Introduction

The floor function \([x] : \mathbb{R} \rightarrow \mathbb{Z}\) discretizes a continuous input by rounding down to the nearest integer. When we discretize input on different scales, it is natural to introduce *dilated floor functions* \(f_\alpha(x) := \lfloor \alpha x \rfloor\). Each \(f_\alpha\) has the same general "staircase" shape:

![Figure 1: Graph of \(f_\varphi = \lfloor \varphi x \rfloor\), where \(\varphi = \frac{1 + \sqrt{5}}{2}\) is the golden ratio](image)

When two such functions are composed, things get complicated—the "staircase" is now uneven:

![Figure 2: Graph of \(f_\alpha \circ f_\beta = \lfloor \varphi x \rfloor\), where \(\varphi = \frac{1 + \sqrt{5}}{2}\)](image)

As one step towards understanding compositions \(f_\alpha \circ f_\beta\), we may ask how dilated floor functions behave under changing the order of composition.

Problem

For which real parameters \((\alpha, \beta)\) do the following hold for all real \(x\):

\[
\lfloor \alpha \beta x \rfloor = \lfloor \beta \alpha x \rfloor \quad (\star)
\]

\[
\lfloor \alpha \beta x \rfloor \geq \lfloor \beta \alpha x \rfloor \quad (\star \star)
\]


In order words, we ask when the commutator

\[
f_\alpha \circ f_\beta - f_\beta \circ f_\alpha = [\alpha \beta x] - [\beta \alpha x]
\]

is either identically zero (\(\star\)), or bounded below by zero (\(\star \star\)). Both are satisfied in the *trivial* cases

- \(\alpha = 0\) or \(\beta = 0\) or \(\alpha = \beta\).

What else is allowed? We obtain a complete classification for both problems.

The classification for (\(\star \star\)) relies on connections to two well-studied problems:
- **Beatty’s problem** [1] to show the existence of “interesting” solutions along green hyperbolic arcs
- **Sylvester’s problem** [3] on numerical semigroups to show the non-existence of additional solutions.

![Theorem 1. The non-trivial solutions to (\(\star\)) occur exactly where both \(\frac{1}{\beta}\) and \(\frac{1}{\alpha}\) are positive integers.](image)

![Theorem 2. The non-trivial solutions to (\(\star \star\)) include the entire second quadrant, and fall in the first and third quadrants in explicitly known families indicated below.](image)

![Figure 3: All solutions to (\(\star\))](image)

![Figure 4: All solutions to (\(\star \star\))](image)

To solve (\(\star\)) and (\(\star \star\)) we first identify all “jump points” of \(f_\alpha \circ f_\beta\), i.e. points where the graph first reaches a given (integer) height. For example in the first quadrant \(\alpha, \beta > 0\)

\[
\lfloor \alpha \beta x \rfloor \geq n \iff x \geq \frac{1}{\beta} \left\lfloor \frac{n}{\alpha} \right\rfloor,
\]

so we deduce (\(\star \star\)) \(\iff \left\lfloor \frac{1}{\beta} \right\rfloor \leq \frac{1}{\alpha} \left\lfloor \frac{n}{\beta} \right\rfloor\) for all integers \(n\).

A complete list of such criteria for all four quadrants is found in [2].

Results

**Poset Structure**

The relation \([f_\alpha, f_\beta] \geq 0\) defines a binary relation on dilation factors \(\alpha, \beta\). For nonzero dilations it is a *transitive* relation since (\(\star \star\)) \(\iff \alpha \leq \beta \iff \beta \leq \alpha\) for all \(n \in \mathbb{Z}\), so it induces a poset structure \((\mathbb{R}_{\geq 0}, \leq)\).

- On positive reals, it says \(\alpha \leq \beta\) iff
  \[
  \alpha = \frac{\beta}{m + n \beta}
  \]
  for some integers \(m, n \geq 0\).
- On negative reals, the *Farey sets* make a natural appearance within this poset structure.

**Coordinate Changes**

The change of coordinates \((u, v) := \left(\frac{1}{\alpha}, \frac{1}{\beta}\right)\) is used to view (\(\star \star\)) in terms of Beatty sets. This reveals a hidden symmetry

\[
(u, v) \mapsto (v, u)
\]

of problem (\(\star \star\)) which holds in the first quadrant.

![Figure 5: First quadrant solutions to (\(\star \star\))]