\text{eff}}(\Gamma \setminus e) = \frac{bc}{b+c}$. When measuring effective resistance between points in the interior of $e$, we can think of $\Gamma$ as a circle of total length $\ell(e) + \ell_{\text{eff}}(\Gamma \setminus e) = \frac{ab+ac+bc}{b+c}$. Thus the canonical measure on this edge is $\mu = \frac{b+c}{ab+ac+bc} dx$, by the computation for a circle in Example 3.17. The total measure on this edge is $\mu(e) = \frac{ab+ac}{ab+ac+bc}$, and by symmetry the total measure on the metric graph is $\mu(\Gamma) = 2$.

**Proposition 3.25.** The canonical measure $\mu$ on a metric graph $\Gamma$ is a piecewise-constant multiple of the Lebesgue measure which vanishes on all bridge segments.

On a non-bridge segment $e$ in $\Gamma$,

$$
\mu|_e = \frac{1}{\ell(e) + \ell_{\text{eff}}(\Gamma \setminus e)} \, dx
$$

where $\ell(e)$ denotes the length of $e$ and $\ell_{\text{eff}}(\Gamma \setminus e)$ denotes the effective resistance between the endpoints of $e$ on the graph after removing the interior of $e$.

For a bridge segment, $\mu|_e = 0$.

**Proof.** See Baker–Faber [6, Theorem 12]; note that our $\mu$ is defined to be the continuous part of Baker–Faber’s $\mu_{\text{can}}$.

(The proof idea should be clear from Example 3.24.)

If a segment $e$ is subdivided into $e_1 \sqcup e_2$, the expression $\mu|_e$ agrees with $\mu|_{e_1} = \mu|_{e_2}$.

**Corollary 3.26.** Let $\Gamma$ be a metric graph with canonical measure $\mu$, and let $e$ be a segment in $\Gamma$ (i.e. $e$ is subspace isometric to a closed interval, whose interior points all have valence 2 in $\Gamma$). Then

(a) $0 \leq \mu(e) \leq 1$;
(b) $\mu(e) = 0 \iff e$ is a bridge edge;
(c) $\mu(e) = 1 \iff e$ is a loop edge.

**Proof.** By Proposition 3.25, $\mu(e) = 0$ for bridges and $\mu(e) = \frac{\ell(e)}{\ell(e) + \ell_{\text{eff}}(\Gamma \setminus e)}$ otherwise.

**Proposition 3.27** (Foster’s theorem). Let $\Gamma$ be a metric graph of genus $g$, and let $\mu$ be the canonical measure on $\Gamma$. Then the total measure on $\Gamma$ is

$$
\mu(\Gamma) = g.
$$

**Proof.** See Baker–Faber [6, Corollary 5 and Corollary 6] and Foster [16].
CHAPTER 4

Weierstrass Points

In this section we define the Weierstrass locus and the stable Weierstrass locus of an arbitrary divisor $D$ on a metric graph $\Gamma$. We first review the notion of Weierstrass point on an algebraic curve.

The results in this section first appeared in the preprint [32].

4.1 Classical Weierstrass points

Recall that for an algebraic curve $X$ of genus $g$, the ordinary Weierstrass points are defined as follows. The canonical divisor $K$ on $X$ determines a canonical map to projective space $\varphi_K : X \to \mathbb{P}^{g-1}$. Generically, a point on $\varphi_K(X)$ will have an osculating hyperplane in $\mathbb{P}^{g-1}$ which intersects $\varphi_K(X)$ with multiplicity $g-1$. For finitely many “exceptional” points on $\varphi_K(X)$, the osculating hyperplane will intersect the curve with higher multiplicity; the preimages of these exceptional points are the ordinary Weierstrass points of $X$. These are also known as the flex points of the embedded curve $\varphi_K(X) \subset \mathbb{P}^{g-1}$.

This notion may be generalized by replacing $K$ with an arbitrary (basepoint-free) divisor. Given a divisor $D$ on $X$, there is an associated map to projective space $\varphi_D : X \to \mathbb{P}^r$, known as the complete linear embedding defined by $D$. The set of flex points of the embedded curve $\varphi_D(X)$, where the osculating hyperplane intersects the curve with multiplicity greater than $r$, are the (higher) Weierstrass points associated to the divisor $D$. If $D$ has degree $n \geq 2g - 1$, the number of Weierstrass points of $D$ counted with multiplicity is $g(n - g + 1)^2$.

The existence of an osculating hyperplane of multiplicity greater than $r$, at the point $\varphi_D(x) \in \varphi_D(X)$, is equivalent to the existence of a non-zero global section of the line bundle $\mathcal{L}(X, D - (r + 1)x)$, i.e. to having $h^0(X, D - (r + 1)x) \geq 1$. 

33
4.2 Tropical Weierstrass points

Given a divisor $D$ on a metric graph, we define the set of Weierstrass points of $D$ using the Baker-Norine rank function $r(D)$, which is the analogue of $h^0(D) - 1$.

**Definition 4.1.** Let $D$ be a divisor on a metric graph $\Gamma$, with rank $r = r(D)$. A point $x \in \Gamma$ is a **Weierstrass point for $D$** if

$$[D - (r + 1)x] \geq 0.$$  

The **Weierstrass locus $W(D) \subset \Gamma$** of $D$ is the set of its Weierstrass points. An **ordinary Weierstrass point** is a Weierstrass point for the canonical divisor $K$.

Note that the Weierstrass locus of $D$ depends only on the divisor class $[D]$.

**Remark 4.2.** If the divisor class $[D]$ is not effective, i.e. $r(D) = -1$, then the set of Weierstrass points of $D$ is empty. Thus we may restrict our attention to studying Weierstrass points for effective divisor classes.

**Example 4.3.** Suppose $\Gamma$ is a genus 1 graph and $D$ is a divisor of degree 6, indicated by the black dots in the figure below with multiplicities. This divisor has rank $r = 5$ since it is in the nonspecial range of Riemann–Roch. The Weierstrass locus of $D$ consists of 6 points evenly spaced around $\Gamma$, indicated in red.

![Figure 4.1: Weierstrass points, in red, on a genus 1 metric graph.](image1)

**Example 4.4.** Suppose $\Gamma$ is a complete graph on 4 vertices, with distinct edge lengths. This graph has genus 3. Consider the canonical divisor $K$ on $\Gamma$, which is supported on the four trivalent vertices. The Weierstrass locus of $K$ consists of 8 distinct points on $\Gamma$, shown in red in Figure 4.2.

![Figure 4.2: Metric graph with finite Weierstrass locus.](image2)

In the following examples, we use “chip firing” language to describe linear equivalence of divisors; see Remark 2.3.
Example 4.5 (Wedge of circles). Suppose $\Gamma$ is a wedge of $g$ circles, and let $x_0$ denote the point of $\Gamma$ lying on all $g$ circles. For a generic divisor class $[D_n]$ of degree $n$ (meaning generic inside of $\text{Pic}^n(\Gamma)$), the $x_0$-reduced representative of $[D_n]$ consists of $n - g$ chips at $x_0$ and one chip in the interior of each circle. The Weierstrass locus $W(D_n)$ contains $n - g + 1$ evenly-spaced points on each circle of $\Gamma$, for a total of $g(n - g + 1)$ points.

Example 4.6 (Failure of $W(D)$ to be finite). Consider the genus 3 graph shown in Figure 4.3. Suppose $D$ is a degree 4 divisor supported on one of the bridge edges as shown. (Note that $D \sim K$.) This divisor has rank $r \leq 2$, since we cannot move the chips in $D$ to lie on three distinct loops freely. However, for any point $x$, the reduced divisor $\text{red}_x[D]$ has at least 3 chips at $x$.

Example 4.7 (Failure of $W(D)$ to be finite, v2). Consider the genus 3 graph shown in Figure 4.4. Suppose $D = K$ is the canonical divisor. By Riemann–Roch, $K$ has rank $r = 2$. It is possible to move all 4 chips to lie on the middle loop, so any point in the middle loop has $\text{red}_x[D] \geq 3x$. The Weierstrass locus $W(K)$ contains the middle loop, but not the two outer loops.

Remark 4.8. For any metric graph with a bridge edge, it can be shown that the entire bridge edge is contained in the Weierstrass locus of the canonical divisor so in particular $W(K)$ is not finite. We omit the details.

4.2.1 Stable tropical Weierstrass points

In this section we define the stable Weierstrass locus $W^{st}(D)$ of a divisor $D$ on a metric graph. This definition is meant to fix undesireable behavior of the naive Weierstrass locus $W(D)$. In particular, $W^{st}(D)$ is always a finite set.
For the definition of break divisor, see Section 2.5.

**Definition 4.9.** Let $D$ be a divisor of degree $n$ on a metric graph $\Gamma$. If $n \geq g$, the *stable Weierstrass locus* $W^{st}(D) \subset \Gamma$ is the set of all points $x \in \Gamma$ such that

$$\text{br}[D - (n - g)x] \geq x$$

where $\text{br}[E]$ is the break divisor representative of the divisor class $[E]$. In other words, $x$ is a stable Weierstrass point of $D$ if there exists a break divisor $E \geq x$ such that $E + (n - g)x \in [D]$.

Note that if $D$ has degree $n = g$, then $W^{st}(D)$ is exactly the support of $\text{br}[D]$.

If $D$ has degree $n < g$, we define $W^{st}(D)$ to be empty.

In the above definition, if $n \geq g$ then $n - g$ is the rank of a generic divisor class in $\text{Pic}^n(\Gamma)$. If a divisor class $[D]$ in $\text{Pic}^n(\Gamma)$ has rank $r(D) = n - g$, then $W^{st}(D) \subset W(D)$; otherwise, this containment may fail to hold. In particular, we have $W^{st}(D) \subset W(D)$ for all divisors of degree $n \geq 2g - 1$.

**Example 4.10** (Divisor with $W^{st}(D) \not\subset W(D)$). Consider the genus 3 metric graph shown in Figure 4.5. The canonical divisor $K$ is indicated in black. This divisor has degree $n = 4$ and rank $r(K) = 2$. The divisor is special, because $r(K) > n - g = 1$.

On the left side, the Weierstrass locus is shown in red; the right side shows the stable Weierstrass locus. The stable Weierstrass locus consists of the midpoint of each edge. The sets $W(K)$ and $W^{st}(K)$ are disjoint.

![Figure 4.5: Divisor with Weierstrass locus and stable Weierstrass locus.](image)

4.3 Finiteness of Weierstrass points

In this section we show that the Weierstrass locus of a generic divisor class $[D]$ on a metric graph is a finite set whose cardinality is $\#W(D) = g(n - g + 1)$. We do so by studying the stable Weierstrass locus $W^{st}(D)$, defined in Section 4.2.1.
4.3.1 Setup

Our main technical tool is to consider the ABKS decomposition of 
\( \text{Pic}^g(\Gamma) \) (see Section 2.5) and the topology of certain branched covering spaces.

As the divisor class \([D]\) varies over \( \text{Pic}^n(\Gamma) \), we realize the stable Weierstrass loci \( W^{st}(D) \) as the fibers of a surjective map \( X \to \text{Pic}^n(\Gamma) \). We are able to study the cardinality of \( W^{st}(D) \) by imposing a nice topology on \( X \) and analyzing topological properties of the map \( X \to \text{Pic}^n(\Gamma) \).

Recall that \( \text{Br}^g(\Gamma) \) denotes the space of break divisors on \( \Gamma \), viewed as a subspace of \( \text{Sym}^g(\Gamma) \).

**Definition 4.11.** Let \( \tilde{\text{Br}}^g(\Gamma) \) denote the space
\[
\tilde{\text{Br}}^g(\Gamma) = \{(x,E) \in \Gamma \times \text{Sym}^{g-1}(\Gamma) : x + E \text{ is a break divisor}\}.
\]
This defines a closed subset of the compact Hausdorff space \( \Gamma \times \text{Sym}^{g-1}(\Gamma) \), so \( \tilde{\text{Br}}^g(\Gamma) \) is compact and Hausdorff.

**Remark 4.12.** We may think of \( \tilde{\text{Br}}^g(\Gamma) \) as the space of “pointed break divisors” on \( \Gamma \), i.e. \( \tilde{\text{Br}}^g(\Gamma) \) is homeomorphic to \( \{(x,D) \in \Gamma \times \text{Br}^g(\Gamma) : x \leq D\} \).

Let \( \sigma : \tilde{\text{Br}}^g(\Gamma) \to \text{Br}^g(\Gamma) \) denote the “summation” map \( (x,E) \mapsto x + E \), and let \( \sigma_m : \tilde{\text{Br}}^g(\Gamma) \to \text{Pic}^{m+g-1}(\Gamma) \) denote the “summation with multiplicity” map defined by
\[
\sigma_m : (x,E) \mapsto [mx + E].
\]

Let \( \pi_1 : \tilde{\text{Br}}^g(\Gamma) \to \Gamma \) denote projection to the first factor, i.e. \( \pi_1(x,E) = x \).

**Lemma 4.13.** Suppose \([D]\) \( \in \text{Pic}^{m+g-1}(\Gamma) \), and let \( \sigma_m \) and \( \pi_1 \) be defined as above.

(a) The stable Weierstrass locus \( W^{st}(D) \) is equal to \( \pi_1(\sigma_m^{-1}[D]) \).
(b) We have \( \#W^{st}(D) = \#\sigma_m^{-1}[D] \).

**Proof.** (a) This follows from the definition of the stable Weierstrass locus.

(b) The claim is that \( \pi_1 \) is injective on the preimage \( \sigma_m^{-1}[D] \). To see this, consider two points \((x,E)\) and \((x',E')\) \( \in \tilde{\text{Br}}^g(\Gamma) \) in the same fiber \( \sigma_m^{-1}[D] \). This means that \([mx+E]=[mx'+E']=[D]\). Suppose \( \pi_1(x,E) = \pi_1(x',E') \), i.e. that \( x = x' \). Then
\[
[D - (m - 1)x] = [x + E] = [x + E'] \in \text{Pic}^g(\Gamma).
\]
Since both \((x+E)\) and \((x+E')\) are break divisors, the uniqueness of break divisor representatives (Theorem 2.8) implies that \( E = E' \). This shows that the restriction of \( \pi_1 \) to \( \sigma_m^{-1}[D] \) is injective, as desired. \( \Box \)
Let \((G, \ell)\) be a combinatorial model for \(\Gamma\), which induces a decomposition of break divisors \(\text{Br}^g(\Gamma)\) into a union of cells
\[
(4.1) \quad \text{Br}^g(\Gamma) = \bigcup_{T \in \mathcal{T}(G)} C_T
\]
indexed by spanning trees of \(G\), where the interior of each cell \(C_T\) is homeomorphic to an open hypercube. (See Section 2.5 or [3].) Note that \(\text{Br}^g(\Gamma)\) is homeomorphic to \(\text{Pic}^g(\Gamma)\). The ABKS decomposition (4.1) of \(\text{Br}^g(\Gamma)\) induces a decomposition
\[
(4.2) \quad \tilde{\text{Br}}^g(\Gamma) = \bigcup_{T \in \mathcal{T}(G)} \left( \bigcup_{e \notin E(T)} \tilde{C}_{T,e} \right)
\]
where the second union is over edges \(e\) of \(G\) not contained in the spanning tree \(T\). There are \(g\) such edges for any \(T\). Namely,
\[
\tilde{C}_{T,e} = \{ (x, E) : x + E \in C_T, x \in e \}
\]
The map \(\tilde{\text{Br}}^g(\Gamma) \to \text{Br}^g(\Gamma)\) sends the cell \(\tilde{C}_{T,e}\) surjectively to \(C_T\). On the interior \(C_T^o\) of each cell, each fiber of \(\tilde{\text{Br}}^g(\Gamma) \to \text{Br}^g(\Gamma)\) contains exactly \(g\) points.

If \(\kappa(G) = \#\mathcal{T}(G)\) denotes the number of spanning trees of \(G\), the ABKS decomposition (4.2) decomposes \(\tilde{\text{Br}}^g(\Gamma)\) into a union of \(g \cdot \kappa(G)\) cells.

**Example 4.14.** In Figure 4.6, we show the decomposition of \(\tilde{\text{Br}}^2(\Gamma)\) into six cells \(\tilde{C}_{T,e}\), where \(\Gamma\) is a theta graph. This graph has genus \(g = 2\) and \(\kappa(G) = 3\) spanning trees. In this case \(\text{Br}^2(\Gamma) \cong \text{Pic}^2(\Gamma) \cong \mathbb{R}^2 / \mathbb{Z}^2\) is a genus 1 surface (cf. Example 2.11, Theorem 2.5), and \(\tilde{\text{Br}}^2(\Gamma)\) is a surface of genus 2. The map \(\tilde{\text{Br}}^2(\Gamma) \to \text{Br}^2(\Gamma)\) is a branched double cover ramified at two points, corresponding to the two break divisors which consist of two chips at a trivalent vertex of \(\Gamma\).

![Figure 4.6: ABKS decomposition of \(\tilde{\text{Br}}^2(\Gamma)\).](image)

In Figure 4.6, each cell \(\tilde{C}_{T,e}\) shows a representative break divisor \(x + E\) where the point \(x \in e\) is marked with an extra outline. Edges of \(\tilde{C}_{T,e}\) which have \(x\) on an endpoint of \(e\) are marked in bold. Edges on the boundary are glued to the parallel boundary edge which has the same weighting (bold or unbold).
4.3.2 Point-set topology

**Definition 4.15.** Let $M$ and $N$ be compact Hausdorff spaces, and let $N$ be path-connected. We say $p : M \to N$ is a branched covering map if

(i) $p$ is continuous and surjective

(ii) $p$ is an open map (the image of an open set is open)

(iii) $p^{-1}(y)$ is finite for each $y \in N$

and there exists a closed subset $R \subset N$ such that

(iv) $N \setminus R$ is path-connected

(v) $R$ has empty interior in $N$

(vi) the restriction of $p$ to $M \setminus p^{-1}(R) \to N \setminus R$ is a topological covering map.

The subspace $R$ is a ramification locus of $p$, and the preimage $p^{-1}(R)$ is a branch locus. (Note that properties (ii) and (v) imply $p^{-1}(R)$ has empty interior in $M$.)

It is straightforward to verify that the map $\widetilde{\text{Br}}^g(\Gamma) \to \text{Br}^g(\Gamma)$ from Section 4.3.1 is a branched covering. We show below, in Proposition 4.19, that in fact each $\sigma_m : \widetilde{\text{Br}}^g(\Gamma) \to \text{Pic}^{m+g-1}(\Gamma)$, for $m \geq 1$, is a branched covering.

Recall that a map is proper if the preimage of a compact set is compact. Recall that a map $f : X \to Y$ is a local homeomorphism if, for any $x \in X$ there is an open neighborhood $U$ containing $x$ such that $f(U)$ is open in $Y$ and the restriction $U \to f(U)$ is a homeomorphism. A covering map is always a local homomorphism, but the converse is not true.

The following lemma will be used to check the last condition (vi) in Definition 4.15, that the restriction $M \setminus p^{-1}(R) \to N \setminus R$ is a covering map.

**Lemma 4.16.** Suppose $p : X \to Y$ is a local homeomorphism between locally compact, Hausdorff spaces. If $p$ is proper and surjective, then $p$ is a covering map.

This is a standard exercise in point-set topology; see e.g. [20, Lemma 2].

**Lemma 4.17.** Suppose $p : M \to N$ is a branched covering with ramification locus $R \subset N$ such that the restriction $p : M \setminus p^{-1}(R) \to N \setminus R$ is a covering map of degree $d$. Then for any $y \in N$, the preimage $p^{-1}(y)$ has cardinality at most $d$.

Note: the restriction of $p$ to $M \setminus p^{-1}(R) \to N \setminus R$ has constant degree $d$ because in the definition of branched cover, $N \setminus R$ is assumed to be path connected.

**Proof.** Let $y \in R$ be a point in the ramification locus, and let $x_1, \ldots, x_k$ be the points in the preimage $p^{-1}(y)$. Since $M$ is Hausdorff, we may choose open neighborhoods $U_1, \ldots, U_k$ with $x_i \in U_i$ which are disjoint, $U_i \cap U_j = \emptyset$. Let $C = M \setminus (U_1 \cup \cdots \cup U_k)$ be
the complement of these neighborhoods, which is closed in $M$. Since $M$ is compact and $N$ is Hausdorff, the image $p(C)$ is closed in $N$. Thus $V = N \setminus p(C)$ is open and nonempty since $y \in V$. Note that by construction $p^{-1}(V) = M \setminus p^{-1}(p(C)) \subset M \setminus C = U_1 \cup \cdots \cup U_k$.

Let $U_i'$ be the intersection of $p^{-1}(V)$ with $U_i$, which is open and nonempty because $x_i \in U_i'$. Since the $U_i$ were chosen to be disjoint, $p^{-1}(V) = U_1' \cup \cdots \cup U_k'$.

Note that $p$ is an open map (by definition of branched cover), so the intersection $p(U_1') \cap \cdots \cap p(U_k')$ is an open neighborhood of $y$ in $N$. Since $R$ has empty interior in $N$, we can choose some point $z \in (p(U_1') \cap \cdots \cap p(U_k')) \setminus R \subset V \setminus R$.

By the assumption that $M \setminus p^{-1}(R) \to N \setminus R$ is a degree $d$ covering map, the preimage $p^{-1}(z)$ contains $d$ points $w_1, \ldots, w_d$. Since $z \in V$ by construction, each $w_i \in p^{-1}(V) = U_1' \cup \cdots \cup U_k'$ so $w_i$ lies within $U_{j_i}'$ for some unique $j_i \in \{1, \ldots, k\}$. This relation defines a map $\pi : \{1, \ldots, d\} \to \{1, \ldots, k\}$. Moreover, the map $\pi$ is surjective because $z \in p(U_j')$ for each $j \in \{1, \ldots, k\}$. This proves that $k \leq d$, so the preimage $p^{-1}(y)$ has cardinality at most $d$ as desired. \hfill\qed

4.3.3 Proofs

**Proposition 4.18.** For any divisor $D$, the stable Weierstrass locus $W^{st}(D)$ is a finite subset of $\Gamma$.

**Proof.** If $D$ has degree $n < g$, the stable Weierstrass locus is defined to be empty. Thus we assume below that $D$ has degree $n \geq g$.

Recall that $\widetilde{\text{Br}}^g(\Gamma) = \{(x, E) \in \Gamma \times \text{Sym}^{g-1}(\Gamma) : x + E \text{ is a break divisor}\}$ and that $\sigma_m : \widetilde{\text{Br}}^g(\Gamma) \to \text{Pic}^m(\Gamma)$ is defined by

$$\sigma_m : (x, E) \mapsto [mx + E].$$

Recall that $\pi_1$ denotes the projection $\pi_1(x, E) = x$. (See Section 4.3.1.) By Lemma 4.13, for a divisor $D$ of degree $m + g - 1$ we have $W^{st}(D) = \pi_1(\sigma_m^{-1}[D])$. Hence it suffices to show that the preimage $\sigma_m^{-1}[D]$ is a finite set.

Let $(G, \ell)$ be a combinatorial model for $\Gamma$, which induces the ABKS decomposition $\text{Br}^g(\Gamma) = \bigcup_{T \in \mathcal{T}(G)} C_T$, where the cells $C_T$ are indexed by spanning trees of $G$. The ABKS decomposition of $\text{Br}^g(\Gamma)$ induces a decomposition

$$\widetilde{\text{Br}}^g(\Gamma) = \bigcup_{T \in \mathcal{T}(G)} \left( \bigcup_{e \in E(T)} \widetilde{C}_{T,e} \right).$$

Let $\sigma_m^{(T,e)} : \widetilde{C}_{T,e} \to \text{Pic}^m(\Gamma)$ denote the restriction of $\sigma_m$ to $\widetilde{C}_{T,e}$.
Claim: The preimage of \([D]\) under \(\sigma_m^{T,e} : \tilde{C}_{T,e} \to \text{Pic}^{m+g-1}(\Gamma)\) is finite. This Claim implies that the preimage \(\sigma_m^{-1}[D]\) is a finite set, since \(\tilde{\text{Br}}^g(\Gamma)\) is covered by finitely many \(\tilde{C}_{T,e}\).

Proof of Claim: The map \(\sigma_m^{T,e} : \tilde{C}_{T,e} \to \text{Pic}^{m+g-1}(\Gamma)\) is locally defined by a linear map, which we show is full rank. For a spanning tree \(T = G \setminus \{e, e_2, \ldots, e_g\}\), there is a natural surjective parametrization \(\prod_{i=1}^g [0, \ell(e_i)] \to \tilde{C}_{T,e}\).

Let \(f_m^{T,e}\) denote the lift of \(\prod_{i=1}^g [0, \ell(e_i)] \to \tilde{C}_{T,e} \to \text{Pic}^{m+g-1}(\Gamma)\) to the universal cover \(\mathbb{R}^g \to \text{Pic}^{m+g-1}(\Gamma)\).

\[
\begin{array}{ccc}
\prod_{i=1}^g [0, \ell(e_i)] & \xrightarrow{f_m^{T,e}} & \mathbb{R}^g \\
\downarrow & & \downarrow \pi \\
\tilde{C}_{T,e} & \xrightarrow{\sigma_m^{T,e}} & \text{Pic}^{m+g-1}(\Gamma)
\end{array}
\]

When \(m = 1\), coordinates may be chosen on \(\mathbb{R}^g\) such that \(f_1^{T,e}\) is represented by the identity matrix. Using these same coordinates on \(\mathbb{R}^g\) (up to a translation from \(\text{Pic}^g\) to \(\text{Pic}^{m+g-1}\)), for \(m \geq 1\) the definition \(\sigma_m(x, E) = [mx + E]\) implies that \(f_m^{T,e}\) is represented by the diagonal matrix

\[
\begin{pmatrix}
m & 1 & \cdots & 1
\end{pmatrix}
\]

This shows that \(f_m^{T,e}\) is locally injective, which implies \(\sigma_m^{T,e}\) is locally injective as well. Thus for any \([D] \in \text{Pic}^{m+g-1}(\Gamma)\), the preimage under \(\sigma_m^{T,e}\) is a discrete subset of \(\tilde{C}_{T,e}\). Since \(\tilde{C}_{T,e}\) is compact, the preimage of \([D]\) is finite as claimed. \(\square\)

In the following proposition, “generic” means the statement holds for \([D] \in \text{Pic}^n(\Gamma)\) outside of a nowhere dense exceptional set.

**Proposition 4.19.** For any divisor class \([D]\) of degree \(n \geq g\), we have

\[\#W^{st}(D) \leq g(n - g + 1)\]

For a generic divisor class \([D]\) of degree \(n \geq g\), the stable Weierstrass locus \(W^{st}(D)\) has cardinality \(\#W^{st}(D) = g(n - g + 1)\).

Proof. Let \(\tilde{\text{Br}}^g(\Gamma), \sigma_m : \tilde{\text{Br}}^g(\Gamma) \to \text{Pic}^{m+g-1}(\Gamma)\), and \(\pi_1 : \tilde{\text{Br}}^g(\Gamma) \to \Gamma\) be defined as in Section 4.3.1. Recall that for a divisor \(D\) of degree \(m + g - 1\), we have \(\#W^{st}(D) = \#(\sigma_m^{-1}[D])\) by Lemma 4.13. Thus it suffices to show that \(\sigma_m : \tilde{\text{Br}}^g(\Gamma) \to \text{Pic}^{m+g-1}(\Gamma)\) is a branched covering map of degree \(gm\), for any \(m \geq 1\). From this, Lemma 4.17...
implies the inequality $\#W^s(D) \leq gm$ and Definition 4.15 implies that equality holds for $[D]$ outside of the ramification locus.

(If $D$ has degree $n = m + g - 1$, then $gm = g(n - g + 1)$.)

Claim 1: The map $\sigma_m : \widetilde{Br}^g(\Gamma) \to \text{Pic}^{m+g-1}(\Gamma)$ is open, for any $m \geq 1$.

Proof of Claim 1: As above, let $(G, \ell)$ be a combinatorial model for $\Gamma$, and

$$\widetilde{Br}^g(\Gamma) = \bigcup_{T \in \mathcal{T}(G)} \bigcup_{e \notin E(T)} \widetilde{C}_{T,e}$$

the induced ABKS decomposition. (See Section 4.3.1.) The map $\sigma_m$ is naturally a piecewise affine map with domains of linearity $\widetilde{C}_{T,e}$.

To show that $\sigma_m$ is open, it suffices to check that for any $(x_0, E_0) \in \widetilde{Br}^g(\Gamma)$, the image of a neighborhood contains points in all tangent directions around $\sigma_m(x_0, E_0) \in \text{Pic}^{m+g-1}(\Gamma)$. To check this, we observe how $\sigma_m$ restricts to each domain of linearity $\widetilde{C}_{T,e}$ containing $(x_0, E_0)$. We will show that the behavior of $\sigma_m$ on tangent directions does not depend on the integer $m$.

For a point $(x_0, E_0)$ in $\widetilde{C}_{T,e}$, let $\text{cone}(\sigma_{m,T,e}(x_0, E_0))$ denote the positive cone in $\mathbb{R}^g$ spanned by

$$\sigma_m(x, E) - \sigma_m(x_0, E_0) \quad \text{for } (x, E) \text{ in a neighborhood of } (x_0, E_0) \text{ in } \widetilde{C}_{T,e}.$$ 

(Here we identify $\mathbb{R}^g$ with the tangent space of $\text{Pic}^0(\Gamma)$ at the identity.) Since $\sigma_m$ is affine on $\widetilde{C}_{T,e}$, this cone does not depend on the neighborhood chosen. Since $m \geq 1$, the positive span of

$$\sigma_m(x, E) - \sigma_m(x_0, E_0) = m[x - x_0] + [E - E_0] \quad \text{for } (x, E) \text{ in } \widetilde{C}_{T,e}$$

is equal to the positive span of

$$\sigma_1(x + E) - \sigma_1(x_0 + E_0) = [x - x_0] + [E - E_0] \quad \text{for } (x, E) \text{ in } \widetilde{C}_{T,e},$$

so $\text{cone}(\sigma_{m,T,e}(x_0, E_0)) = \text{cone}(\sigma_{1,T,e}(x_0, E_0))$. This holds for all cells $\widetilde{C}_{(T,e)}$ containing $(x_0, E_0)$.

Hence to show that $\sigma_m$ is open, it suffices to show that $\sigma_1 : \widetilde{Br}^g(\Gamma) \to \text{Pic}^g(\Gamma)$ is open. This is clear from the construction of $\widetilde{Br}^g(\Gamma)$ as a branched cover $\widetilde{Br}^g(\Gamma) \to \text{Br}^g(\Gamma)$, and from Theorem 2.8 which states that $\text{Br}^g(\Gamma) \to \text{Pic}^g(\Gamma)$ is a homeomorphism.

Claim 2: The map $\sigma_m : \widetilde{Br}^g(\Gamma) \to \text{Pic}^{m+g-1}(\Gamma)$ is a branched cover, for any $m \geq 1$.

Proof of Claim 2: In the definition of branched cover, Definition 4.15, condition (ii) was verified by Claim 1 and condition (iii) was verified by Proposition 4.18. Condition (i) is clear.\(^1\)

\(^1\) The map $\sigma_m$ is surjective because it is an open map from a compact space to a connected, Hausdorff space.
We first identify a ramification locus $R$ for $\sigma_m$, and then apply Lemma 4.16 to show that the restriction of $\sigma_m$ away from $R$ is a covering map.

Let $\Br^g(\Gamma) = \bigcup_{T \in T(G)} C_T$ be the ABKS decomposition induced by a combinatorial model $\Gamma = (G, \ell)$ (see Section 2.5). Let $Z^{(2)} \subset \Br^g(\Gamma)$ denote the union of faces of $C_T$ of codimension at least 2, and let $U^{(2)} = \Br^g(\Gamma) \setminus Z^{(2)}$. In other words,

$$U^{(2)} = \bigcup_{T \in T(G)} \{\text{interior } C^\circ_T \text{ of } C_T\} \cup \{\text{interiors of facets of } \partial C_T\}.$$ 

More concretely in terms of break divisors, given a set of edges $e_1, \ldots, e_g$ in $G$ whose complement is a spanning tree, $U^{(2)}$ contains break divisors which are a sum of $g$ points taken from the interior of each $e_1, e_2, \ldots, e_g$, and divisors which are a sum of one endpoint of $e_1$ and a point in the interior of each $e_2, \ldots, e_g$. We assume our combinatorial model $(G, \ell)$ is chosen to have no loops, so that each cell $C_T$ in the ABKS decomposition has $2g$ distinct boundary facets.

Note that for a break divisor $E$,

$$\sigma_m(R) = R = \sigma_m(\bar{Z}^{(2)})$$

(4.3) if $E \in U^{(2)}$, the support of $E$ consists of $g$ distinct points.

We let $\bar{Z}^{(2)}$ and $\bar{U}^{(2)}$ denote the preimages of $Z^{(2)}$ and $U^{(2)}$ under $\sigma: \bar{\Br}^g(\Gamma) \to \Br^g(\Gamma)$. Note that with respect to the ABKS decomposition

$$\bar{\Br}^g(\Gamma) = \bigcup_{T \in T(G)} \bigcup_{e \not\in E(T)} \bar{C}_{T,e},$$

$\bar{Z}^{(2)}$ is the union of codimension 2 faces of $\bar{C}_{T,e}$, and $\bar{U}^{(2)} = \bar{\Br}^g(\Gamma) \setminus \bar{Z}^{(2)}$. Thus $\bar{Z}^{(2)}$ is a closed subset of codimension 2 and $\bar{U}^{(2)}$ is a dense open subset of $\bar{\Br}^g(\Gamma)$.

Next, let $R = R_m = \sigma_m(\bar{Z}^{(2)})$. We will show that $R$ is a valid ramification locus for the branched cover $\sigma_m$. The conditions (iv) and (v) hold because $R$ is a codimension 2 submanifold of the connected manifold $\Pic^{m+g-1}(\Gamma)$. It remains to check condition (vi), that the restriction

$$\sigma_m|_{\bar{\Br}^g(\Gamma) \setminus \sigma_m^{-1}(R)}: \bar{\Br}^g(\Gamma) \setminus \sigma_m^{-1}(R) \to \Pic^{m+g-1}(\Gamma) \setminus R$$

(4.4) away from ramification is a covering map. To check this condition, we apply Lemma 4.16. It is clear that the domain and codomain of (4.4) are locally compact Hausdorff spaces. The map in (4.4) is surjective by construction; it is proper because $\sigma_m$ is a map from a compact space to a Hausdorff space, hence proper. It remains to check

\[\frac{2}{2}\] The domain is locally compact and Hausdorff because it is an open subspace of $\bar{\Br}^g(\Gamma)$ which is a finite CW complex, hence compact and Hausdorff. The same holds for the codomain, as an open subspace of $\Pic^{m+g-1}(\Gamma) \cong \mathbb{R}^g/\mathbb{Z}^g$. 

43
that (4.4) is a local homeomorphism, which we leave for the next claim. Note that the domain of (4.4) is contained in \( \widetilde{U}^{(2)} \):

\[
\widetilde{\text{Br}}^g(\Gamma) \setminus \sigma_m^{-1}(R) = \widetilde{\text{Br}}^g(\Gamma) \setminus \sigma_m^{-1}(\sigma_m(\widetilde{Z}^{(2)})) \subset \widetilde{\text{Br}}^g(\Gamma) \setminus Z^{(2)} = \widetilde{U}^{(2)}.
\]

Assuming Claim 3, Lemma 4.16 implies that \( \sigma_m \) is a covering map away from the ramification locus \( R \), which completes the proof of Claim 2.

Claim 3: The restriction of \( \sigma_m \) to \( \widetilde{U}^{(2)} \to \text{Pic}^{m+g-1}(\Gamma) \) is a local homeomorphism, for any \( m \geq 1 \).

Proof of Claim 3: First consider \( m = 1 \). Observation (4.3) implies that (4.5) the restriction \( \sigma_1|_{\widetilde{U}^{(2)}} : \widetilde{U}^{(2)} \to U^{(2)} \) is a (unbranched) covering of degree \( g \).

Since \( U^{(2)} \subset \text{Pic}^g(\Gamma) \) is open, it follows that \( \sigma_1 : \widetilde{U}^{(2)} \to \text{Pic}^g(\Gamma) \) is a local homeomorphism.

Recall that \( \widetilde{U}^{(2)} \) is the union of the interior of \( \widetilde{C}_{T,e} \) and the interiors of facets of \( \partial \widetilde{C}_{T,e} \), over all \( (T,e) \). In the interior of \( \widetilde{C}_{T,e} \), \( \sigma_m \) can be expressed as a full-rank linear map so it is a local homeomorphism. Now consider how \( \sigma_m \) acts near the interior of a facet of \( \partial \widetilde{C}_{T,e} \). We claim that each facet is shared by exactly two cells.

Suppose \( T = G \setminus \{ e = e_1, e_2, \ldots, e_g \} \). There are \( 2g \) facets of the boundary \( \partial \widetilde{C}_{T,e} \), indexed by choosing an edge \( e_j \) and choosing one of its two endpoints. For a fixed index \( j \) in \( \{1, \ldots, g\} \) and \( v(e_j) \) a fixed endpoint of \( e_j \), the corresponding facet of \( \partial \widetilde{C}_{T,e} \) consists of pairs \( (x, E) \in \widetilde{\text{Br}}^g(\Gamma) \) of the form

\[
(4.6) \quad \widetilde{F}^{(j,v)}_{(T,e)} = \{(x = x_1, E = x_2 + \cdots + x_g) : x_j = v(e_j), x_i \in e_i^c \text{ for } i = 1, \ldots, g, i \neq j\}
\]

Let \( G_j = T \cup e_j \). Since \( e_j \notin T \), the graph \( G_j \) contains a unique cycle, which must contain \( v(e_j) \in e_j \). Let \( e_j' \) be the unique edge \( \neq e_j \) in this cycle which also borders \( v(e_j) \), and let \( T' = G_j \setminus e_j' = (T \cup e_j) \setminus e_j' \). Then \( \widetilde{C}_{T',e'} \) is the only other cell containing the facet (4.6), where \( e' = e'_1 \) if \( j = 1 \), and \( e' = e \) otherwise. The facet (4.6) is then the relative interior of \( \widetilde{C}_{T,e} \cap \widetilde{C}_{T',e'} \).

As before, let \( f^{T,e}_m \) denote the lift of \( \widetilde{C}_{T,e} \to \text{Pic}^{m+g-1}(\Gamma) \) in the diagram

\[
\begin{array}{ccc}
\prod_{i=1}^g [0, \ell(e_i)] & \longrightarrow & \mathbb{R}^g \\
\downarrow & & \downarrow \pi \\
\widetilde{C}_{T,e} & \longrightarrow & \text{Pic}^{m+g-1}(\Gamma)
\end{array}
\]

and define \( f^{T',e'}_m \) analogously.
We may choose coordinates (depending on $T$) on $\mathbb{R}^g$ such that

\[
\text{the matrix representing } f^T_e \text{ is } \begin{pmatrix} m & 1 \\ & \ddots \\ & & 1 \end{pmatrix}.
\]

In these same coordinates, the matrix representing $f^T_{e'}$ is

\[
\begin{pmatrix} -m \\ * & 1 \\ * & \ddots \\ * & \ldots & 1 \end{pmatrix} \quad \text{if } j = 1, \quad \text{or} \quad \begin{pmatrix} m \\ \ddots \\ & * \\ & -1 \end{pmatrix} \quad \text{if } j \in \{2, \ldots, g\}.
\]

(Recall that $j$ is the index specifying which edge $e_j \in G \setminus T$ has a break divisor chip on one of its endpoints; $e_j$ is the unique edge in $T' \setminus T$.) This shows that $\sigma_m$ is a local homeomorphism in a neighborhood of the chosen facet of $\partial \tilde{C}_{T,e}$.

Claim 4: The branched cover $\sigma_m : \tilde{Br}^g(\Gamma) \to \text{Pic}^{m+g-1}(\Gamma)$ has degree $g_m$.

Proof of Claim 4: When $m = 1$, it is clear that $\sigma_1 : \tilde{Br}^g(\Gamma) \to \text{Pic}^g(\Gamma) \cong \text{Br}^g(\Gamma)$ is a degree $g$ branched cover. When $m > 1$, we note that $\sigma_m$ differs from $\sigma_1$ by a scaling factor of $m$, i.e. on a sufficiently small neighborhood $U \subset \tilde{Br}(\Gamma)$, the Haar measure of $\sigma_m(U)$ is $m$-times as large as the Haar measure of $\sigma_1(U)$. (The space $\text{Pic}^{m+g-1}(\Gamma)$ carries a Haar measure since it is a torsor for the compact topological group $\text{Pic}^0(\Gamma)$.) This implies that the degree of $\sigma_m$ as a branched cover must be $m$ times the degree of $\sigma_1$, so $\sigma_m$ must have degree $g_m$ as desired. \hfill \square

**Theorem 4.20.** Let $\Gamma$ be a compact, connected metric graph of genus $g$.

(a) For a generic divisor class of degree $n \geq g$, the Weierstrass locus $W(D)$ is finite with cardinality $\#W(D) = g(n - g + 1)$. For a generic divisor class of degree $n < g$, $W(D)$ is empty.

(b) For an arbitrary divisor class of degree $n \geq g$, the stable Weierstrass locus $W^{st}(D)$ is finite with cardinality

\[\#W^{st}(D) \leq g(n - g + 1),\]

and equality holds for a generic divisor class.

Proof. Part (b) is a restatement of Proposition 4.19.

For part (a), first suppose $n < g$. The space $\text{Pic}^n(\Gamma)$ has dimension $g$, while the subspace of effective divisor classes has dimension at most $n$. Thus a generic divisor
class in $\text{Pic}^n(\Gamma)$ is not effective, assuming $n < g$. By Remark 4.2, the Weierstrass locus is empty for a non-effective divisor class.

Now suppose $n \geq g$. To prove (a), it suffices to show that $W(D) = W_{\text{st}}(D)$ for a generic divisor class, since then part (b) applies. To compare $W(D)$ with $W_{\text{st}}(D)$, we construct a map $X \to \text{Pic}^n(\Gamma)$ whose fiber over $[D]$ is the Weierstrass locus $W(D)$; this parallels our construction in Section 4.3.1 for $W_{\text{st}}(D)$.

For $m \geq 1$, let $s_m : \Gamma \times \text{Sym}^{g-1}(\Gamma) \to \text{Pic}^{m+g-1}(\Gamma)$ denote the map

$$s_m(x, E) = [mx + E].$$

Let $\pi_1 : \Gamma \times \text{Sym}^{g-1}(\Gamma) \to \Gamma$ denote projection to the first factor.

The Riemann–Roch formula, Theorem 2.13, implies that a generic divisor class $[D] \in \text{Pic}^{m+g-1}(\Gamma)$ has rank $r(D) = (m + g - 1) - g = m - 1$. For such a divisor,

$$W(D) = \{x \in \Gamma : [D - mx] \geq 0\} = \pi_1(s_m^{-1}[D]).$$

Recall that $W_{\text{st}}(D) = \pi_1(\sigma_m^{-1}[D])$, where $\sigma_m$ is defined to be the restriction of $s_m$ to the subset $\widetilde{Br}(\Gamma) \subset \Gamma \times \text{Sym}^{g-1}(\Gamma)$; note that

$$(4.7) \quad \sigma_m^{-1}[D] = s_m^{-1}[D] \cap \widetilde{Br}(\Gamma) \subset s_m^{-1}[D].$$

Under the genericity assumption on $[D]$, we have

$$W_{\text{st}}(D) = \pi_1(\sigma_m^{-1}[D]) \subset \pi_1(s_m^{-1}[D]) = W(D).$$

Using part (b), this observation implies that a generic Weierstrass locus $W(D)$ contains at least $g(n - g + 1)$ points.

We consider when $W(D)$ can be strictly larger than $W_{\text{st}}(D)$. By (4.7), this happens only if $s_m^{-1}[D]$ is not contained in $\widetilde{Br}(\Gamma)$; equivalently, only if $[D]$ lies in the image of $(\Gamma \times \text{Sym}^{g-1}(\Gamma)) \setminus \widetilde{Br}(\Gamma)$ under $s_m$.

Claim: The image $s_m((\Gamma \times \text{Sym}^{g-1}(\Gamma)) \setminus \widetilde{Br}(\Gamma))$ has dimension $g - 1$ in $\text{Pic}^{m+g-1}(\Gamma)$.

It is clear that $s_m$ is piecewise affine on $\Gamma \times \text{Sym}^{g-1}(\Gamma)$, with domains of linearity indexed by $g$-tuples of edges $(e_1; e_2, \ldots, e_g)$, up to reordering the edges $e_2, \ldots, e_g$. (Here we choose an arbitrary combinatorial model $(G, \ell)$ for $\Gamma$.) The edges $e_i$ are not necessarily distinct.

If the edges $(e_1; e_2, \ldots, e_g)$ form the complement of a spanning tree $T$ in $G$, then the corresponding domain is in $\widetilde{Br}(\Gamma)$; namely, it is the cell $\tilde{C}_{T,e_1}$ in the notation of Section 4.3.1. Conversely, if the edges $(e_1; e_2, \ldots, e_g)$ are not the complement of a spanning tree in $G$, then either some edge is repeated or the edges contain a cut set of $G$. In either case, the fibers of $s_m : \Gamma \times \text{Sym}^{g-1}(\Gamma) \to \text{Pic}^{m+g-1}(\Gamma)$ have dimension at least 1 over the interior of the corresponding domain (see [19, Proposition 13]).
so the image of this domain under $s_m$ has dimension at most $g - 1$. This proves the claim.

The claim implies that for a generic divisor class $[D]$, the preimage $s_m^{-1}[D]$ is contained in $\widehat{Br}^g(\Gamma)$. By (4.7) this implies $W(D) = W^{st}(D)$, as desired. \hfill \square

4.4 Distribution of Weierstrass points

In this section we prove Theorem 4.24. We show that for a degree-increasing sequence of generic divisors on a metric graph, the Weierstrass points become distributed with respect to the Zhang canonical measure (defined in Section 3.4). We also give a quantitative version of this distribution result, Theorem ??.

Our proofs of Theorems 4.24 and ?? work unchanged when $W(D)$ is replaced by the stable Weierstrass locus $W^{st}(D)$.

4.4.1 Examples

First we consider some low genus examples of Weierstrass points converging to a limiting distribution.

**Example 4.21** (Genus 0 metric graph). Let $\Gamma$ be a genus 0 metric graph. For any divisor $D_n$, the associated Weierstrass locus $W(D_n)$ is empty so $\delta_n = 0$. All edges are bridges, so the canonical measure is $\mu = 0$.

**Example 4.22** (Genus 1 metric graph). Let $\Gamma$ be a genus 1 metric graph which consists of a loop of length $L$. For a divisor $D_n$ of degree $n$, the Weierstrass locus $W_n = W(D_n)$ consists of $n$ evenly-spaced points ("torsion points") around the loop. The distance between adjacent points is $L/n$, so on a segment $e$ of length $\ell(e)$ the number of Weierstrass points is bounded by

$$\frac{\ell(e)}{L/n} - 1 \leq \#(W_n \cap e) \leq \frac{\ell(e)}{L/n} + 1.$$  

This means the associated discrete measure $\delta_n = \frac{1}{n} \sum_{x \in W_n} \delta_x$ satisfies

$$\delta_n(e) = \frac{\#(W_n \cap e)}{n} \Rightarrow \frac{\ell(e)}{L} - \frac{1}{n} \leq \delta_n(e) \leq \frac{\ell(e)}{L} + \frac{1}{n}.$$  

Hence $\delta_n(e) \to \frac{\ell(e)}{L} = \mu(e)$ as $n \to \infty$.

4.4.2 Proofs

We now address the limiting distribution of Weierstrass points $W(D_n)$ as $n \to \infty$ in the case of an arbitrary metric graph $\Gamma$.

**Lemma 4.23.** Suppose the Weierstrass locus $W(D)$ is finite. Let $r = r(D)$.
(a) If $x$ is in the interior of a segment, $\text{red}_x[D]$ contains at most $r + 1$ chips at $x$.

(b) If $x$ is in the interior of a segment $e \subset \Gamma$, $\text{red}_x[D]$ contains at most $r + 1$ chips on $e$ (including its endpoints).

**Proof.** (a) Suppose $\text{red}_x[D]$ contains $r + 2$ chips at $x$. Then for sufficiently small $\epsilon$ we can move $r + 1$ of these chips together for a distance $\epsilon$ in one direction, while moving 1 chip a distance $(r + 1)\epsilon$ in the other. This gives a positive-length interval in $W(D)$, a contradiction.

(b) Suppose $\text{red}_x[D]$ contains $r + 2$ chips on the closed segment $e$. Note that at least $r$ of these chips must be at $x$, in the interior of $e$. By chip-firing, we may move all $r + 2$ chips to a single point $x'$ in the interior of $e$. Then part (a) applies. \qed

**Theorem 4.24.** Let $\{D_n : n \geq 1\}$ be a sequence of divisors on $\Gamma$ with $\deg D_n = n$. Let $W_n$ be the Weierstrass locus of $D_n$. Suppose each $W_n$ is a finite set, and let

$$\delta_n = \frac{1}{n} \sum_{x \in W_n} \delta_x$$

denote the normalized discrete measure on $\Gamma$ associated to $W_n$. Then as $n \to \infty$, the measures $\delta_n$ converge weakly to the Zhang canonical measure $\mu$ on $\Gamma$.

Recall that by definition of weak convergence, Theorem 4.24 says that for any continuous function $f : \Gamma \to \mathbb{R}$, as $n \to \infty$ we have convergence

$$\frac{1}{n} \sum_{x \in W_n} f(x) =: \int_{\Gamma} f(x) \delta_n(dx) \to \int_{\Gamma} f(x) \mu(dx).$$

**Proof of Theorem 4.24.** To show weak convergence of measures on $\Gamma$ it suffices to show convergence when integrated against step functions. Hence it suffices to integrate the measures against the indicator function of an arbitrary segment of $\Gamma$.

Let $e$ be a segment in the metric graph $\Gamma$ of length $\ell(e)$, with endpoints $s$ and $t$. Let $W_n \cap e$ denote the set of Weierstrass points of $D_n$ lying on the segment $e$. It suffices to show that

$$(4.8) \quad \lim_{n \to \infty} \frac{\#(W_n \cap e)}{n} = \mu(e).$$

Recall that by Proposition 3.25,

$$\mu(e) = \frac{\ell(e)}{\ell(e) + \ell_{\text{eff}}(\Gamma \setminus e)}$$

where $\ell_{\text{eff}}(\Gamma \setminus e)$ denotes the effective resistance between the endpoints of $e$ when the interior of $e$ is removed from $\Gamma$. (If $\Gamma \setminus e$ is disconnected, $\ell_{\text{eff}}(\Gamma \setminus e) = +\infty$ and
\( \mu(e) = 0. \) We prove (4.8) by relating each side to the slope of a piecewise linear function on \( \Gamma. \)

For the right-hand side of (4.8), consider the voltage function \( j^s_t : \Gamma \to \mathbb{R} \) (see Section 3.1). The voltage drop in \( \Gamma \) between endpoints of \( e \) is the effective resistance

\[
j^s_t(s) - j^s_t(t) = r(s, t) = \frac{\ell(e) \ell_{\text{eff}}(\Gamma \setminus e)}{\ell(e) + \ell_{\text{eff}}(\Gamma \setminus e)},
\]

by the parallel rule for effective resistance. Thus we have

\[
(4.9) \quad \frac{j^s_t(s) - j^s_t(t)}{\ell(e)} = \frac{\ell_{\text{eff}}(\Gamma \setminus e)}{\ell(e) + \ell_{\text{eff}}(\Gamma \setminus e)} = 1 - \frac{\ell(e)}{\ell(e) + \ell_{\text{eff}}(\Gamma \setminus e)} = 1 - \mu(e).
\]

(Recall that this slope can be interpreted as the current flowing along the segment \( e \) from \( s \) to \( t \), since current = voltage drop/resistance.)

To connect \( j^s_t \) to the left-hand side of (4.8), we consider a sequence of piecewise-linear functions which are “discrete approximations” of \( j^s_t \), and show that certain slopes in these functions are related to the number of Weierstrass points.

Let \( f_n \) be the piecewise \( \mathbb{Z} \)-linear function on \( \Gamma \) satisfying

\[
\Delta(f_n) = \text{red}_t[D_n] - \text{red}_s[D_n] \quad \text{and} \quad f_n(t) = 0.
\]

(Recall that \( \text{red}_x[D] \) denotes the \( x \)-reduced divisor linearly equivalent to \( D \).) By Proposition 3.7, as \( n \to \infty \) we have uniform convergence

\[
(4.10) \quad \frac{1}{n} f_n \to j^s_t.
\]

Thus to show (4.8) using (4.9) and (4.10), it suffices to show that

\[
(4.11) \quad \lim_{n \to \infty} \frac{1}{n} \left( \frac{f_n(s) - f_n(t)}{\ell(e)} \right) = 1 - \lim_{n \to \infty} \frac{\#(W_n \cap e)}{n}.
\]

We first give an intuitive explanation for (4.11): the slope of the function \( f_n \) on a directed segment is equal to the net flow of chips across the segment, as we move from \( \text{red}_s[D_n] \) to \( \text{red}_t[D_n] \) along any path in the linear system \( |D_n| \). If we follow \( \text{red}_x[D_n] \) as \( x \) varies from \( s \) to \( t \), we have \( n - g \) chips moving in the “forward” direction of \( e \) (following \( x \)) and some number of chips moving in the reverse direction one-by-one. The number of “reverse-moving” chips is equal to \( \#(W_n \cap e) \), since \( x \) is in \( W_n \) exactly when \( \text{red}_x[D_n] \) has an “extra” chip at \( x \), i.e. when the \( n - g \) chips on \( x \) collide with a reverse-moving chip. Thus the net number of chips moving across the segment \( e \) is equal to \( (n - g) - \#(W_n \cap e) \), up to some bounded error due to boundary behavior. This yields (4.11) after dividing by \( n \) and taking \( n \to \infty \).
Now we give a rigorous argument. Let \( w_1, w_2, \ldots, w_m \) denote the Weierstrass points on \( e \), ordered from \( s \) to \( t \), so that \( m = \#(W_n \cap e) \). Here we use the hypothesis that \( W_n \) is finite. (Note that \( m = m_n \) depends on \( n \).)

We partition the segment \( e = [s, t] \) into subintervals \( [s, w_1], [w_1, w_2], \ldots, [w_m, t] \). (It is possible that the intervals \( [s, w_1] \) and \( [w_m, t] \) are degenerate.) Let \( \ell([w_i, w_{i+1}]) \) denote the length of the segment \( [w_i, w_{i+1}] \subset e \). We have

\[
\ell(e) = \ell([s, w_1]) + \ell([w_1, w_2]) + \cdots + \ell([w_{m-1}, w_m]) + \ell([w_m, t]).
\]

For each \( i = 1, 2, \ldots, m - 1 \), let \( g_n^{(i)} \) denote the function in \( \text{PL}_{\mathbb{Z}}(\Gamma) \) satisfying

\[
\Delta(g_n^{(i)}) = \text{red}_{w_{i+1}}[D_n] - \text{red}_{w_i}[D_n],
\]

and let \( g_n^{(0)} \) and \( g_n^{(m)} \) denote functions satisfying

\[
\Delta(g_n^{(0)}) = \text{red}_{w_1}[D_n] - \text{red}_{s}[D_n], \quad \text{and} \quad \Delta(g_n^{(m)}) = \text{red}_{t}[D_n] - \text{red}_{w_m}[D_n].
\]

By adding an appropriate constant, we may assume that \( g_n^{(i)}(t) = 0 \) for each \( i = 0, 1, \ldots, m \). By telescoping of poles and zeros, we have

\[
\Delta(f_n) = \Delta(g_n^{(0)}) + \Delta(g_n^{(1)}) + \cdots + \Delta(g_n^{(m)}).
\]

With the additional constraint that \( f_n(t) = \sum_i g_n^{(i)}(t) = 0 \), this implies that

\[
(4.12) \quad f_n = g_n^{(0)} + g_n^{(1)} + \cdots + g_n^{(m)}.
\]

Thus we can compute \( f_n(s) - f_n(t) \) by summing \( \sum_{i=0}^{m} (g_n^{(i)}(s) - g_n^{(i)}(t)) \).

To analyze the slopes of \( g_n^{(i)} \) on segment \( e \), we make use of Lemma 4.23. This information is sufficient to deduce all slopes over \( e \). We may assume without loss of generality that \( r(D_n) = n - g \), since this holds for \( n \geq 2g - 1 \).

For \( i = 1, 2, \ldots, m - 1 \), the function \( g_n^{(i)} \) has slope \(-(n-g)\) on the interval \([w_i, w_{i+1}]\), and slope 1 on \( e \) outside of this interval. See Figure 4.7.

![Figure 4.7](image)

Function \( g_n^{(i)} \) having zeros \( \text{red}_{w_{i+1}}[D_n] \) and poles \( \text{red}_{w_i}[D_n] \), with slopes are indicated above each affine part.
Thus we have
\[ g_n^{(i)}(s) - g_n^{(i)}(t) = (n - g)\ell([w_i, w_{i+1}]) - \ell([s, w_i]) - \ell([w_{i+1}, t]) \]
\[ (4.13) \]
\[ = (n - g + 1)\ell([w_i, w_{i+1}]) - \ell(e). \]

For \( i = 0 \) and \( i = m \), to write an expression for \( g_n^{(i)}(s) - g_n^{(i)}(t) \) we need to set additional notation. If \( \text{red}_s[D_n] \) has a chip in the interior of \( e \), let \( y \) be the position of this chip (which is unique by Lemma 4.23); otherwise, let \( y = t \). Similarly, let \( z \) be the position of the unique chip of \( \text{red}_t[D_n] \) in the interior of \( e \) if it exists; otherwise let \( z = s \). We have
\[ g_n^{(0)}(s) - g_n^{(0)}(t) = (n - g)\ell([s, w_1]) - \ell([w_1, y]) \]
\[ (4.14) \]
\[ = (n - g + 1)\ell([s, w_1]) - \ell([s, y]) \]
and
\[ g_n^{(m)}(s) - g_n^{(m)}(t) = (n - g)\ell([w_m, t]) - \ell([z, w_m]) \]
\[ (4.14') \]
\[ = (n - g + 1)\ell([w_m, t]) - \ell([z, t]) \]

![Figure 4.8: Function \( g_n^{(0)} \) having zeros red\( w_1[D_n] \) and poles red\( w[D_n] \).](image)

Thus adding the expressions (4.13) and (4.14) together, by (4.12) we have
\[ f_n(s) - f_n(t) = (n - g + 1)(\ell([s, w_1]) + \ell([w_1, w_2]) + \cdots + \ell([w_{m-1}, w_m]) + \ell([w_m, t])) \]
\[ - \ell([s, y]) - (m - 1)\ell(e) - \ell([z, t]) \]
\[ = (n - g + 1)\ell(e) - (m - 1)\ell(e) - \ell([s, y]) - \ell([z, t]) \]
\[ = (n - g - m + 2)\ell(e) - \ell([s, y]) - \ell([z, t]) \]
\[ = (n - g - m)\ell(e) + (\ell(e) - \ell([s, y]) + (\ell(e) - \ell([z, t])) \]
\[ = (n - g - m)\ell(e) + \ell([y, t]) + \ell([s, z]). \]

Since \( 0 \leq \ell([y, t]) + \ell([s, z]) \leq 2\ell(e) \) and \( m = \#(W_n \cap e) \), this shows that
\[ n - g - \#(W_n \cap e) \leq \frac{f_n(s) - f_n(t)}{\ell(e)} \leq n - g + 2 - \#(W_n \cap e). \]

Dividing by \( n \) and taking the limit \( n \to \infty \) yields (4.11) as desired. \( \square \)

(a) Suppose each $[D_n]$ is generic in $\text{Pic}^n(\Gamma)$. Then each $W_n$ is finite and we have weak convergence $\delta_n \to \mu$.

(b) Let $W^*_n = W^*(D_n)$ be the stable Weierstrass locus, and define $\delta^*_n$ analogously to $\delta_n$. For any divisors $\{D_n : n \geq 1\}$ we have weak convergence $\delta^*_n \to \mu$.

Proof. (a) This is part of Theorem 1.6.

(b) We may follow the same argument used in Theorem 4.24, except in place of $\text{red}_x[D_n]$ we consider the “stable reduced divisor”

$$\text{red}^*_x[D_n] := (n - g)x + \text{br}[D_n - (n - g)x].$$

With this change in the definitions of $f_n$ and $g_n^{(i)}$, equations (4.13) and (4.14) still hold, as does the convergence (4.10). \qed

Theorem 4.26 (Quantitative distribution of $W(D)$). Let $\Gamma$ be a metric graph of genus $g$, let $D_n$ be a divisor class of degree $n > g$ and let $W_n$ denote the Weierstrass locus of $D_n$. Suppose $W_n$ is finite. Let $\mu$ denote the Zhang canonical measure on $\Gamma$.

(a) For any segment $e$ in $\Gamma$,

$$n\mu(e) - 2g \leq \#(W_n \cap e) \leq n\mu(e) + g + 2.$$

(b) If $e$ is a segment of $\Gamma$ with $\mu(e) > \frac{2g}{n}$, then $e$ contains at least one Weierstrass point of $D_n$.

(c) For a fixed continuous function $f : \Gamma \to \mathbb{R}$,

$$\frac{1}{n} \sum_{x \in W_n} f(x) = \int_{\Gamma} f(x)\mu(dx) + O\left(\frac{1}{n}\right).$$

Proof. It is clear that part (b) follows from part (a), since $\#(W_n \cap e)$ must be an integer. Part (c) is a straightforward extension of (a).

We now prove part (a). Let $f_n$ be the piecewise linear function satisfying $\Delta(f_n) = \text{red}^*_t[D_n] - \text{red}^*_s[D_n]$ and $f_n(t) = 0$, where $s$ and $t$ are the endpoints of $e$. By Proposition 3.8, we have

$$|(f_n - (n - g)j^*_t)'(x)| \leq g$$

so

$$|f_n'(x)| \leq (n - g)|j'(x)| + g.$$

Recall that for $x$ on the segment $e$, $|j'(x)| = 1 - \mu(e)$. Thus we have the bound

$$|f_n'(x)| \leq n - n\mu(e) + \mu(e)g.$$
Moreover the proof of Theorem 4.24 shows that
\[ n - g - \#(W_n \cap e) \leq |f'_n(x)|. \]
Combining these inequalities gives
\[ n \mu(e) - (1 + \mu(e))g \leq \#(W_n \cap e). \]
Finally, the inequality \( \mu(e) \leq 1 \) from Corollary 3.26 yields the lower bound in (a).
We similarly obtain the upper bound
\[ \#(W_n \cap e) \leq n \mu(e) + g + 2 \]
by combining the inequalities
\[ n - n \mu(e) - (2 - \mu(e))g \leq |f'_n(x)| \quad \text{and} \quad |f'_n(x)| \leq n - g - \#(W_n \cap e) + 2 \]
and \( \mu(e) \geq 0 \) from Corollary 3.26.

\section{4.5 Tropicalizing Weierstrass points}

In this section, we describe how the Weierstrass locus for a tropical curve can be related to the Weierstrass locus for an algebraic curve. The key result is Baker’s Specialization Lemma [5, Lemma 2.8]; here we use a more general version given by Jensen–Payne [21] in the language of Berkovich analytic spaces. The results of this section are not needed for any later sections of the paper.

Throughout this section, let \( K \) denote an algebraically closed field equipped with a nontrivial non-Archimedean valuation \( v : K^\times \to \mathbb{R} \); we assume \( K \) is complete with respect to \( v \).

\textbf{Theorem 4.27} (Specialization Lemma [21, Lemma 2.4]). Suppose \( X \) is a smooth projective algebraic curve over \( K \). Let \( \Gamma \) be a skeleton on the Berkovich analytification \( X^\text{an} \), let \( \rho : X^\text{an} \to \Gamma \) be the retration to the skeleton and let \( \rho_* : \text{Div}(X) \to \text{Div}(\Gamma) \) denote the induced map on divisors. Then for any divisor \( D \in \text{Div}(X) \),
\[ r_X(D) \leq r_\Gamma(\rho_*(D)). \]

Here \( r_X \) denotes the dimension of a complete linear system \( |D| \) on \( X \), and \( r_\Gamma \) denotes the Baker–Norine rank on \( \Gamma \) (see Section 2.6).

\textbf{Theorem 4.28.} Consider the setup of Theorem 4.27. For any divisor \( D \in \text{Div}(X) \) such that \( \rho_*(D) \in \text{Div}(\Gamma) \) is Riemann–Roch nonspecial, we have
\[ \rho_*(W_X(D)) \subseteq W_\Gamma(\rho_*(D)). \]
Proof. The map $\rho_*$ respects degree; let $n = \deg(D) = \deg(\rho_*(D))$. Recall that $\rho_*(D)$ is nonspecial means that
\[ r_\Gamma(\rho_*(D)) = \max\{n - g, -1\}. \]
In this case, Theorem 4.27 implies $r_X(D) \leq \max\{n - g, -1\}$ while Riemann–Roch implies $r_X(D) \geq \max\{n - g, -1\}$ for any divisor. Thus $r_X(D) = r_\Gamma(\rho_*(D))$.

Let $r$ denote the rank in either sense. If $x \in W_X(D)$, we have
\[ r_X(D - (r + 1)x) \geq 0. \]
By Theorem 4.27 and linearity of $\rho_*$, this implies
\[ r_\Gamma(\rho_*(D - (r + 1)x)) = r_\Gamma(\rho_*(D) - (r + 1)\rho_*(x)) \geq 0. \]
This means $\rho_*(x) \in W_\Gamma(\rho_*(D))$ as claimed. \hfill \Box

The conclusion of Theorem 4.28 also holds for $D = K_X$ the canonical divisor, and $\rho_*(K_X) \sim K_\Gamma$. This was observed by Baker in [5, Corollary 4.9].
CHAPTER 5

Torsion Points of the Jacobian

In this section we study torsion points in the Jacobian of a tropical curve.

5.1 The classical Manin–Mumford conjecture

Given an algebraic curve $X$ and choice of basepoint $x_0$, we say that $x \in X$ is a torsion point if the divisor $n(x - x_0)$ is linearly equivalent to 0 for some positive integer $n$. Equivalently, $x$ is a torsion point if the Abel–Jacobi embedding (with respect to $x_0$) sends $x$ to the torsion subgroup of the Jacobian. The Jacobian of a genus $g$ smooth algebraic curve over $\mathbb{C}$ is a compact abelian group, isomorphic to $\mathbb{C}^g/\mathbb{Z}^g \cong H^1(X, \mathbb{C})/H_1(X, \mathbb{Z})^\vee$.

Faltings's theorem (previously known as Mordell’s conjecture) states that a smooth curve of genus $g \geq 2$ has finitely many rational points, i.e. points whose coordinates are all rational numbers.

By analogy with Mordell’s conjecture, Manin and Mumford conjectured that an algebraic curve of genus 2 or more has finitely many torsion points. The Manin–Mumford Conjecture was proved by Michel Raynaud [31], which inspired several generalizations concerning torsion points in abelian varieties.

For a metric graph $\Gamma$, there is an analogous Jacobian which encapsulates all possible sums of points, generalizing the addition of angles on a circle. For a metric graph, the Jacobian is isomorphic to $\mathbb{R}^g/\mathbb{Z}^g \cong H^1(\Gamma, \mathbb{R})/H_1(\Gamma, \mathbb{Z})^\vee$.

5.2 The Manin–Mumford conjecture for tropical curves

In light of the Manin–Mumford conjecture (Raynaud’s theorem [31]) on torsion points of an algebraic curve of genus $g \geq 2$, we may ask the same question for a tropical curve.

We say points $x, y$ on a metric graph $\Gamma$ are torsion equivalent if $n[x - y] = 0$ in $\text{Jac}(\Gamma)$ for some positive integer $n$. Given a point $x$ on a metric graph $\Gamma$, the torsion
packet of $x$ is the set of points $y$ which are torsion equivalent to $x$.

**Definition 5.1.** We say a metric graph $\Gamma$ satisfies the *Manin–Mumford condition* if every torsion packet of $\Gamma$ has finite image in $\text{Jac}(\Gamma)$ under the Abel–Jacobi map. (The Abel–Jacobi map $\Gamma \to \text{Jac}(\Gamma)$ is defined in Section 2.3).

Suppose $\Gamma = (G, \ell)$ is a metric graph of genus $g \geq 2$ whose edge lengths are all integers, i.e. $\ell(e) \in \mathbb{Z}_{>0}$ for all $e \in E(G)$. Then $\Gamma$ does not satisfy the Manin–Mumford condition.

This observation is a consequence of the fact that on a graph with unit edge lengths, the degree-0 divisor classes supported on vertices form a finite abelian group, known as the *critical group* of the graph. In other words, vertex-supported divisor classes are always torsion. This implies that all vertices of $G$ lie in the same torsion packet.

If we take the $k$-th subdivison graph $G^{(k)}$ of $G$, meaning every edge if $G$ is subdivided into $k$ edges of equal length, then the same reasoning says that these new vertices are also in the same torsion packet.

$$\#V(G^{(k)}) = \#V(G) + (k-1)\#E(G).$$

Taking $k \to \infty$ shows that $\Gamma$ has an infinite torsion packet.

**Example 5.2.** Let $G$ be the theta graph, which has two vertices $x$ and $y$ connected by three edges. If all edge lengths are 1, then the divisor class $[x - y]$ has order three in the Jacobian.

![Figure 5.1: Critical group of order 3.](image)

**Example 5.3.** Let $G$ be the theta graph with vertices $x$ and $y$, and let $\Gamma = (G, \ell)$ be the metric graph which assigns edge lengths 1, 2, and 3. Then the divisor class $[x - y]$ has order 11 in $\text{Jac}(\Gamma)$.

We say that a property holds for a *very general* point of some real parameter space if it holds outside of a countable collection of proper Zariski-closed subsets. A Zariski-closed subset is the set of zeros of a polynomial function. Note that the zero locus of a non-constant polynomial on $\mathbb{R}^n$ has codimension at least 1. Thus the complement of a very general subset of $\mathbb{R}^n$ has Lebesgue measure 0.
Definition 5.4. We say a combinatorial graph $G$ satisfies the Manin–Mumford condition if the metric graph $(G, \ell)$ satisfies the Manin–Mumford condition for a very general choice of edge lengths.

Recall that a graph $G$ is biconnected (or two-connected) if $G$ is connected after deleting any vertex. Our main theorem of this section is that a biconnected graph $G$ of genus $g \geq 2$ satisfies the Manin–Mumford condition.

5.2.1 Higher-degree Manin–Mumford

The classical Manin–Mumford conjecture concerns the map $X \to \text{Jac}(X)$ from a smooth algebraic curve to its Jacobian. The image of this map is the set of degree 1 effective divisors (up to some translation). It is natural to ask the analogous question for all degree $d$ effective divisors. Namely, given some effective divisor $D_0 \in \text{Div}^d(X)$ we may ask whether the image of $\text{Sym}^d(X) \to \text{Jac}(X)$ defined by

$$(x_1, \ldots, x_d) \mapsto \left[ \sum_{i=1}^d x_i - D_0 \right]$$

contains finitely many torsion points of $\text{Jac}(X)$.

Definition 5.5. We say a metric graph $\Gamma$ satisfies the degree-$d$ Manin–Mumford condition if the image of the $d$-dimensional Abel–Jacobi map

$$AJ_D^{(d)} : \Gamma^d \to \text{Jac}(\Gamma)$$

$$(x_1, \ldots, x_d) \mapsto \left[ \sum_{i=1}^d x_i - D \right]$$

intersects finitely many torsion points of $\text{Jac}(\Gamma)$, for any choice of base-divisor $D \in \text{Pic}^d(\Gamma)$. We abbreviate this condition as $\text{MM}(d)$.

Definition 5.6. We say a graph $G$ satisfies the Manin–Mumford condition in degree $d$ if the metric graph $(G, \ell)$ satisfies the Manin–Mumford condition in degree $d$ for very general edge lengths $\ell : E(G) \to \mathbb{R}_{>0}$. By abuse of notation, we also use $\text{MM}(d)$ to abbreviate the Manin–Mumford condition in degree $d$ for a combinatorial graph.
When $d = 1$ the condition MM(1) is the usual Manin–Mumford condition on $\Gamma$. When $g = g(\Gamma) \geq 1$ and $d \geq g$, then MM($d$) cannot hold, since the higher Abel–Jacobi map $A J^{(d)}_{D}$ is surjective and $\text{Jac}(\Gamma)_{\text{tors}}$ is infinite. If a metric graph $\Gamma$ satisfies MM($d$), then it also satisfies MM($d'$) for any $1 \leq d' \leq d$.

5.3 Tropical Manin–Mumford results

5.4 Setup

We say points $x, y \in \Gamma$ are torsion equivalent if there exists a positive integer $n$ such that $n[x - y] = 0$ in $\text{Jac}(\Gamma)$.

Lemma 5.7. Torsion equivalence defines an equivalence relation on the points of $\Gamma$

Proof. It is clear that torsion equivalence is reflexive and symmetric. Suppose $n, m$ are positive integers such that $n[x - y] = 0$ and $m[y - z] = 0$ in $\text{Jac}(\Gamma)$. Then $mn[x - z] = mn([x - y] + [y - z]) = 0$. This shows that torsion equivalence is transitive.

5.4.1 Stabilization of metric graphs

A metric graph $\Gamma$ is semistable if every point $x \in \Gamma$ has valence at least 2. A combinatorial graph $G$ is stable if every vertex $v \in V(G)$ has valence at least 3.

Proposition 5.8 (Metric graph stabilization). Let $\Gamma$ be a metric graph of genus $g \geq 1$. Then there is a canonical subgraph $\Gamma' \subset \Gamma$ which is semistable and homotopy equivalent to $\Gamma$.

If a semistable metric graph $\Gamma'$ has genus $g \geq 2$, then $\Gamma'$ has a unique stable model $(G', \ell)$.

add figure of stabilization

Figure 5.3: Metric graph and its stabilization
Proposition 5.9. Suppose $G$ is a stable graph of genus $g$. Then the number of edges is at most $3g - 3$.

Proof. Since every vertex has valence at least 3, we have

$$\#E(G) = \frac{1}{2} \sum_{v \in V(G)} \text{val}(v) \geq \frac{3}{2} \#V(G).$$

By the genus formula $g = \#E(G) - \#V(G) + 1$, this implies

$$\#E(G) = g + \#V(G) - 1 \leq g - 1 + \frac{2}{3} \#E(G)$$

which is equivalent to the desired inequality $\#E(G) \leq 3g - 3$. \qed

5.4.2 Girth and independent girth

Recall that the girth of a graph is the minimal length of a simple cycle.

Let $G$ be a finite connected graph with girth $\gamma$. Then for any choice of edge lengths the metric graph $\Gamma = (G, \ell)$ does not satisfy the generalized Manin–Mumford condition in degree $d \geq \gamma$.

Note that if $G$ has girth 1, i.e. $G$ has a loop edge, then the loop is a biconnected component of genus 1.

We define the independent girth $\gamma^{\text{ind}}$ of a graph as

$$\gamma^{\text{ind}}(G) = \min_C(\#E(C) + 1 - h_0(G \setminus E(C)))$$

where the minimum is taken over all closed cycles $C$ in $G$, and $h_0$ denotes the number of connected components. Since the (usual) girth is defined as $\gamma(G) = \min_C(\#E(C))$ and $h_0 \geq 1$, we have the inequality $\gamma^{\text{ind}} \leq \gamma$. The independent girth is invariant under subdivision of edges, so it is well-defined for a metric graph.

5.5 Proofs

Lemma 5.10. Let $\Gamma = (G, \ell)$ be a metric graph. If an edge $e \in E(G)$ contains two distinct points in the same torsion packet, then the torsion packet contains infinitely many points.

Proof. Suppose that an edge $e$ contains distinct points $x, y$ such that $[x - y]$ is torsion. Let $z$ denote the midpoint of $x$ and $y$; we claim $[x - z]$ is also torsion. The midpoint satisfies $[2z] = [x + y]$. If $n$ is a positive integer such that $n[x - y] = 0$, then $2n[x - z] = n[2x - 2z] = n[x - y] = 0$. This proves the claim that $[x - z]$ is torsion. By repeating this argument on the midpoint of $x$ and $z$, we obtain infinitely many points on $e$ in the same torsion packet as $x$. \qed
Theorem 5.11. Suppose \( \Gamma \) is a metric graph of genus \( g \geq 2 \). If a torsion packet is finite, then its cardinality is at most \( 3g - 3 \).

Proof. The map from a metric graph to its stabilization induces an isomorphism on Jacobians, so we may assume that \( \Gamma \) is a stable metric graph.

A stable metric graph of genus \( g \geq 2 \) has a combinatorial model \((G, \ell)\) such that every vertex of \( G \) has degree at least 3. In such a model, the number of edges is at most \( 3g - 3 \). By Lemma 5.10, a finite torsion packet has at most one point on a given edge of \( G \). This proves that the size of a finite torsion packet is at most \( 3g - 3 \).

Theorem 5.12. Suppose \( G \) is a biconnected metric graph of genus \( g \geq 2 \). For a very general choice of edge lengths \( \ell : E(G) \to \mathbb{R}_{>0} \), the metric graph \( \Gamma = (G, \ell) \) has at most \( 3g - 3 \) torsion points.

Proof. Let \( G' \) be the stabilization of \( G \). Given a choice of edge lengths \( \ell : E(G) \to \mathbb{R}_{>0} \) there are associated edge lengths \( \ell' : E(G') \to \mathbb{R}_{>0} \), which induces a map \( \mathbb{R}^{E(G)} \to \mathbb{R}^{E(G')} \) which is component-wise linear and surjective. Under such a map the preimage of a very general set is also very general; this means that if the theorem statement is true for \( G' \), then it is also true for \( G \). Thus it suffices to prove the result when \( G \) is a stable graph.

Choose some edge \( e \) in \( E(G) \). We claim that for very general edge lengths, there is at most one point of \( e \) in a given torsion class.

Consider two distinct points \( x, y \in e \). Let \( z, w \) be the endpoints of \( e \), and let \( G_0 = G \setminus e \) denote the graph with edge \( e \) deleted.

By assumption that \( G \) is biconnected and has genus \( g \geq 2 \), in the deletion \( G_0 \) the vertices \( z, w \) are joined by at least two distinct simple paths. Thus \( [z - w] \neq 0 \) in the deleted metric graph \( \Gamma_0 \). By Corollary 5.16, this implies that \( [z - w] \) is non-torsion on \( \Gamma_0 \) for very general edge lengths. By Proposition 5.14, this implies that \( [x - y] \) is not torsion on \( \Gamma \) for very general edge lengths.

Thus assuming very general edge lengths, a torsion packet contains at most one point on any edge of \( G \).

Proposition 5.13. Suppose \( x, y \) are two points on a metric graph \( \Gamma \). Then \( [x - y] \) is torsion in the Jacobian of \( \Gamma \) if and only if all slopes of the voltage function \( j^\tau_{xy} \) are rational.

Proof. If a \( \mathbb{R} \)-piecewise-linear function \( f \) has rational slopes, then for some positive integer \( n \) the multiple \( nf \) has integer slopes. This means that the divisor \( \Delta(nf) = n\Delta(f) \) is a zero-representative in the Jacobian. These implications are all reversible. Thus \( f \) having rational slopes is equivalent to \( \Delta(f) \) being torsion.
Proposition 5.14. Suppose $\Gamma$ is a biconnected metric graph and points $x, y \in \Gamma$ lie on the same edge. Let $\Gamma_0$ denote the metric graph with the open segment between $x$ and $y$ removed. If $[x - y]$ is torsion on $\Gamma$, then $[x - y]$ is torsion on $\Gamma_0$.

(Remark: in the other direction, $[x - y]$ is torsion on $\Gamma$ only if $[x - y]$ is torsion on $\Gamma_0$, and $\ell(x, y)$ is commensurate with $\ell_{\text{eff}}(\Gamma_0)$.)

Proof. Suppose $[x - y]$ is torsion on $\Gamma$. Let $j^y_x$ denote the voltage function on $\Gamma$ when one unit of current is sent from $y$ to $x$. By Proposition 1 [above], all slopes of $j^y_x$ are rational. In particular, the slope of $j^y_x$ on the segment between $x$ and $y$ is rational; let $q$ denote this slope. Since $G$ is biconnected, we have $0 < q < 1$.

Let $\Gamma_0$ denote the metric graph obtained from $\Gamma$ by deleting the interior of edge $e$. It is clear that the restriction of $j^y_x$ to $\Gamma_0$ has Laplacian $\Delta(j^y_x|_{\Gamma_0}) = (1 - q)x - (1 - q)y$.

Let $j^y_{x,0}$ denote the voltage function on $\Gamma_0$ when one unit of current is sent from $y$ to $x$. Since $j^y_{x,0} = (1 - q)^{-1}j^y_x$, all slopes of $j^y_{x,0}$ are rational. By Proposition 5.13, this implies $[x - y]$ is torsion on $\Gamma_0$ as desired.

Proposition 5.15. Suppose $x, y$ are two vertices on a graph $G$. Let $j^y_x$ be the voltage function on $\Gamma = (G, \ell)$, depending on variable edge lengths $\ell : E(G) \to \mathbb{R}$. Either:

1. There exists some edge $e$ such that the slope of $j^y_x$ along $e$ is a non-constant function of the edge lengths, or
2. All slopes of $j^y_x$ are 1 or 0, independent of edge lengths.

Proof. Suppose there is a unique simple path in $G$ from $x$ to $y$. Then the slope of $j^y_x$ is 1 along this path, and 0 away from this path, since all current flowing from $y$ to $x$ must follow this path. Thus we are in case (2).

On the other hand, suppose there are two distinct simple paths $\gamma_1, \gamma_2$ in $G$ from $x$ to $y$. Let $e$ be an edge of $G$ which lies on $\gamma_1$ but not $\gamma_2$. If we fix the lengths of edges in $\gamma_1$ and send all other edge lengths to infinity, then the slope of $j^y_x$ along $e$ approaches 1. (For a general choice of edge lengths, the slope of $j^y_x$ is nonzero on $e$.) If we send the length $\ell(e)$ to infinity while keeping all other edge lengths fixed, then the slope of $j^y_x$ along $e$ approaches zero. Thus the slope of $j^y_x$ along $e$ is a non-constant function of the edge lengths.

Corollary 5.16. Suppose $x, y$ are two vertices on a graph $G$. Then either

1. $[x - y]$ is non-torsion on $(G, \ell)$ for very general edge lengths, or
2. $[x - y] = 0$ on $(G, \ell)$ for any edge lengths.

Proof. If the slope of $j^y_x$ along some edge $e$ is a non-constant rational function of the edge lengths, then for very general edge lengths this slope is irrational. By Proposition 5.13, this implies that $[x - y]$ is nontorsion.
On the other hand, if the slopes of $j_{x}^{y}$ do not vary as a function of edge lengths, then by Proposition 5.15 all slopes of $j_{x}^{y}$ are zero or one. This implies that $[x - y] = 0.$

5.5.1 Higher-degree Manin–Mumford

Lemma 5.17. Suppose $D = x_1 + \cdots + x_d$ and $E = y_1 + \cdots + y_d$ are effective divisors of degree $d$ on a metric graph $\Gamma$. Let $f \in \mathcal{PL}_\mathbb{R}(\Gamma)$ be a function satisfying $\Delta(f) = D - E$. (Up to an additive constant, $f = \sum_{i=1}^{d} j_{x_i}^{y_i}$.)

(a) We have $[D - E] = 0$ in $\text{Jac}(\Gamma)$ if and only if all slopes of $f$ are integers.

(b) We have $[D - E]$ is torsion in $\text{Jac}(\Gamma)$ if and only if all slopes of $f$ are rational.

Proof. Part (a) is a restatement of the definition of linear equivalence. Part (b) follows from part (a) and the definition of torsion. More precisely, the torsion order of $[D - E]$ divides $n$ if and only if all slopes of $f$ like in $\frac{1}{n}\mathbb{Z}$.

Lemma 5.18. Suppose $0 \leq d \leq g$. An effective divisor of degree $d$ is linearly equivalent to a semibreak divisor of degree $d$.

Proof. This is a result of Gross–Shokrieh–Tóthmérész; see [18, Theorem A].

Lemma 5.19. Suppose $G$ is a graph with edges $e_1, \ldots, e_k \in E(G)$. Let $r$ be the rank of $\{e_1, \ldots, e_k\}$ in the cographic matroid of $G$. (In other words, $r = k - h^0(G \setminus \{e_1, \ldots, e_k\}) + 1$.) Then the set of divisor classes $[x_1 + \cdots + x_k]$, with $x_i \in e_i$, has dimension $r$ in the Jacobian of $G$.

Proof. This was stated in Section 2.7 as Theorem 2.19.

Proposition 5.20. Suppose $G$ has a simple cycle with $d$ edges. Then for any edge lengths, the metric graph $\Gamma = (G, \ell)$ fails to satisfy the degree $d$ Manin–Mumford condition.

Proof. Suppose $c$ is a simple cycle in $G$ with edges $e_1, e_2, \ldots, e_d$ and vertices $v_1, v_2, \ldots, v_d$ in cyclic order, where edge $e_i$ has endpoints $v_i$ and $v_{i+1}$ modulo $d$.

On the metric graph $\Gamma = (G, \ell)$, let $D_0 = v_1 + \cdots + v_d$ be a degree $d$ base-divisor, and consider the effective divisor $E_n = x_1 + \cdots + x_d$ where $x_i$ is the point on edge $e_i$ whose distance from endpoint $v_i$ is $\frac{1}{n}\ell(e_i)$.

We claim

1. $[E_n - D_0]$ is torsion of order $n$ in $\text{Jac}(\Gamma)$,

2. $[E_n - D_0] \neq [E_m - D_0]$ for $n \neq m$.

Together, these claims imply that the image of the higher Abel–Jacobi map $A_{f_{D_0}^{(d)}} : \Gamma^d \to \text{Jac}(\Gamma)$ intersects infinitely many torsion points.
Proof of Claim (1): Let \( f_n \in \text{PL}_\mathbb{R}(\Gamma) \) be a function satisfying \( \Delta(f_n) = E_n - D_0 \). It can be verified that on edge \( e_i, f_n \) has slopes of magnitude \( \frac{1}{n} \) and \( \frac{n-1}{n} \). By Lemma 5.17, it follows that \([E_n - D_0]\) is torsion of order \( n \) as claimed.

Proof of Claim (2): It suffices to show that when \( n \neq m \), the divisor class \([E_n - E_m]\) \( \neq [0] \). The function \( f_n - f_m \) has \( \Delta(f_n - f_m) = E_n - E_m \). The slopes of \( f_n - f_m \) along \( e_i \), modulo 1, are \( \pm \frac{1}{n} \pm \frac{1}{m} \). This value is nonzero if \( n \neq m \). By Lemma 5.17, this implies the claim.

**Proposition 5.21.** Suppose \( G \) is a graph and \( e_1, \ldots, e_k \) are edges which do not contain a cut of \( G \). Let \( \text{Div}(e_1, \ldots, e_k) \) denote the set of effective divisors of the form \( D = x_1 + \cdots + x_k \) where \( x_i \in e_i \). Then either:

1. for very general edge lengths, distinct divisors in \( \text{Div}(e_1, \ldots, e_k) \) are in distinct torsion classes, or
2. In the deletion \( G_0 = G\setminus\{e_1, \ldots, e_k\} \), there are bridge edges \( b_1, \ldots, b_j \) such that the edges \( \{e_1, \ldots, b_j, \ldots\} \) contain a cycle of \( G \).

(Remark: the condition (2) may be restated in matroid language as, the closure of \( \{e_1, \ldots, e_k\} \) with respect to the cographic matroid \( \mathcal{M}_G^\perp \) contains a cycle of the graphic matroid \( \mathcal{M}_G \).

**Proof.** Let \( D = x_1 + \cdots + x_k \) and \( E = y_1 + \cdots + y_k \). Suppose \( f \) is a piecewise linear function such that \( \Delta(f) = D - E \).

Let \( G_0 \) denote the graph obtained from deleting edges \( e_1, \ldots, e_k \) from \( G \). Consider let \( f_0 = f|_{G_0} \) denote the restriction of \( f \) to the subgraph \( G_0 \). We have
\[
\Delta(f_0) = \lambda_1w_1 + \cdots + \lambda_\ell w_\ell,
\]
where \( w_1, \ldots, w_\ell \) are the set of endpoints of edges \( e_1, \ldots, e_k \) (so in particular \( \ell \leq 2k \)), and \( \lambda_i \) are rational coefficients.

Case 1: all \( \lambda_i = 0 \). Then we claim that the edges \( \{e_1, \ldots, e_k\} \) must contain a cycle.

Since the edges \( \{e_1, \ldots, e_k\} \) do not contain a cut, the deleted graph \( G_0 \) has a single connected component. The assumption that all \( \lambda_i = 0 \) implies that all slopes of \( f \) are zero on \( G_0 \). Thus for every edge \( e_i \), the voltage drop \( f(e_i^+ - e_i^-) \) across the edge is zero. Since \( \Delta(f) \) restricted to \( e_i \) has the form \( x_i - y_i \), the slopes along \( e_i \) are as shown in Figure 5.4, where slope \( s = \ell([x_i, y_i])/\ell(e_i) \).

This implies the current is non-zero in a neighborhood of the endpoints \( e_i^+ \) and \( e_i^- \). By conservation of current, there must be a cycle in \( \{e_1, \ldots, e_k\} \) as claimed.

Case 2: some \( \lambda_i \neq 0 \). Then the restriction \( f_0 \) to the deleted graph \( \Gamma_0 \) is non-constant. If current flows across a non-bridge edge \( e \), then we claim this current is a non-constant function of edge lengths. To verify this, suppose that we fix the length
of all edges \( e' \neq e \), and vary \( \ell(e) \). As \( \ell(e) \to \infty \), we claim that the current across \( e \) approaches zero.

Let \( \Lambda = \frac{1}{2} \sum_i |\lambda_i| \). (By conservation of current, \( \Lambda = \sum_{i:\lambda_i > 0} |\lambda_i| = \sum_{i:\lambda_i < 0} |\lambda_i| \).) The slope-current principle, Proposition 3.5, states that the current across any edge of \( \Gamma_0 \) is bounded above in magnitude by \( \Lambda \). Since \( e \) is not a bridge edge, there is some simple path \( \pi \) from \( e^- \) to \( e^+ \) which does not contain \( e \). Thus the voltage drop \( f_0(e^-) - f_0(e^+) \) is bounded above by \( \Lambda \cdot \ell(\pi) \). This means that the current through \( e \) is bounded above by

\[
\left| \frac{df_0}{dx} \right|_e \leq \frac{\Lambda \ell(\pi)}{\ell(e)}.
\]

This bound approaches zero as \( \ell(e) \to \infty \), assuming \( \ell(\pi) \) is held constant. This verifies the claim that the slope of \( f_0 \) along \( e \) approaches zero.

Since we assumed that the slope of \( f_0 \) along \( e \) is non-zero for some choice of \( \ell \), the slope (or current) along \( e \) is a non-constant rational function of the edge lengths. (It is a rational function by Kirchhoff’s formulas, Theorem 3.10.) Thus for very general edge lengths, this slope is an irrational number.

In the first case, for any distinct divisors \( D_1, D_2 \in \text{Div}(e_1, \ldots, e_k) \), the voltage function \( j_{D_1}^{D_2} \) has non-zero slope on some edge not in \( \{e_1, \ldots, e_k\} \).

In the second case, there exist distinct divisors \( D_1, D_2 \in \text{Div}(e_i) \) such that the voltage function \( j_{D_1}^{D_2} \) has zero slope outside of a cycle contained in \( \{e_1, \ldots, b_1, \ldots\} \).

**Theorem 5.22.** Let \( G \) be a graph of genus \( g \gg 0 \). Suppose \((G, \ell)\) satisfies the generalized Manin–Mumford condition in degree \( d \), for some lengths function \( \ell : E(G) \to \mathbb{R}_{> 0} \). Then

\[
d < C \log(g)
\]

for some universal constant \( C \).

**Proof.** Recall that the *girth* \( \gamma \) of a graph \( G \) is the length of the smallest (simple) cycle in \( G \). By theorem [ref above], it suffices to show that for \( G \) with no valence 2 vertices* the girth satisfies \( \gamma(G) < C \log g(G) \). (has a simple cycle of length \( d < C \log g \).

First we consider when \( G \) is a trivalent graph. If the girth is \( \gamma \), then for any vertex \( v \in V(G) \) the neighborhood of \( v \) of radius \( r = \lceil \gamma/2 \rceil - 1 \) is isomorphic to a
neighborhood of that radius in an infinite trivalent tree. The number of vertices in this neighborhood is

\[ 1 + 3 + 6 + 12 + \cdots + 3 \cdot 2^{r-1} = 3 \cdot 2^r - 2. \]

In a trivalent graph, the genus satisfies

\[ 2g - 2 = \#V(G) \geq \#V(N_G(v, r)) = 3 \cdot 2^r - 2. \]

Hence

\[ g \geq 3 \cdot 2^{r-1} = \frac{3}{4} \cdot 2^{[\gamma/2]} \]

so

\[ \gamma \leq \frac{2}{\log(2)} (\log g + \log(4/3)) \]

as desired.

\[ \square \]

**Theorem 5.23.** Let \( \Gamma = (G, \ell) \) be a connected metric graph of genus \( g \geq 2 \). If \( \Gamma \) satisfies the Manin–Mumford condition in degree \( d \), then

\[ \#(AJ_D^{(d)}(\Gamma) \cap \text{Jac}(\Gamma)_{\text{tors}}) \leq \left( \frac{3g - 3}{d} \right). \]

**Proof.** The number \( \#(AJ_D^{(d)}(\Gamma) \cap \text{Jac}(\Gamma)_{\text{tors}}) \) doesn’t change under replacing \( \Gamma \) with its stabilization, so we may assume \( \Gamma \) is stable and \( (G, \ell) \) is a stable model. This means that the number of edges \( E(G) \) is bounded above by \( 3g - 3 \).

The image of \( AJ_D^{(d)}(\Gamma) \) is homeomorphic to \( \text{Eff}^d(\Gamma) \). (They differ by a translation sending \( \text{Pic}^d(\Gamma) \) to \( \text{Pic}^0(\Gamma) \).) The maximal cells in \( \text{Eff}^d(\Gamma) \) are indexed by \( d \)-tuples of edges of \( G \), which are independent with respect to the cographic matroid \( M_G^\perp \).

The number of maximal cells is clearly bounded above by \( (\#E^d(G)) \), the number of all \( d \)-tuples of edges. Since we assumed \( G \) is stable, we have \( (\#E^d(G)) \leq (3g-3)^d \).

We claim that the hypotheses on \( \Gamma \) imply that there is at most one torsion point on each maximal cell of \( AJ_D^{(d)}(\Gamma) \). To verify this claim, consider a maximal cell \( \text{Eff}(e_1, \ldots, e_d) \) which contains two distinct torsion points \( [E_1 - D] \) and \( [E_2 - D] \). Then the divisor class \( [E_1 - E_2] \) is also torsion, as is any rational multiple of \( [E_1 - E_2] \).

The upshot is that given two distinct torsion points, the set of all torsion points is dense on the line segment connecting them. Since the maximal cell \( \text{Eff}(e_1, \ldots, e_d) \) is convex, it contains this line segment, hence contains infinitely many torsion points. This proves the claim.

\[ \square \]

Recall that \( \text{Eff}(e_1, \ldots, e_k) \) denotes the set of divisor classes of the form \( [x_1 + \cdots + x_k] \), where \( x_i \) is a point on the edge \( e_i \).
Theorem 5.24. Let $G$ be a connected graph with genus $g \geq 1$ and independent girth $\gamma^{\text{ind}}$. Then $G$ is Manin–Mumford finite in degree $d$ if and only if $1 \leq d < \gamma^{\text{ind}}$.

Proof. If $d \geq \gamma^{\text{ind}}$, then there is a $d$-tuple of edges $\{e_1, \ldots, e_d\}$ whose closure in the cographic matroid contains a cycle of $G$. Let $\{e_1, \ldots, e_d, \ldots, e_k\}$ denote the closure. By Proposition 5.20, the cell $\text{Eff}(e_1, \ldots, e_k)$ contains infinitely many torsion points for any choice of edge lengths. Moreover, the torsion points are dense along a 1-dimensional (affine) subspace $S$ corresponding to the cycle of $G$. Since the cells $\text{Eff}(e_1, \ldots, e_d) \subset \text{Eff}(e_1, \ldots, e_k)$ have equal dimension, the subspace $S$ also intersects $\text{Eff}(e_1, \ldots, e_d)$ along a positive-measure segment. Hence $\text{Eff}(e_1, \ldots, e_d)$ contains infinitely many torsion points.

If $d < \gamma^{\text{ind}}$, then Proposition 5.21 implies that each maximal cell in $AJ_D^{(d)}(\Gamma)$ has at most one torsion point, for a very general choice of edge lengths. Since there are finitely many maximal cells, this implies that for very general edge lengths there are finitely many torsion points. \qed
APPENDIX A

Theta Intersection

A.1 Appendix: Theta intersections

In this appendix we give an alternate description of the Weierstrass locus $W(D)$ as the intersection of two polyhedral subcomplexes of complementary dimension in $\text{Pic}^{g-1}(\Gamma)$. This allows us to give an alternate proof that $W(D)$ is finite for a generic divisor class $[D]$. In this perspective, the stable Weierstrass locus $W^{st}(D)$ naturally appears as the stable tropical intersection of these two subsets.

Throughout this section (including the above paragraph), we assume that the divisor class $[D]$ is (Riemann–Roch) nonspecial, meaning that its rank satisfies

$$r(D) = \begin{cases} 
\deg(D) - g & \text{if } \deg(D) \geq g, \\
-1 & \text{otherwise.}
\end{cases}$$

A generic divisor class in $\text{Pic}^n(\Gamma)$ is nonspecial. If $n \geq 2g - 1$, all divisors in $\text{Pic}^n(\Gamma)$ are nonspecial.

A.1.1 Intersection with $\Theta$

Recall that the theta divisor $\Theta \subset \text{Pic}^{g-1}(\Gamma)$ is the space of degree $g - 1$ divisor classes which have an effective representative;

$$\Theta = \{[D] \in \text{Pic}^{g-1}(\Gamma) : [D] \geq 0\}.$$  

Given a divisor $D$ of degree $n \geq g$, let $\Phi_D : \Gamma \to \text{Pic}^{g-1}(\Gamma)$ denote the map

$$\Phi_D : x \mapsto [D - (n - g + 1)x].$$

If $D$ has degree $n < g$ let $\Phi_D : x \mapsto [D]$ be the constant map. Note that the map $\Phi_D$ depends only on the divisor class $[D]$. If $D$ is nonspecial, the Weierstrass locus of $D$ is equal to the intersection $\Phi_D(\Gamma) \cap \Theta$, pulled back to $\Gamma$ from $\text{Pic}^{g-1}(\Gamma)$.  

68
Proposition A.1. Let $D$ be a divisor of degree $n \geq g$, and let $\Phi_D : \Gamma \to \text{Pic}^{g-1}(\Gamma)$ be the map $\Phi_D(x) = [D - (n - g + 1)x]$. If $D$ is a nonspecial,

$$W(D) = \Phi_D^{-1}(\Phi_D(\Gamma) \cap \Theta).$$

Proof. This follows from the definition of Weierstrass locus, if $D$ has rank $n - g$. \[\square\]

Proposition A.2. Suppose $\Gamma$ is a bridgeless metric graph. If $D$ has degree $n \geq g$, the map $\Phi_D : \Gamma \to \text{Pic}^{g-1}(\Gamma)$ is locally injective (i.e. an immersion).

Proof. The map $\Phi_D$ may be expressed as a composition of three maps

$$\Phi_D : \Gamma \xrightarrow{\alpha} \text{Pic}^1(\Gamma) \xrightarrow{\beta} \text{Pic}^{n-g+1}(\Gamma) \xrightarrow{\gamma} \text{Pic}^{g-1}(\Gamma),$$

where $\alpha$ sends $x \mapsto [x]$, $\beta$ sends $[E] \mapsto [(n - g + 1)E]$, and $\gamma$ sends $[E] \mapsto [D - E]$. The map $\gamma = \gamma_D$ is a homeomorphism. The map $\beta$ is a $(n - g + 1)^g$-fold covering map, so it is a local homeomorphism if $n \geq g$. Thus it suffices to verify that the first map $\alpha$ is locally injective.

This follows from the Abel–Jacobi theorem for metric graphs, see e.g. Baker–Faber [7, Theorem 4.1 (3)(4)]. Note that $\text{Pic}^1(\Gamma)$ is (non-canonically) isomorphic to the Jacobian $\text{Jac}(\Gamma) = \text{Pic}^0(\Gamma)$ by choosing a basepoint $x_0$ to subtract. \[\square\]

If $\Gamma$ contains bridge segments, let $\Gamma_{/(\text{br})}$ denote the metric graph obtained from $\Gamma$ by contracting all bridges. Let $S_{(\text{br})} \subset \Gamma_{/(\text{br})}$ denote the set of points which were bridges in $\Gamma$.

Lemma A.3. Let $\pi : \Gamma \to \Gamma_{/(\text{br})}$ denote the canonical map contracting all bridge segments of $\Gamma$, which induces $\pi_* : \text{Pic}^n(\Gamma) \to \text{Pic}^n(\Gamma_{/(\text{br})})$ for all $n$. For any divisor $D$ on $\Gamma$,

$$W(D) = \pi^{-1}W(\pi_*D).$$

Proof. On $\Gamma$ the linear equivalence map $x \mapsto [x]$ factors through $\pi : \Gamma \to \Gamma_{/(\text{br})}$; i.e. we have a commuting diagram

$$\begin{array}{ccc}
\Gamma & \xrightarrow{\pi} & \Gamma_{/(\text{br})} \\
[x] \downarrow & & \downarrow [x] \\
\text{Pic}^1(\Gamma) & \xrightarrow{\sim} & \text{Pic}^1(\Gamma_{/(\text{br})}).
\end{array}$$

Using this, the result is clear from the definition of $W(D)$. \[\square\]

Lemma A.4. Suppose $S \subset \Gamma$ is a finite set of points in a metric graph $\Gamma$. For a generic divisor class $[D]$, the intersection $W(D) \cap S$ is empty.
Proof. It suffices to consider when \( S = \{s\} \) contains one point. Assuming \( D \) is nonspecial, which holds for generic \([D] \in \text{Pic}^n(\Gamma)\), we have \( s \in W(D) \) if and only if

\[
[D - (n - g + 1)s] \text{ is effective } \iff [D] = [(n - g + 1)s + E] \text{ for some } [E] \in \Theta.
\]

Since \( \Theta \) has dimension \( g - 1 \), the space \([D] = [(n - g + 1)s + E] : [E] \in \Theta\) also has dimension \( g - 1 \). Hence a generic class \([D]\) has \( s \notin W(D) \).

\( \square \)

**Theorem A.5.** For a generic divisor class \([D]\) in \( \text{Pic}^n(\Gamma) \), the Weierstrass locus \( W(D) \) is finite.

**Proof.** If \( n < g \), then a generic divisor class in \( \text{Pic}^n(\Gamma) \) is not effective because the image of \( \text{Sym}^n(\Gamma) \to \text{Pic}^n(\Gamma) \) has dimension at most \( n \), while \( \text{Pic}^n(\Gamma) \) has dimension \( g \). For a non-effective divisor class \([D]\), the Weierstrass locus \( W(D) \) is empty.

Now suppose \( n \geq g \). By Riemann–Roch, a generic divisor class in \( \text{Pic}^n(\Gamma) \) has rank \( r(D) = n - g \). (By the above paragraph, \( r(K - D) = -1 \) generically.) Thus, it suffices to show that \( W(D) \) is finite for a generic nonspecial divisor class.

**Case 1:** \( \Gamma \) is bridgeless. As above, let \( \Phi_D : \Gamma \to \text{Pic}^{g-1}(\Gamma) \) be the map \( \Phi_D(x) = [D - (n - g + 1)x] \). Recall that the Weierstrass locus \( W(D) \) is equal to

\[
W(D) = \Phi_D^{-1}(\Phi_D(\Gamma) \cap \Theta) \subset \Gamma
\]

where \( \Theta = \{[E] \in \text{Pic}^{g-1}(\Gamma) : [E] \geq 0\} \) is the theta divisor. Note that as \([D]\) varies, the image \( \Phi_D(\Gamma) \) varies by translation inside \( \text{Pic}^{g-1}(\Gamma) \).

Recall that \( \Theta \) is a \((g - 1)\)-dimensional polyhedral complex with finitely many facets, and \( \Phi_D(\Gamma) \) is a 1-dimensional polyhedral complex with finitely many segments. This implies that the space of translations which cause \( \Phi_D(\Gamma) \) to intersect \( \Theta \) non-transversally has dimension at most \( g - 1 \). Hence for a generic divisor class \([D]\), the intersection \( \Phi_D(\Gamma) \cap \Theta \) is transverse.

Suppose all intersections in \( \Phi_D(\Gamma) \cap \Theta \) are transverse, and occur in the interiors of the respective segment and facet. Recall that \( \Phi_D \) is locally injective by Proposition A.2. If \( \Phi_D \) sends \( x \in \Gamma \) to a transverse intersection, then \( x \) must have some neighborhood \( U \subset \Gamma \) such that \( \Phi_D(U \setminus \{x\}) \) is disjoint from \( \Theta \). This means that \( W(D) = \Phi_D^{-1}(\Phi_D(\Gamma) \cap \Theta) \) is a discrete subset of \( \Gamma \). Because \( \Gamma \) is compact, this implies \( W(D) \) is finite.

**Case 2:** \( \Gamma \) has bridge segments. Let \( \pi : \Gamma \to \Gamma_{/\text{(br)}} \) denote the map contracting all bridge segments of \( \Gamma \). Let \( S_{/\text{(br)}} \subset \Gamma_{/\text{(br)}} \) denote the image of all bridges, which is a finite subset of \( \Gamma_{/\text{(br)}} \). Note that \( \pi \) restricts to an injection away from \( \pi^{-1}S_{/\text{(br)}} \).

By Lemma A.4, a generic divisor class \([D] \in \text{Pic}^n(\Gamma_{/\text{(br)}}) \) has \( W(D) \) disjoint from \( S_{/\text{(br)}} \). Since \( \pi \) induces a homeomorphism \( \pi_* : \text{Pic}^n(\Gamma) \to \text{Pic}^n(\Gamma_{/\text{(br)}}) \), this implies that a generic class \([D] \in \text{Pic}^n(\Gamma) \) has \( W(\pi_*[D]) \) disjoint from \( S_{/\text{(br)}} \). The result then follows from Lemma A.3 and Case 1.

\( \square \)
A.1.2 Stable Weierstrass locus

In this section we describe the relation of the current setup, involving the theta divisor $\Theta$, and the stable Weierstrass locus defined in Section 4.2.1.

**Proposition A.6.** Suppose $\Gamma$ is a bridgeless metric graph of genus $g$. Let $D$ be a divisor of degree $g$, and let $\Phi_D : \Gamma \to \text{Pic}^{g-1}(\Gamma)$ send $\Phi_D(x) = [D - x]$. Then the break divisor $\text{br}[D]$ is equal to

$$\text{br}[D] = \Phi_D^{-1}(\Phi_D(\Gamma) \cap^{\text{st}} \Theta)$$

where $\Theta$ is the theta divisor and $\cap^{\text{st}}$ denotes stable tropical intersection.$^1$

**Proof.** Let us denote $\text{br}^*[D] := \Phi_D^{-1}(\Phi_D(\Gamma) \cap^{\text{st}} \Theta)$. For a generic divisor class $[D] \in \text{Pic}^g(\Gamma)$, the intersection $\Phi_D(\Gamma) \cap \Theta$ is transverse so

$$\text{br}^*[D] = \{ x \in \Gamma : [D - x] \geq 0 \},$$

i.e. $\text{br}^*[D]$ contains the support of any effective representative of $[D]$. Generically, the class $[D]$ contains a single effective representative so $\text{br}^* : \text{Pic}^g(\Gamma) \to \text{Sym}^g(\Gamma)$ defines a generic section of the linear equivalence map $\text{Sym}^g(\Gamma) \to \text{Pic}^g(\Gamma)$.

By general properties of stable tropical intersection, the map $\text{br}^* : \text{Pic}^g(\Gamma) \to \text{Sym}^g(\Gamma)$ is continuous. But by Theorem 2.8, the break divisor map $\text{br}$ is the unique continuous section of $\text{Sym}^g(\Gamma) \to \text{Pic}^g(\Gamma)$ so we must have $\text{br}^*[D] = \text{br}[D]$. \qed

$^1$ The stable tropical intersection may have multiplicites, so here we interpret the preimage to be a multiset in $\Gamma$ carrying the same multiplicities.
BIBLIOGRAPHY


