1 Logistics

Next meetings: Wednesday February 27, 9-9:45am and Friday March 1, 12:15 - 1pm

Midterm presentations: Tuesday February 26, 5 - 6:30pm

Expectations for next meetings:

- Problem 7: If I am given a hexagon configuration in terms of the 6 crease vectors, what are the 6 dihedral angles in this configuration?
- Problem 8: mountain / valley labellings
- [**Writing**] Continue writing up relevant discussion from this week in draft of final report
- [**Visualization**] Share code for updated visualization program

2 Hexagon configuration space

2.1 Changing coordinates

In the hexagon configuration space, we discussed two choices for how to put "coordinates" on this space: crease vectors and dihedral fold angles. How do we change coordinates between these, from crease vectors to dihedral angles?

For example, if we have consecutive crease vectors

$$v_1 = (1, 0, 0),\ v_2 = (0, 1, 0),\ v_3 = (-1, 0, 0)$$

then the corresponding dihedral angle is $\theta_2 = 0$. If we have consective crease vectors

$$v_2 = (0, 1, 0),\ v_3 = (-1, 0, 0),\ v_4 = (0, 0, 1)$$

then the corresponding dihedral angle is $\theta_3 = \frac{\pi}{2}$.

Problem 7. Given three nonzero vectors v_0, v_1, $v_2 \in \mathbb{R}^3$, what is a formula to express the dihedral angle between the planes $p_{01} = \mathbb{R}v_0 + \mathbb{R}v_1$ and $p_{12} = \mathbb{R}v_1 + \mathbb{R}v_2$?
(Possible hint: `https://en.wikipedia.org/wiki/Dihedral_angle#Mathematical_background`)

Our goal will be to use this expression for the angles to express the *energy*

$$E(\Phi) = \left(\sum_i |\theta_i - \pi|^2 \right)^{1/2}$$

of a given fold configuration in terms of the crease vectors, and to study configurations of constant energy.

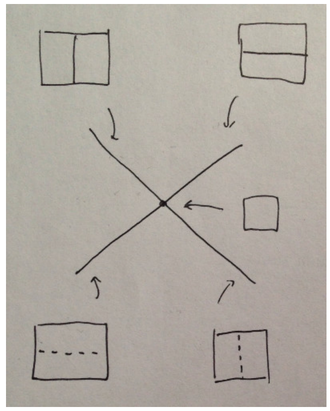
2.2 Montain / valley diagrams

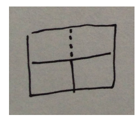
To understand a topological space it often helps to cut it up into smaller, more manageable pieces.

For a fold configuration which is "near flat," we say a crease is a *mountain fold* if it is higher than a secant line between its two adjacent flat regions, and a *valley fold* if it is lower. (We take the nearby flat configuration as reference for what "higher" and "lower" mean.)

Suppose we label each crease in a configuration with "mountain" or "valley" or neither. Visually we can indicate these respectively by a solid line, a dotted line, or no line.

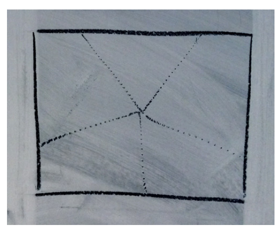
Example 1. For two perpendicular creases (so there are 4 total crease vectors), the following shows possible mountain/valley labellings:



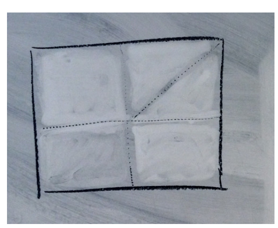
The following mountain/valley labelling is not possible: 

Problem 8. In the following diagrams with 5 creases coming from a central vertex, which mountain/valley labellings are possible? How do these fit together in configuration space near the unfolded state?

1. creases with equal spacings



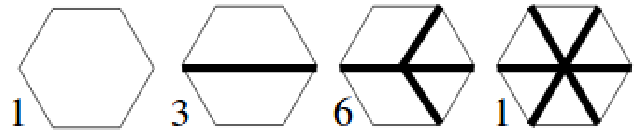
2. creases perpendicular plus one at 45° angle



2.3 Neighborhoods of flat configurations

The paper "Hodge Theory and the Art of Paper Folding" claims that for every point in the configuration space that does not lie completely in a plane (i.e. corresponds to a non-degenerate linkage), a small neighborhood around this point looks like an open ball in $\mathbb{R}^3$.

The 11 flat configurations of a folded hexagon are discussed in the "Discrete Folding" paper, and shown in Figure 9:



Problem 5. [see "Hodge Theory" paper] For each of the 11 flat configurations of the hexagon,

(a) What are the parameters f, b, and w as used in Theorem 1.1?

(b) What is the signature of the null cone describing this neighborhood of configuration space?

Discussion: For the unfolded configuration, we have $f = 6$, $b = 0$, and $w = 1$ (OR $f = 0$, $b = 6$, $w = -1$.) The corresponding null cone has signature $(3, 1)$. A small neighborhood in configuration space is a cone over two disjoint 2-spheres $S^2 \times S^0$.

For the other 10 configurations, $f = 3$, $b = 3$ and $w = 0$. The corresponding null cone has signature $(2, 2)$. A small neighborhood in configuration space is a cone over the 2-torus $S^1 \times S^1$.

3 Quadratic forms

Given an $n \times n$ (real) symmetric matrix Q we can define a *quadratic form* on $\mathbb{R}^n$ by

$$(u, v) \quad \mapsto u^T Q v.$$

We denote this map $f_Q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. The *null cone* of a quadratic form is the set

$$Z(Q) = \{x \in \mathbb{R}^n \text{ such that } f_Q(x, x) = x^T Q x = 0\}.$$

The *unit null cone* (nonstandard terminology) is the set

$$\hat{Z}(Q) = \{x \in \mathbb{R}^n : x^T Q x = 0 \text{ and } |x|^2 = 1\}.$$

Problem 6. For each case below, the null cone $Z(Q)$ is the cone over the "unit null cone" $\hat{Z}(Q)$; we want a nice geometric description of $\hat{Z}(Q)$.

(a) Describe the null cone of a quadratic form with signature $(1, 3)$.

(b) Describe the null cone of a quadratic form with signature $(2, 2)$.

(c) [Challenge(?)] Describe the null cone of a general quadratic form with signature (p, q).

Discussion (b) When the signature is $(2, 2)$, the unit null cone is described by

$$\hat{Z}(Q) = \{(a, b, c, d) : a^2 + b^2 - c^2 - d^2 = 0,\ a^2 + b^2 + c^2 + d^2 = 1\}.$$

By adding and subtracting the two constraints, this is equivalent to

$$\hat{Z}(Q) = \{(a, b, c, d) : a^2 + b^2 = \frac{1}{2},\ c^2 + d^2 = \frac{1}{2}\}.$$

This shows that $\hat{Z}(Q)$ is a *product* of the spaces

$$\{(a, b) : a^2 + b^2 = \frac{1}{2}\} \times \{(c, d) : c^2 + d^2 = \frac{1}{2}\}.$$

Each of these factors is a circle of radius $\sqrt{1/2}$, so topologically $\hat{Z}(Q) \cong S^1 \times S^1$.

(c) In general, when the signature is (p, q) the null cone

$$\hat{Z}(Q) = \{a_1^2 + \cdots + a_p^2 - b_1^2 - \cdots - b_q^2 = 0,\ a_1^2 + \cdots + b_q^2 = 1\}$$

can be expressed equivalently as

$$\{a_1^2 + \cdots + a_p^2 = \frac{1}{2},\ b_1^2 + \cdots + b_q^2 = \frac{1}{2}\}$$

so $\hat{Z}(Q) \cong S^p \times S^q$ is topologically a product of two spheres.

Example 2. The null cone of of a quadratic form of signature $(1, 1)$ is two lines passing through the origin. The unit null cone $\hat{Z}(Q)$ is a set of 4 discrete points, which is equal to the product of spheres $S^0 \times S^0$.

Example 3. The null cone of a quadratic form of signature $(2, 1)$ is a "doubled cone". The unit null cone $\hat{Z}(Q)$ is two disjoint circles, which is $S^1 \times S^0$.