Dilated floor functions having non-negative commutator

Jeffrey C. Lagarias, Takumi Murayama, D. Harry Richman

University of Michigan

Introduction

The floor function $\lfloor x \rfloor : \mathbb{R} \to \mathbb{Z}$ discretizes a continuous input by rounding down to the nearest integer. When we discretize input on different scales, it is natural to introduce *dilated floor functions* $f_\alpha(x) := \lfloor \alpha x \rfloor$. Each f_α has the same general "staircase" shape:

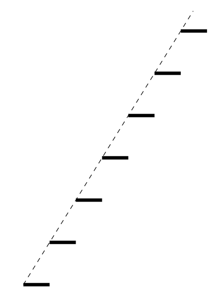

Figure 1: Graph of $f_\varphi = \lfloor \varphi x \rfloor$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio

When two such functions are composed, things get complicated—the "staircase" is now uneven:

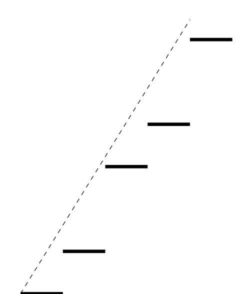

Figure 2: Graph of $f_\varphi \circ f_1 = \lfloor \varphi \lfloor x \rfloor \rfloor$ where $\varphi = \frac{1+\sqrt{5}}{2}$

As one step towards understanding compositions $f_\alpha \circ f_\beta$, we may ask how dilated floor functions behave under changing the order of composition.

Problem

For which real parameters (α, β) do the following hold for all real x?

$$\lfloor \alpha \lfloor \beta x \rfloor \rfloor = \lfloor \beta \lfloor \alpha x \rfloor \rfloor \qquad (*)$$

$$\lfloor \alpha \lfloor \beta x \rfloor \rfloor \geq \lfloor \beta \lfloor \alpha x \rfloor \rfloor \qquad (**)$$

In order words, we ask when the commutator

$$[f_\alpha, f_\beta] = \lfloor \alpha \lfloor \beta x \rfloor \rfloor - \lfloor \beta \lfloor \alpha x \rfloor \rfloor$$

is either identically zero $(*)$, or bounded below by zero $(**)$. Both are satisfied in the *trivial* cases

- $\alpha = 0$ or $\beta = 0$ or $\alpha = \beta$.

What else is allowed? We obtain a complete classification for both problems.

Results

The classification for $(**)$ relies on connections to two well-studied problems:

- **Beatty's problem** [1] to show the existence of "interesting" solutions along green hyperbolic arcs
- **Sylvester's problem** [3] on numerical semigroups to show the non-existence of additional solutions.

Theorem 1. The non-trivial solutions to $(*)$ occur exactly where both $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ are positive integers.

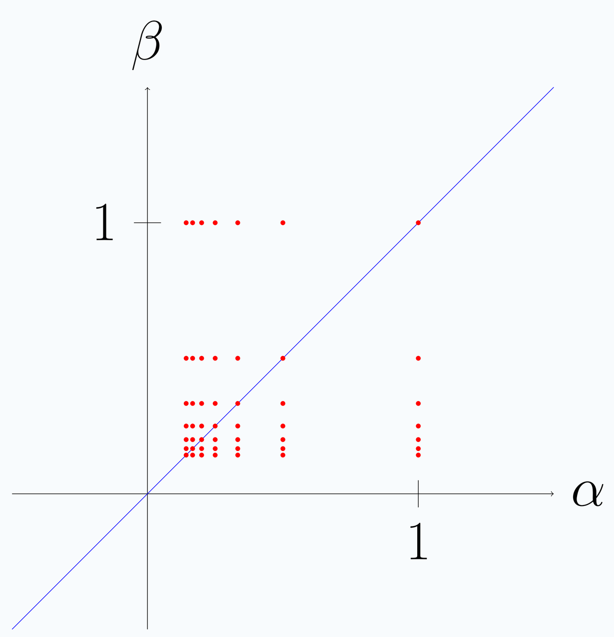

Figure 3: All solutions to $(*)$

Theorem 2. The non-trivial solutions to $(**)$ include the entire second quadrant, and fall in the first and third quadrants in explicitly known families indicated below.

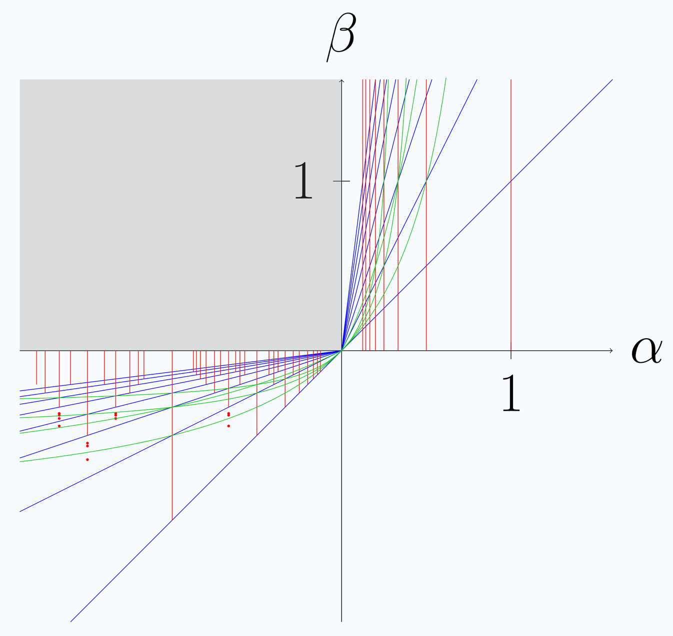

Figure 4: All solutions to $(**)$

To solve $(*)$ and $(**)$ we first identify all "jump points" of $f_\alpha \circ f_\beta$, i.e. points where the graph first reaches a given (integer) height. For example in the **first quadrant** $\alpha, \beta > 0$,

$$\lfloor \alpha \lfloor \beta x \rfloor \rfloor \geq n \iff x \geq \frac{1}{\beta}\left\lceil \frac{1}{\alpha} n \right\rceil, \quad \text{so we deduce} \quad (**) \iff \frac{1}{\beta}\left\lceil \frac{1}{\alpha} n \right\rceil \leq \frac{1}{\alpha}\left\lceil \frac{1}{\beta} n \right\rceil \text{ for all integers } n.$$

A complete list of such criteria for all four quadrants is found in [2].

Poset Structure

The relation $[f_\alpha, f_\beta] \geq 0$ defines a binary relation on dilation factors α, β. For nonzero dilations it is a *transitive* relation since

$$(**) \quad \Leftrightarrow \quad \alpha \left\lceil \frac{1}{\alpha} n \right\rceil \leq \beta \left\lceil \frac{1}{\beta} n \right\rceil \text{ for all } n \in \mathbb{Z},$$

so it induces a poset structure $(\mathbb{R}_{\neq 0}/\sim, \preceq)$.

- On positive reals, it says $\alpha \preceq \beta$ iff
  $$\alpha = \frac{\beta}{m + n\beta}$$
  for some integers $m, n \geq 0$.
- On negative reals, the *Farey sets* make a natural appearance within this poset structure.

Coordinate Changes

The change of coordinates $(u, v) := \left(\frac{1}{\alpha}, \frac{\beta}{\alpha}\right)$ is used to view $(**)$ in terms of Beatty sets. This reveals a hidden symmetry

$$(u, v) \mapsto (v, u)$$

of problem $(**)$ which holds in the first quadrant.

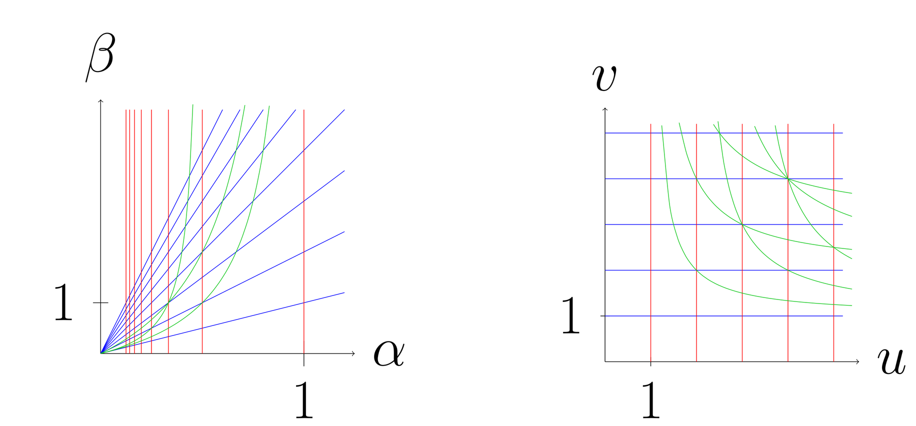

Figure 5: First quadrant solutions to $(**)$

Beatty Sequences

Given $u \geq 1$, the *Beatty sequence* $\mathcal{B}(u)$ takes floors of all multiples of u:

$$\mathcal{B}(u) := \{\lfloor u \rfloor, \lfloor 2u \rfloor, \lfloor 3u \rfloor, \ldots\} \subset \mathbb{N}_+.$$

Beatty [1] observed that under the conditions

$$\frac{1}{u} + \frac{1}{v} = 1 \quad \text{and } u, v \text{ irrational}$$

the corresponding sets $\mathcal{B}(u)$ and $\mathcal{B}(v)$ are **complementary**, i.e.

$$\mathcal{B}(u) \sqcup \mathcal{B}(v) = \mathbb{N}_+$$

This explains the green solutions in the first and third quadrants.

Sylvester Symmetry

Given coprime integers a, b, which numbers can be represented as non-negative integer combinations? The *gap set* contains the non-representable numbers:

$$G(a, b) = \mathbb{N} \setminus \{am + bn : m, n \in \mathbb{N}\}.$$

For example, $G(3, 5) = \{1, 2, 4, 7\}$. The gap set obeys the following **symmetry**: for any integer x in the range $[0, ab]$, not divisible by a or b,

$$x \in G(a, b) \quad \Leftrightarrow \quad ab - x \notin G(a, b).$$

This is used to show our classification is complete.

Applications

Floor functions arise in the following areas:

- digital straight lines
- algebraic singularities, minimal model program
- ergodic theory, dynamics on nilmanifolds

References

[1] Samuel Beatty, *Problem 3173*, Amer. Math. Monthly **33** (1926), no. 3, 159.

[2] J. C. Lagarias, T. Murayama, and D. H. Richman, *Dilated floor functions that commute*, Amer. Math. Monthly **123** (2016), no. 10, 1033–1038.

[3] J. J. Sylvester, *Problem 7382*, Educational Times, New Ser. **37** (1884), no. 266, 177.