1 Logistics

Next meeting: Friday February 22, 9:30 - 11am.

Midterm presentations: Tuesday February 26, 5 - 6:30pm

Expectations for next meeting:

- Problem 5: We discussed why the hexagon crease pattern has a total of 11 flat configurations. What do neighborhoods of other flat configurations look like?
- Problem 6: To apply Theorem 1.1 of "Hodge Theory and the Art of Paper Folding" we should understand what the null cone of a general quadratic form looks like geometrically
- Problem 4: Once we understand the neighborhood of each flat configuration, we can ask: How do these neighborhoods fit together? One approach: divide configuration space into regions labelled by whether each crease is a "mountain" or "valley" fold (or neither)
- Problem 7: If I am given a hexagon configuration in terms of the 6 crease vectors, what are the 6 dihedral angles in this configuration? (Possible hint: `https://en.wikipedia.org/wiki/Dihedral_angle#Mathematical_background`)
- [**Writing**] Continue writing up relevant discussion from this week in draft of final report
- [**Visualization**] In visualization of fold configuration, fill in the sheet with shading / color; when sheets intersect indicate that in visualization

2 Hexagon configuration space

Problem 4. What is the space of folding configurations of a hexagon with creases along its three axes?

Discussion: (For now we consider just the massless case.)
The paper "Branches of Triangulated Origami" proves that in a neighborhood of the unfolded configuration, the configuration space looks like the null cone of the quadratic form

$$\{(a, b, c, d) \in \mathbb{R}^4 : a^2 + b^2 + c^2 - d^2 = 0\}.$$

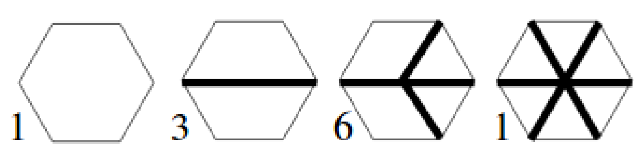
In particular, the unfolded state is the "vertex" $(0, 0, 0, 0)$ of this cone.
In the paper "Hodge Theory and Paper Folding," Theorem 1.1 describes the neighborhood of each flat configuration of the hexagon. The neighborhood is described as the null cone of a quadratic form of a certain signature.

Problem 5. [see "Hodge Theory" paper] For each of the 11 flat configurations of the hexagon,

(a) What are the parameters f, b, and w as used in Theorem 1.1?

(b) What is the signature of the null cone describing this neighborhood of configuration space?

The 11 flat configurations of a folded hexagon are discussed in the "Discrete Folding" paper, and shown in Figure 9:



3 Quadratic forms

Given an $n \times n$ (real) symmetric matrix Q we can define a *quadratic form* on $\mathbb{R}^n$ by

$$(u, v) \quad \mapsto u^T Q v.$$

We denote this map $f_Q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. The *null cone* of a quadratic form is the set

$$Z(Q) = \{x \in \mathbb{R}^n \text{ such that } f_Q(x, x) = x^T Q x = 0\}.$$

If x is in the null cone, then for any $\lambda \in \mathbb{R}$

$$(\lambda x)^T Q(\lambda x) = \lambda^2 (x^T Q x) = 0$$

so $\{\lambda x : \lambda \in \mathbb{R}\}$ is contained in the null cone. Why is this called a "cone"? See `https://en.wikipedia.org/wiki/Cone_(topology)`; this makes $Z(Q)$ (essentially) the cone over the "unit distance" null vectors

$$\hat{Z}(Q) = \{x \in \mathbb{R}^n : x^T Q x = 0 \text{ and } |x|^2 = 1\}.$$

A quadratic form f_Q is *nondegenerate* if the matrix Q is invertible (i.e. all eigenvalues are nonzero). The *nullity* of f_Q is the dimension of the 0-eigenspace of Q. The *spectral theorem* says that up to a change of coordinates, a nondegenerate quadratic form f_Q is equivalent to a form associated to a matrix

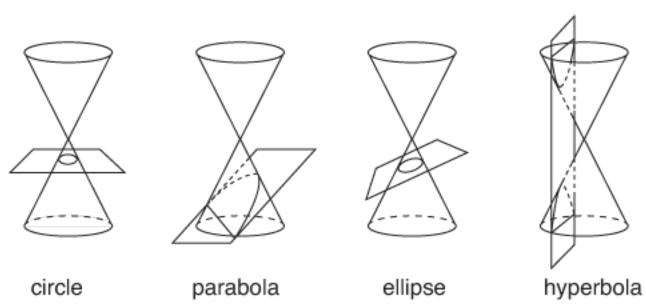
$$Q = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 \\ \vdots & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{pmatrix}.$$

The quadratic form has *signature* (p, q) if there are p-many $+1$'s and q-many -1's in this diagonal form. We described the null cone of a quadratic form of signature $(1, 1)$, $(2, 1)$, and $(3, 1)$.

Example 1. The null cone of of a quadratic form of signature $(1, 1)$ is two lines passing through the origin. This counts as having two branches at the origin because the equation $\{a^2 - b^2 = 0\}$ defining the null cone factors into two linear terms $\{a - b = 0\} \cup \{a + b = 0\}$.

The unit null cone $\hat{Z}(Q)$ is a set of 4 discrete points.

Example 2. The null cone of a quadratic form of signature $(2, 1)$ is a "doubled cone":



Source: cliffnotes.com

The unit null cone $\hat{Z}(Q)$ is two disjoint circles.

Problem 6. For each case below, the null cone $Z(Q)$ is the cone over the "unit null cone" $\hat{Z}(Q)$; we want a nice geometric description of $\hat{Z}(Q)$.

(Hint: in the general case your answer should involve n-spheres; see `https://en.wikipedia.org/wiki/N-sphere`)

(a) Describe the null cone of a quadratic form with signature $(1, 3)$.

(b) Describe the null cone of a quadratic form with signature $(2, 2)$.

(c) [Challenge(?)] Describe the null cone of a general quadratic form with signature (p, q).