1. Let $E \subset \mathbb{R}^1$. Show that the characteristic function $\chi_E(x)$ is the limit of a sequence of continuous functions if and only if $E$ is both $F_\sigma$ and $G_\delta$.

Proof. $(\Rightarrow)$ First, suppose $\chi_E(x)$ is the (pointwise) limit of a sequence $\{f_n\}$ of continuous functions. Let $U_n$ be the open set $\{x : f_n(x) > 1/2\} \subset \mathbb{R}$, and let $V_n$ be the closed set $\{x : f_n(x) \geq 1/2\}$. Then define

$$
\tilde{U}_n = \bigcup_{i \geq n} U_i = \{x : f_i(x) > 1/2 \text{ for some } i \geq n\},
$$

$$
\tilde{V}_n = \bigcap_{i \geq n} V_i = \{x : f_i(x) \geq 1/2 \text{ for all } i \geq n\}.
$$

Note that each $\tilde{U}_n$ (resp. $\tilde{V}_n$) is open (closed) because it is a union of open sets (intersection of closed sets). We claim that $E = \bigcap_{n \geq 1} \tilde{U}_n$, which shows that $E$ is $G_\delta$. Indeed,

$$
\bigcap_{n \geq 1} \tilde{U}_n = \bigcap_{n \geq 1} \{x : f_i(x) > 1/2 \text{ for some } i \geq n\}
$$

$$
= \{x : \limsup_{n \to \infty} f_n(x) > 1/2\}
$$

$$
= \{x : \chi_E(x) > 1/2\} = E
$$

by our assumption that $f_n \to \chi_E$ pointwise. We claim also that $E = \cup_{n \geq 1} \tilde{V}_n$, which shows $E$ is $F_\sigma$. Indeed,

$$
\bigcup_{n \geq 1} \tilde{V}_n = \bigcup_{n \geq 1} \{x : f_i(x) \geq 1/2 \text{ for all } i \geq n\}
$$

$$
= \{x : \liminf_{n \to \infty} f_n(x) \geq 1/2\}
$$

$$
= \{x : \chi_E(x) \geq 1/2\} = E.
$$

$(\Leftarrow)$ Now suppose that $E$ is both $F_\sigma$ and $G_\delta$. Let $V_n$ be a sequence of closed sets such that $E = \cup_{n \geq 1} V_n$ and let $U_n$ be a sequence of open sets such that $E = \cap_{n \geq 1} U_n$. Without loss of generality, we may assume that the $V_n$ are increasing, i.e. $V_n \subset V_{n+1}$, by replacing the sequence with $\tilde{V}_n = \bigcup_{i=1}^n V_i$. Similarly, we may assume that $U_n \supset U_{n+1}$. For each $n$, we have

$$
V_n \subset E \subset U_n.
$$

We claim that there exists a continuous function $f_n$, for each $n$, such that

$$
f_n(x) = \begin{cases} 
1 & \text{if } x \in V_n, \\
0 & \text{if } x \notin U_n.
\end{cases}
$$

This follows from Urysohn’s lemma, since $V_n$ and $U_n^c$ are disjoint closed subsets of $\mathbb{R}$, and $\mathbb{R}$ is a metric space and thus normal (i.e. disjoint closed sets are separated by disjoint open neighborhoods). The sequence $\{f_n\}$ converges pointwise to $\chi_E$, as desired. $\square$
2. Let \( \{g_n\} \) be a sequence of measurable functions on \([a, b]\), satisfying

(a) \( |g_n(x)| \leq M \), a.e. \( x \in [a, b] \);
(b) for every \( c \in [a, b] \), \( \lim_{n \to \infty} \int_a^c g_n(x)dx = 0 \).

Show that for any \( f \in L^1[a, b] \),
\[
\lim_{n \to \infty} \int_a^b f(x)g_n(x)dx = 0.
\]

**Proof.** We first observe that condition (b) implies that for any interval \([c_1, c_2] \subset [a, b] \), we have \( \lim_{n \to \infty} \int_{c_1}^{c_2} g_n(x)dx = 0 \) since
\[
\lim_{n \to \infty} \left| \int_{c_1}^{c_2} g_n(x)dx \right| = \lim_{n \to \infty} \left| \int_a^{c_2} g_n(x)dx - \int_a^{c_1} g_n(x)dx \right| \\
\leq \lim_{n \to \infty} \left| \int_a^{c_2} g_n(x)dx \right| + \lim_{n \to \infty} \left| \int_a^{c_1} g_n(x)dx \right| = 0.
\]

We next claim that for any measurable set \( E \subset [a, b] \), \( \lim_{n \to \infty} \int_E g_n(x)dx = 0 \). Indeed, the previous observation implies this is true for any finite union of intervals. For any \( \epsilon > 0 \), there is some finite union of intervals \( \tilde{E} \supset E \) such that \( \mu(\tilde{E} - E) < \epsilon \), so
\[
\lim_{n \to \infty} \left| \int_E g_n(x)dx \right| = \lim_{n \to \infty} \left| \int_{\tilde{E}} g_n(x)dx - \int_{\tilde{E} - E} g_n(x)dx \right| \\
\leq \lim_{n \to \infty} \left| \int_{\tilde{E}} g_n(x)dx \right| + \lim_{n \to \infty} \int_{\tilde{E} - E} |g_n(x)|dx \leq M\epsilon
\]
so as we let \( \epsilon \to 0 \) we see that the limit must be 0.

Now we show that the desired equality holds for any simple function \( \tilde{f} \in L^1[a, b] \). Indeed, if \( \tilde{f} = \sum_{k=1}^m a_k\chi_{E_k} \) for some measurable sets \( E_k \subset [a, b] \) and \( a_k \in \mathbb{R} \) then
\[
\lim_{n \to \infty} \left| \int_a^b \tilde{f}(x)g_n(x)dx \right| = \lim_{n \to \infty} \left| \int_a^b \left( \sum_{k=1}^m a_k\chi_{E_k}(x) \right) g_n(x)dx \right| = \lim_{n \to \infty} \sum_{k=1}^m |a_k| \int_{E_k} g_n(x)dx \\
\leq \lim_{n \to \infty} \sum_{k=1}^m |a_k| \int_{E_k} g_n(x)dx = \sum_{k=1}^m |a_k| \lim_{n \to \infty} \int_{E_k} g_n(x)dx = 0.
\]

Finally, since simple functions are dense in \( L^1[a, b] \), for any \( \delta > 0 \) there is a simple function \( \tilde{f} \) such that \( \|f - \tilde{f}\|_1 < \delta \). Then
\[
\lim_{n \to \infty} \left| \int_a^b f(x)g_n(x)dx \right| = \lim_{n \to \infty} \left| \int_a^b \tilde{f}(x)g_n(x)dx + \int_a^b (f - \tilde{f})(x)g_n(x)dx \right| \\
\leq \lim_{n \to \infty} \left| \int_a^b \tilde{f}(x)g_n(x)dx \right| + \lim_{n \to \infty} \int_a^b |(f - \tilde{f})(x)| \cdot |g_n(x)|dx \\
\leq 0 + M \int_a^b |(f - \tilde{f})(x)|dx < M\delta,
\]
so the claim follows from letting \( \delta \to 0 \). \( \square \)
3. Let \( f_k(x), k = 1, 2, \ldots \) be increasing functions on \([a, b]\). Assume
\[
\sum_{k=1}^{\infty} f_k(x)
\]
is convergent on \([a, b]\). Show that
\[
\left( \sum_{k=1}^{\infty} f_k(x) \right)' = \sum_{k=1}^{\infty} f_k'(x), \quad \text{a.e. } x \in [a, b].
\]

Proof. Let \( F(x) = \sum_{k=1}^{\infty} f_k(x) \), and let \( T_n(x) = \sum_{k=n}^{\infty} f_k(x) \) denote the tail of this summation, so that
\[
F(x) = \sum_{k=1}^{n-1} f_k(x) + T_n(x)
\]
for any \( n \geq 1 \). Taking derivatives, we have
\[
F'(x) = \sum_{k=1}^{n-1} f_k'(x) + T'_n(x),
\]
so it suffices to show that as \( n \to \infty \), \( T'_n(x) \to 0 \) for a.e. \( x \).

Note that since each \( f_k \) is increasing all the derivatives \( f_k', F', T'_n \) are non-negative. Thus for fixed \( x \), the sequence \( \{T'_n(x)\}_n \) is monotonically decreasing and bounded below by 0, so for each \( x \), \( \lim_{n \to \infty} T'_n(x) = \liminf_{n \to \infty} T'_n(x) \) exists. (We include the possibility that this limit is \( +\infty \).) By Fatou’s lemma,
\[
\int_a^b (\liminf_{n \to \infty} T'_n(x)) dx \leq \liminf_{n \to \infty} \int_a^b T'_n(x) dx,
\]
and by the fundamental theorem of calculus, since \( T_n \) is increasing we have
\[
\int_a^b T'_n(x) dx \leq T_n(b) - T_n(a).
\]
We are given that the infinite summation defining \( F(x) \) converges, so for all \( x \in [a, b] \) we have \( T_n(x) \to 0 \) as \( n \to \infty \). Thus
\[
\int_a^b (\lim_{n \to \infty} T'_n(x)) dx \leq \liminf_{n \to \infty} \int_a^b T'_n(x) dx \leq \liminf_{n \to \infty} (T_n(b) - T_n(a)) = 0,
\]
and since the first function is non-negative this implies \( \lim_{n \to \infty} T'_n(x) \equiv 0 \) a.e., as desired. \( \Box \)

4. (a) Assume that \( f \in L^\infty(\mathbb{R}) \), and \( f \) is continuous at 0. Show that
\[
\lim_{n \to \infty} \int_a^b \frac{n}{(1 + nx)^2} f(x) dx = f(0).
\]
(b) Assume that $f \in L^\infty(\mathbb{R})$. Show that
\[
\lim_{n \to \infty} \int \frac{n}{\pi(1 + n^2(x - y)^2)} f(y) dy = f(x) \quad \text{a.e. } x \in \mathbb{R}.
\]
(Hint: $\int \frac{1}{\pi(1+y^2)} dy = 1.$)

Proof. (a) By the change of variables $u = nx$,
\[
\lim_{n \to \infty} \int \frac{n}{\pi(1 + (nx)^2)} f(x) dx = \lim_{n \to \infty} \int \frac{f(u/n)}{\pi(1 + u^2)} du,
\]
so it suffices to show that the right-hand limit is equal to $f(0)$. Since $f \in L^\infty(\mathbb{R})$, there is some $M < \infty$ such that $|f(x)| < M$ a.e., and since $f$ is continuous at 0, for any $\epsilon > 0$ there is some $\delta$ such that $|f(x) - f(0)| < \epsilon$ for any $-\delta < x < \delta$. (In particular, this implies $|f(0)| < M$.) Following the hint, we have $\int \frac{f(u/n)}{\pi(1+y^2)} dy = f(0)$ so
\[
\left| f(0) - \int \frac{f(u/n)}{\pi(1 + u^2)} du \right| = \left| \int f(0) - f(u/n) \frac{1}{\pi(1 + u^2)} du \right|
\leq \int_{|u| < n\delta} \left| \frac{f(0) - f(u/n)}{\pi(1 + u^2)} \right| du + \int_{|u| \geq n\delta} \left| \frac{f(0) - f(u/n)}{\pi(1 + u^2)} \right| du
\leq \int_{|u| < n\delta} \frac{\epsilon}{\pi(1 + u^2)} du + \int_{|u| \geq n\delta} \frac{2M}{\pi(1 + u^2)} du
\leq \epsilon + 2M \int_{|u| \geq n\delta} \frac{1}{\pi(1 + u^2)} du,
\]
which holds for arbitrary $n$. As $n \to \infty$, it is clear that the last integral approaches 0, so we have
\[
\lim_{n \to \infty} \left| f(0) - \int \frac{f(u/n)}{\pi(1 + u^2)} du \right| = \left| f(0) - \lim_{n \to \infty} \int \frac{f(u/n)}{\pi(1 + u^2)} du \right| \leq \epsilon.
\]
Since this holds for arbitrary $\epsilon$, this shows $f(0) = \lim_{n \to \infty} \int \frac{f(u/n)}{\pi(1+y^2)} du$ as desired.

(b) Since $f$ is measurable, by Luzin’s theorem for any $\epsilon > 0$ there is a continuous function $\hat{f}$ and a closed set $E \subset \mathbb{R}$ such that $f(x) = \hat{f}(x)$ for all $x \in E$ and $\mu(E^c) < \epsilon$. Since $|f(x)| < M$ a.e. we may also choose $\hat{f}$ to have this same bound. Then for any $x$
\[
\lim_{n \to \infty} \int \frac{n}{\pi(1 + n^2(x - y)^2)} \hat{f}(y) dy = \hat{f}(x)
\]
by continuity of $\hat{f}$ (using the same argument as in part (a)), so for any $x \in E$,
\[
\left| f(x) - \lim_{n \to \infty} \int \frac{n}{\pi(1 + n^2(x - y)^2)} f(y) dy \right| \leq \lim_{n \to \infty} \int \frac{n}{\pi(1 + n^2(x - y)^2)} |\hat{f}(y) - f(y)| dy
\]
\[ \lim_{n \to \infty} \left( \int_E \frac{n|\tilde{f}(y) - f(y)|}{\pi(1 + n^2(x-y)^2)} \, dy + \int_{E^c} \frac{n|\tilde{f}(y) - f(y)|}{\pi(1 + n^2(x-y)^2)} \, dy \right) \leq \lim_{n \to \infty} \left( 0 + 2M \int_{E^c} \frac{n}{\pi(1 + n^2(x-y)^2)} \, dy \right) \]

If we choose \( x \in \text{int}(E) \), the interior of \( E \), so \( E \) contains the open interval \((x - \delta, x + \delta)\) for some \( \delta > 0 \), then
\[ \lim_{n \to \infty} \int_{E^c} \frac{n}{\pi(1 + n^2(x-y)^2)} \, dy = 0, \]
so \( f(x) = \lim_{n \to \infty} \int \frac{n}{\pi(1 + n^2(x-y)^2)} f(y) \, dy \) for any \( x \in \text{int}(E) \). Then the measure of the points in \( \mathbb{R} \) where this equality does not hold is bounded above by \( \mu(E^c) = \mu(E^c) < \epsilon \), and since \( \epsilon \) was arbitrary this equality is true almost everywhere.

5. Let \( \{f_n\} \) be a sequence of functions in \( L^p(\mathbb{R}^n) \), \( 1 < p < \infty \), which converge almost everywhere to a function \( f \in L^p(\mathbb{R}^n) \), and suppose that there is a constant \( M \) such that \( \|f_n\|_p \leq M \) for all \( n \). Show that for every \( g \in L^q(\mathbb{R}^n) \), \( q \) the conjugate of \( p \),
\[ \int fg = \lim_{n \to \infty} \int f_ng. \]
Is the statement true for \( p = 1 \)?
(Hint: you may want to use Egorov’s theorem.)

**Proof.** We first show the statement is false for \( p = 1 \). Indeed, consider the functions \( f_n \) on \( \mathbb{R} \) defined by
\[ f_n(x) = \begin{cases} 1/n & \text{if } 0 < x < n \\ 0 & \text{otherwise.} \end{cases} \]
Then \( \|f_n\|_1 = 1 \) for all \( n \), and the sequence \( \{f_n\} \) converges pointwise to the zero function \( f = 0 \). Taking the constant function \( g = 1 \in L^\infty(\mathbb{R}) \), we have
\[ 0 = \int fg = \lim_{n \to \infty} \int f_ng = 1. \]

Now we prove the claim for \( 1 < p < \infty \). The proof relies on the fact that for any \( h \in L^1(\mathbb{R}^n) \),
\[ \int_{\mathbb{R}^n} h = \lim_{R \to \infty} \int_{B_R} h \Leftrightarrow \lim_{R \to \infty} \int_{\mathbb{R}^n - B_R} h = 0 \]
where \( B_R \subset \mathbb{R}^n \) denotes the ball of radius \( R \) around the origin in \( \mathbb{R}^n \), and
\[ \lim_{\epsilon \to 0} \int_{E_\epsilon} h = 0 \]
where \( E_\epsilon \) is a measureable set whose measure is bounded by \( \epsilon \).
By Egorov’s theorem, for any $\epsilon > 0$ (and $R > 0$ fixed) there is a subset $E_\epsilon \subset B_R$ such that $\mu(E_\epsilon) \leq \epsilon$ and the convergence $f_n \to f$ is uniform on $B_R - E_\epsilon$. Then

$$\left| \int f g - \int f_n g \right| \leq \int |f - f_n| |g|$$

$$= \int_{B_R - E_\epsilon} |f - f_n||g| + \int_{E_\epsilon} |f - f_n||g| + \int_{R^n - B_R} |f - f_n||g|.$$ 

We may use Hölder’s inequality to replace the second and third integrals above with expressions independent of $n$, namely

$$\int_A |f - f_n||g| \leq \|f - f_n\|_p \|g\|_q = \left( \int_A |f - f_n|^p \right)^{1/p} \left( \int_A |g|^q \right)^{1/q}$$

$$\leq (\|f\|_p + \|f_n\|_p) \|g\|_q \leq (\|f\|_p + M) \left( \int_A |g|^q \right)^{1/q},$$

and as $n \to \infty$ the first integral (on $B_R - E_\epsilon$) goes to zero by uniform convergence on a finite measure space. Thus

$$\lim_{n \to \infty} \left| \int f g - \int f_n g \right| \leq (\|f\|_p + M) \left( \int_{E_\epsilon} |g|^q \right)^{1/q} + (\|f\|_p + M) \left( \int_{R^n - B_R} |g|^q \right)^{1/q}.$$ 

This holds for arbitrary $R, \epsilon$ so taking $\epsilon \to 0$ and $R \to \infty$, we have (since $h = |g|^q \in L^1$)

$$\lim_{n \to \infty} \left| \int f g - \int f_n g \right| = 0,$$

so $\int f g = \lim_{n \to \infty} \int f_n g$ as desired. \qed
1. Construct an explicit analytic bijection from
\[ \{ z \in \mathbb{C} : |z| > 1, z \text{ not real and positive} \} \]
to
\[ \{ z \in \mathbb{C} : \text{Re } z > 0 \}. \]
(You may write your mapping as a composition of simpler explicit mappings.)

**Proof.** Take \( f(z) = (-iz)^{1/2}(z + 1/z) \circ (\sqrt{z}) = \frac{1}{2i}(z^{1/2} + z^{-1/2}) \), where we choose the branch of the square root that is positive on positive real numbers. \( \Box \)

2. Let \( A = \{ z \in \mathbb{C} : 5 \leq |z| \leq 10 \} \).

(a) Prove or disprove: there is a function \( f \) analytic on a neighborhood of \( A \) and satisfying \( |f(z)| < 1 \) for \( |z| = 10 \), \( |f(z)| > 1000 \) for \( |z| = 5 \).

(b) Prove or disprove: there is a function \( f \) analytic on a neighborhood of \( A \) and satisfying \( \text{Re } f(z) < 1 \) for \( |z| = 10 \), \( \text{Re } f(z) > 1000 \) for \( |z| = 5 \).

**Proof.** (a) We show such an \( f \) exists: consider \( f(z) = (9/|z|)^n \) for some positive integer \( n \). On \( |z| = 10 \), \( |f(z)| = (9/10)^n < 1 \), and on \( |z| = 5 \), \( |f(z)| = (9/5)^n \) will be larger than 1000 for sufficiently large \( n \) (e.g. \( n \geq 18 \)).

(b) We claim no such \( f \) exists. Indeed, since \( A \) is connected its image \( f(A) \) will be a connected subset of \( \mathbb{C} \). Let \( L \subset \mathbb{C} \) denote the complex numbers with real part = 500. Since \( A \) is compact, its image \( f(A) \) is also compact and thus closed. This implies the intersection \( L \cap f(A) \) is closed. By the open mapping theorem, the interior \( \text{int}(A) \) must be sent by \( f \) to an open set \( f(\text{int}(A)) \subset \text{int}(f(A)) \). Thus \( L \cap f(\text{int}(A)) \) is open in the induced subspace topology of \( L \). We identify two possible cases: either \( L \cap f(\text{int}(A)) \subset L \cap f(A) \), or \( L \cap f(\text{int}(A)) = L \cap f(A) \).

In the first case,
\[ L \cap f(\partial A) = L \cap f(A - \text{int}(A)) \supset (L \cap f(A)) - (L \cap f(\text{int}(A))) \]
is non-empty so some point \( z \) in the boundary of \( A \) (so \( |z| = 5 \) or 10) must be sent to \( f(z) \) with real part = 500, contradicting the given hypotheses for \( f \). In the second case, the intersection \( L \cap f(\text{int}(A)) = L \cap f(A) \) is both open and closed so it is either empty or the entire line \( L \). It cannot be the entire line because \( f(A) \) is compact so it must be empty. Then since \( f(A) \) is connected, it must lie entirely on one side of \( L \) (e.g. \( \text{Re } f(z) < 500 \)) or on the other (\( \text{Re } f(z) > 500 \)), which also contradicts the hypotheses. \( \Box \)

3. For \( f \) analytic on \( \mathbb{D} \) let
\[ \sigma(f) = \sup\{|f^{-1}(w) : w \in \mathbb{C}|. \]
4. Let $f$ be an analytic function defined on a neighborhood of $\mathbb{D}$ and satisfying
- $f(0) = 0$.
- $f(0) = 0$.

Let $f^n = f \circ \cdots \circ f$, $n$ times. Show that $f^n(z) \to 0$ for $z \in \mathbb{D}$.

**Proof.** Consider the function $g(z) = f(z)/z$. Since $f(0) = 0$, $g$ is analytic on $\overline{\mathbb{D}}$, and on the boundary $\partial \overline{\mathbb{D}} = \{z : |z| = 1\}$, the magnitudes of $f(z)$ and $g(z)$ coincide. Thus

$$\max_{|z|=1} |f(z)/z| = \max_{|z|=1} |f(z)| = \max_{|z|=1} |f(z)|.$$

Call this maximum $\lambda$ (which exists by compactness of $\partial \overline{\mathbb{D}}$); since $f(0) = 0$ we have $0 \leq \lambda < 1$. By the maximum modulus principle,

$$|g(z)| = |f(z)/z| \leq \lambda \quad \text{for all } z \in \overline{\mathbb{D}}.$$
But this is equivalent to \(|f(z)| \leq \lambda |z|\) for all \(z \in \mathbb{D}\), and by induction this implies
\[
|f^{\circ n}(z)| \leq \lambda^n |z| \to 0 \quad \text{as } n \to \infty, \quad (z \in \mathbb{D} \text{ fixed})
\]
so \(f^{\circ n}(z) \to 0\) as desired. \(\square\)

5. Suppose \(\{f_n\}\) is a uniformly bounded sequence of analytic functions on a domain \(\Omega\) such that \(\{f_n(z)\}\) converges for every \(z \in \Omega\).

(a) Show that the convergence is uniform on every compact subset of \(\Omega\).

(b) Must \(\{f'_n\}\) converge uniformly on every compact subset of \(\Omega\)? Prove or disprove.

**Proof.** (a) Fix a compact subset \(K \subset \Omega\). Suppose for a contradiction that the convergence \(f_n \to f\) is not uniform on \(K\). Then for some \(\epsilon > 0\), there are infinitely many \(f_n\) such that
\[
\sup_{z \in K} |f_n(z) - f(z)| > \epsilon.
\]
Let \(\{f_{n_k}\}\) consist of all such functions. Then \(\{f_{n_k}\}\) is also uniformly bounded, so by Montel’s theorem there is a subsequence \(f_{n_{kj}}\) that converges uniformly to some function \(g\) analytic on \(K\), i.e.
\[
\lim_{j \to \infty} \left( \sup_{z \in K} |f_{n_{kj}}(z) - g(z)| \right) = 0.
\]
Then \(f_{n_{kj}} \to g\) pointwise, and since \(f_{n_{kj}}\) is also a subsequence of \(\{f_n\}\), which converges pointwise to \(f\), we must have \(g = f\). But this contradicts our assumption that
\[
\sup_{z \in K} |f_{n_{kj}}(z) - g(z)| = \sup_{z \in K} |f_{n_{kj}}(z) - f(z)| > \epsilon
\]
for all \(j\), so we must have uniform convergence on \(K\) as desired.

(b) Yes; \(\{f'_n\}\) must converge uniformly on compact subsets as well. Indeed, for a fixed compact subset \(K \subset \Omega\) we may choose an open neighborhood \(K \subset \Omega_1 \subset \Omega\) such that \(\Omega_1\) has compact closure; let \(K_1\) denote this closure. Then take another open neighborhood \(K_1 \subset \Omega_2 \subset \Omega\) with compact closure \(K_2\). (We may assume \(\Omega_1\) and \(\Omega_2\) are connected.) Then the distance \(|z - w|\) for \(z \in \Omega_1, w \in \partial K_2\) is bounded below by some constant \(\epsilon > 0\).

Moreover, the hypotheses on \(f_n\) gives a uniform bound \(|f_n(z)| < M\) for all \(z \in K_2\) and all \(n\) (where \(M\) may depend on \(K_2\)). By Cauchy’s integral formula,
\[
f'_n(z) = \frac{1}{2\pi i} \oint_{\partial K_2} \frac{f_n(\xi)}{\xi - z} d\xi
\]
for any \(z \in \Omega_1\), so taking magnitudes
\[
|f'_n(z)| \leq \frac{1}{2\pi} \oint_{\partial K_2} \frac{|f_n(\xi)|}{|\xi - z|^2} d\xi \leq \frac{1}{2\pi} |\partial K_2| \frac{M}{\epsilon^2} =: M'
\]
which shows that \( \{ f_n' \} \) is uniformly bounded on \( \Omega_1 \). Also by Cauchy’s formula,

\[
|f_n'(z) - f'(z)| = \left| \frac{1}{2\pi i} \oint_{\partial K_2} \frac{f_n(\xi) - f(\xi)}{(\xi - z)^2} d\xi \right|
\leq \frac{1}{2\pi} |\partial K_2| \sup_{K_2} \left| f_n(\xi) - f(\xi) \right| \to 0 \quad \text{as } n \to \infty
\]

by uniform convergence of \( f_n \to f \), so \( f_n' \to f' \) pointwise on \( \Omega_1 \). Thus we have uniform convergence of \( f_n' \) on \( K \) by the argument in (a).