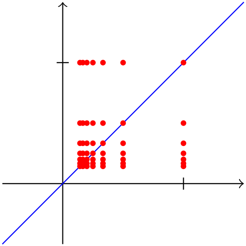
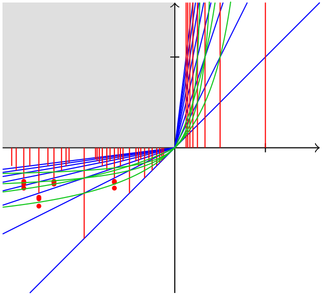
Dilated floor functions and their commutators

Harry Richman

joint w/ Jeff Lagarias and Takumi Murayama
University of Michigan

October 5, 2018





Floor functions

The **floor function** sends continuous input to discrete output

$$\lfloor x \rfloor : \mathbb{R} \to \mathbb{Z}$$

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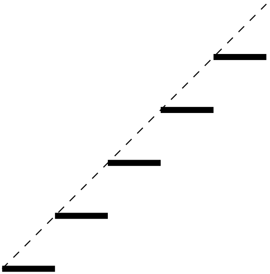

Figure: Graph of $f(x) = \lfloor x \rfloor$

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$$f_{\alpha}(x) := \lfloor \alpha x \rfloor : \mathbb{R} \to \mathbb{Z}$$

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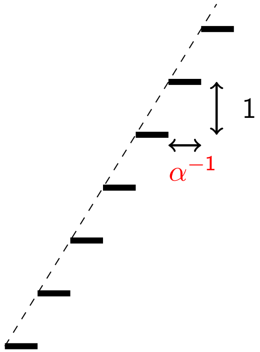

Figure: Graph of $f_\varphi(x) = \lfloor \varphi x \rfloor$, where $\varphi = \frac{1+\sqrt{5}}{2}$

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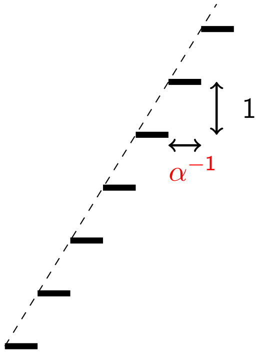
$\rightsquigarrow$ f_α discretizes $\mathbb{R}$
"at length scale α^{-1}"



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Dilated floor functions: Why care?

Elementary number theory:

$$\mathrm{val}_p(n!) = \left\lfloor \frac{1}{p} n \right\rfloor + \left\lfloor \frac{1}{p^2} n \right\rfloor + \left\lfloor \frac{1}{p^3} n \right\rfloor + \cdots$$

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$$\zeta(s) := \sum_{n \geq 1} \frac{1}{n^s} \qquad \text{satisfies} \qquad \int_0^\infty \lfloor x \rfloor \, x^{-s} \frac{dx}{x} = \frac{1}{s} \zeta(s)$$

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What happens when we compose f_α and f_β?

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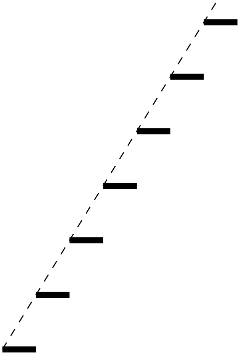

Figure: Graph of $f_1 \circ f_\varphi = \lfloor \lfloor \varphi x \rfloor \rfloor$ where $\varphi = \frac{1+\sqrt{5}}{2}$

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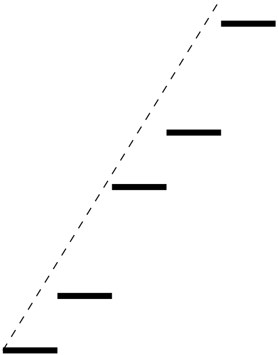

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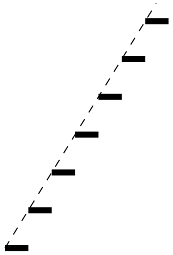

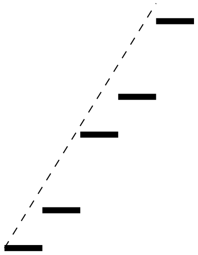

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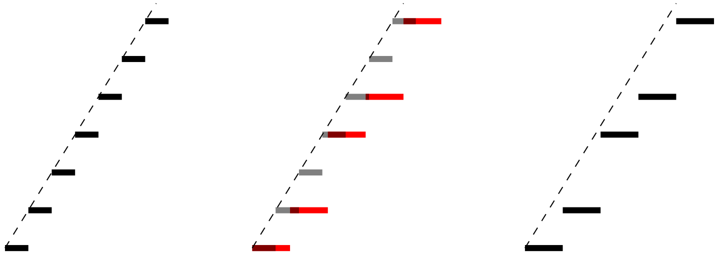

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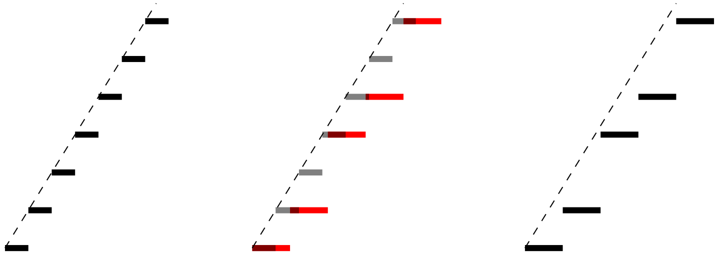

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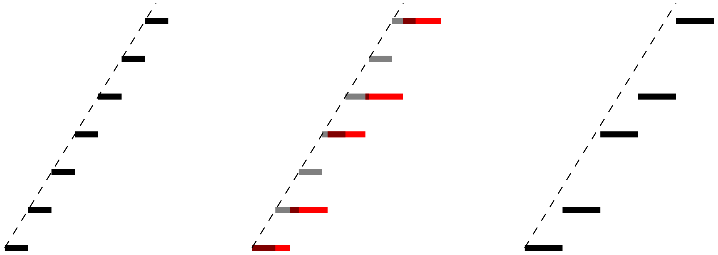

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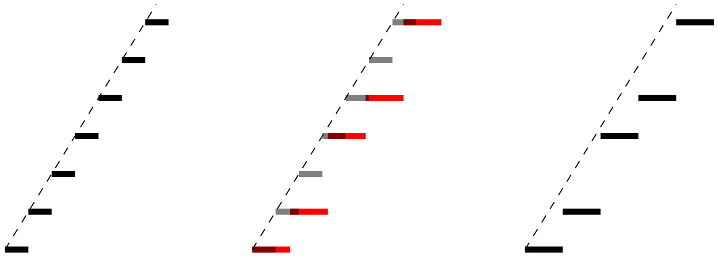

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For which (α, β) do we have

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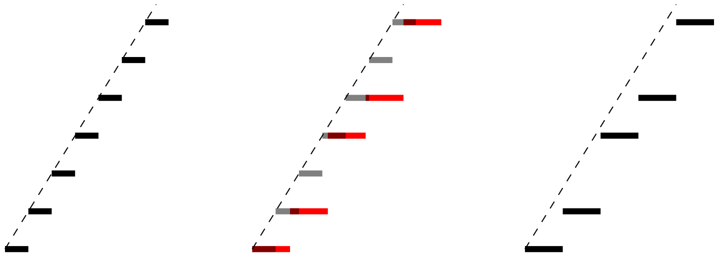

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Problem B

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Composing floor functions: results

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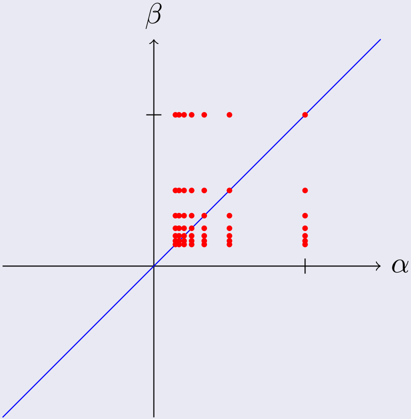
Theorem (Lagarias–Murayama–R)

All solutions to (A) are:



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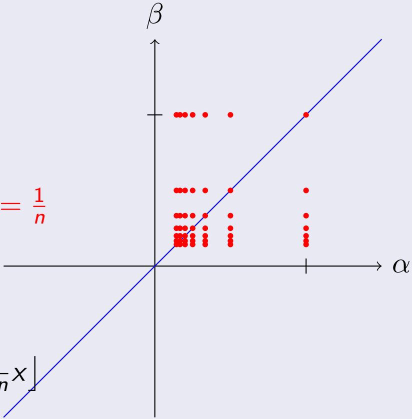
All solutions to (A) are:

$\alpha = \frac{1}{m}, \ \beta = \frac{1}{n}$



In fact: $\left\lfloor \frac{1}{m} \left\lfloor \frac{1}{n} \right\rfloor x \right\rfloor = \left\lfloor \frac{1}{mn} x \right\rfloor$

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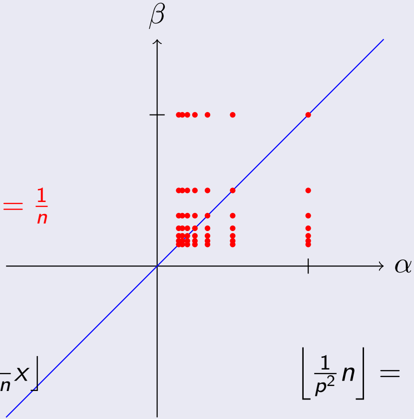

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$\left\lfloor \frac{1}{p^2} n \right\rfloor = \left\lfloor \frac{1}{p} \left\lfloor \frac{1}{p} n \right\rfloor \right\rfloor$!

Composing floor functions: results

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For which (α, β) do we have $f_\alpha \circ f_\beta \geq f_\beta \circ f_\alpha$?

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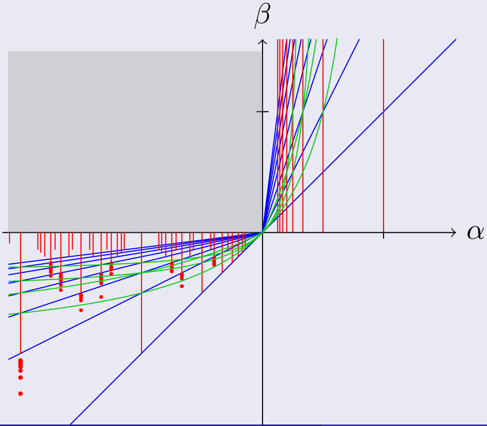

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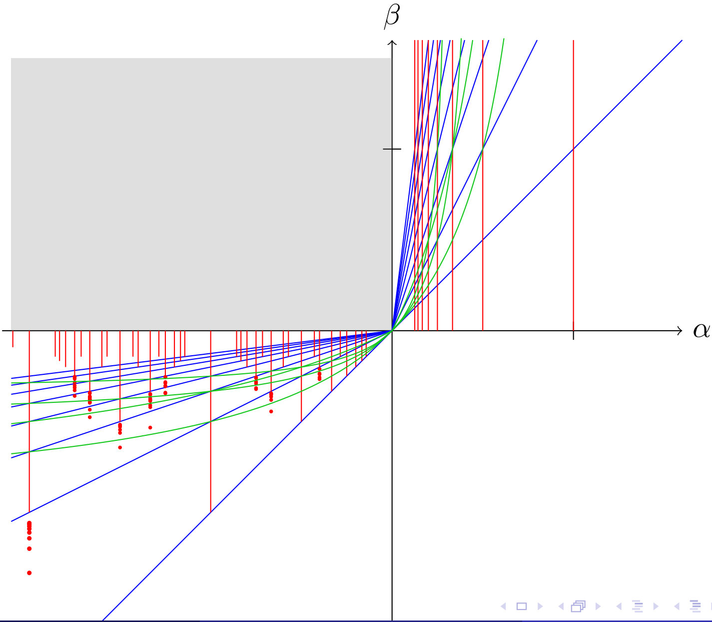
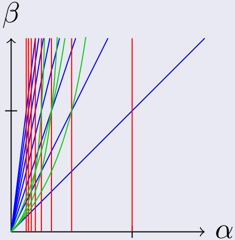


$\lfloor \alpha \lfloor \beta x \rfloor \rfloor \geq \lfloor \beta \lfloor \alpha x \rfloor \rfloor$: positive-dilation results

Theorem (Lagarias–R)

All positive solutions to (B) are:



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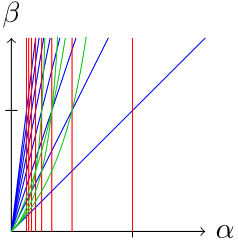

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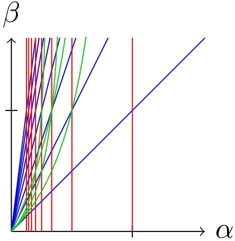
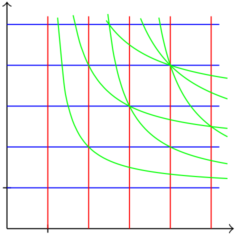


$\rightsquigarrow$



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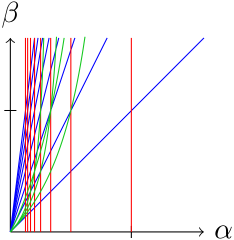

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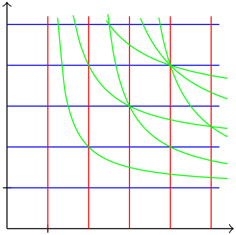
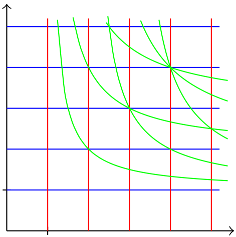


Symmetries:



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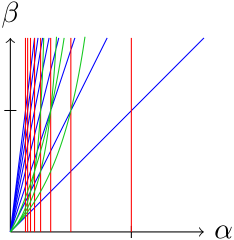

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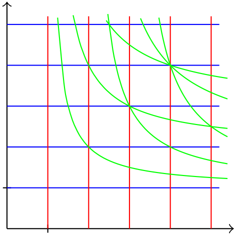

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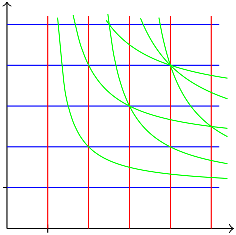

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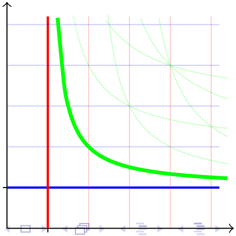

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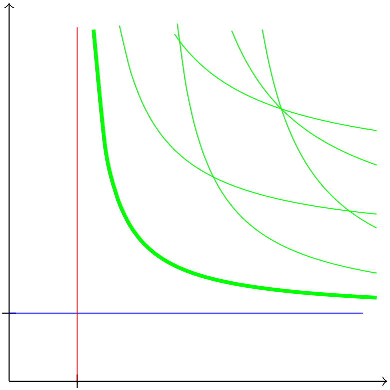
Where do **green** solution curves come from?



Proof ingredient: Beatty sequences

Parameter $\mu \geq 1$,

$$\mathcal{B}(\mu) = \{\lfloor \mu \rfloor, \lfloor 2\mu \rfloor, \lfloor 3\mu \rfloor, \ldots\} \subset \mathbb{N}$$

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Theorem ("Beatty's Theorem," Ostrowski, Hyslop, Aitken, ..)

If μ and ν are irrational and satsify $\frac{1}{\mu} + \frac{1}{\nu} = 1$, then

$$\mathcal{B}(\mu) \cap \mathcal{B}(\nu) = \emptyset \qquad \text{and} \qquad \mathcal{B}(\mu) \cup \mathcal{B}(\nu) = \mathbb{N}$$

i.e. their Beatty sequences **partition** $\mathbb{N}$.

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Theorem ("Beatty's Theorem" 1926)

Beatty sequences $\mathcal{B}(\mu)$, $\mathcal{B}(\nu)$ *partition* $\mathbb{N}$, *i.e.*

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Proposition 1 (Lagarias–R)

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Proof ingredient: Beatty sequences

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For parameters $(\alpha, \beta) > 0$,

$$f_\alpha \circ f_\beta \geq f_\beta \circ f_\alpha \qquad \textbf{iff} \qquad \mathcal{B}(\mu) \cap \mathcal{B}^{<}(\nu) = \emptyset$$

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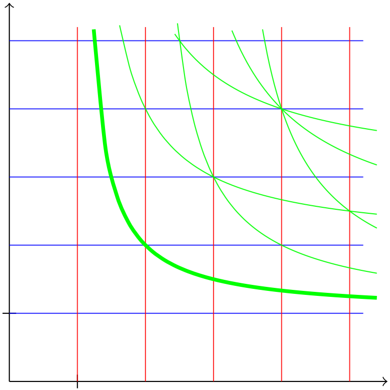

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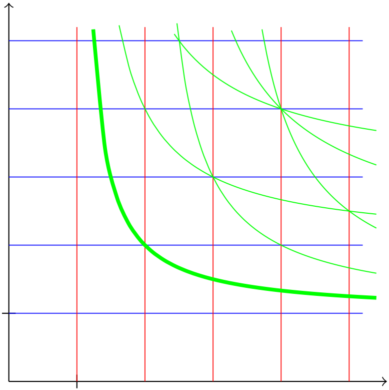

How do we know there are **no more** solutions?

Proof ingredient: Torus subgroups

Torus surface $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$
A point $(\sigma, \tau) \in \mathbb{T}$ generates a **cyclic subgroup** of $\mathbb{T}$

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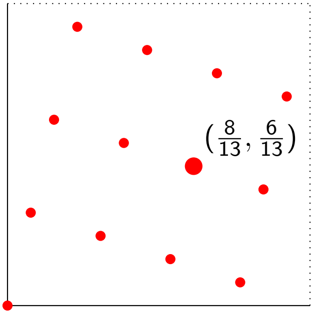

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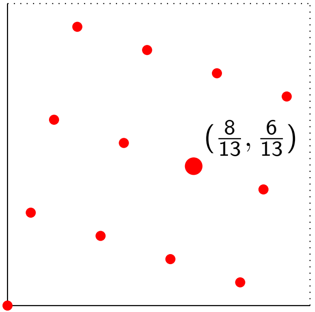

Vague Question

When is $(\sigma, \tau) \in \mathbb{T}$ a "minimal" generator for its subgroup?

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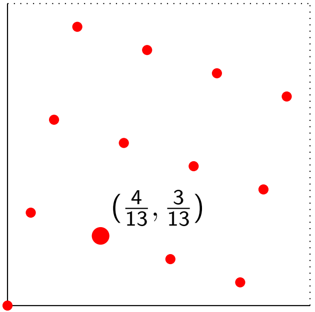

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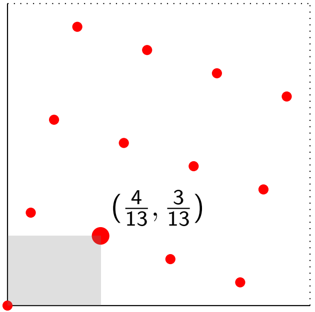

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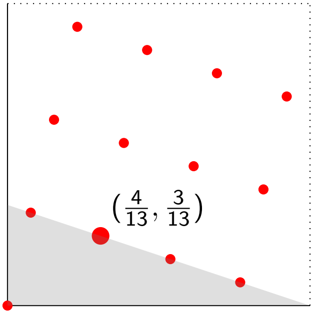

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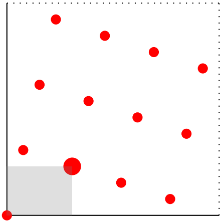

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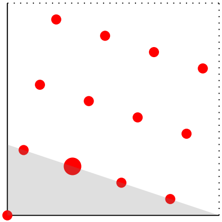

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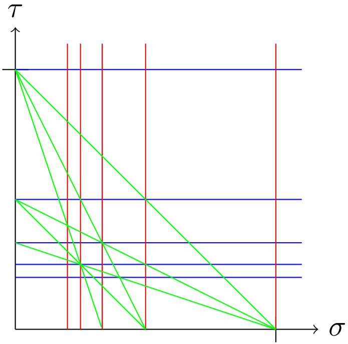
All minimal generators of cyclic subgroups, in $\mathbb{T}$:



Composing floor functions: Why **really** care?

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(Recall: P.N.T. says $\pi(x) = \#\{\text{prime } p \leq x\} \approx \frac{x}{\log x}$)

Riemann hypothesis: $\Pi(x) := \sum_{p \leq x} \log p$

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Jean-Paul Cardinal (2010) defined a "2-dimensional analogue" of the Mertens function

Composing floor functions: Why **really** care?

Let $\{d_i\} = \{n, \left\lfloor \frac{1}{2}n \right\rfloor, \left\lfloor \frac{1}{3}n \right\rfloor, \left\lfloor \frac{1}{4}n \right\rfloor, \ldots, 1\}$ be the "almost divisors" of n.

In Cardinal's matrix $\mathcal{M}_n$, the entry in position i, j is

$$\mathcal{M}_n(i,j) = M\left(\left\lfloor \frac{1}{d_i d_j} n \right\rfloor\right) = M\left(\left\lfloor \frac{1}{d_i} \left\lfloor \frac{1}{d_j} n \right\rfloor \right\rfloor\right) = M\left(\left\lfloor \frac{1}{d_j} \left\lfloor \frac{1}{d_i} n \right\rfloor \right\rfloor\right)$$

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(Note: "almost divisors of almost divisors are almost divisors"!)

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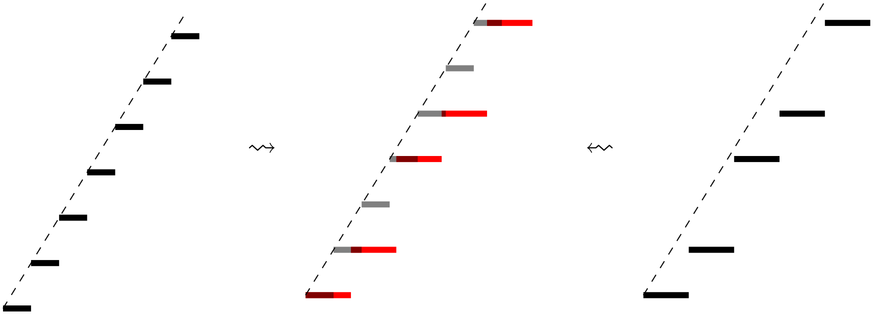
Dilated floor function commutators



Thank you!