Morning session

1. Construct a measurable subset $A$ of $(0, 1)$ such that $m(A) < 1$ and $m(A \cap (a, b)) > 0$ for any $(a, b) \subset (0, 1)$.

Proof. Let $\{q_n\} = \{q_1, q_2, \ldots\}$ be an enumeration of the rational numbers in the interval $(0, 1)$, and let $E_n$ be the open interval

$$E_n = (q_n - 5^{-n}, q_n + 5^{-n}) \cap (0, 1).$$

Then $A = \bigcup_n E_n \subset (0, 1)$ is measurable, as a countable union of measurable sets, and

$$m(A) \leq \sum_{n \geq 1} m(E_n) \leq \sum_{n \geq 1} 2 \cdot 5^{-n} = 1/2.$$

Finally any open interval $(a, b) \subset (0, 1)$ contains some rational $q_n$, so the intersection $E_n \cap (a, b)$ has positive measure and thus $m(A \cap (a, b))$ is positive as well. Thus $A$ satisfies the specified conditions.

2. Let $\{f_k(x)\}$ be a sequence of non-negative measurable functions on $E$ and $m(E) < \infty$. Show that $\{f_k(x)\}$ converges in measure to 0 if and only if

$$\lim_{k \to \infty} \int_E \frac{f_k(x)}{1 + f_k(x)} \, dx = 0.$$

Proof. ($\Rightarrow$): Suppose $f_k \to 0$ in measure, meaning that for any $\epsilon > 0$

$$m(\{x : |f_k(x)| \geq \epsilon\}) \to 0 \quad \text{as} \quad k \to \infty.$$

Then since $y \mapsto \frac{y}{1+y} : \mathbb{R}_{\geq 0} \to [0, 1)$ is monotonically increasing,

$$\int_E \frac{f_k(x)}{1 + f_k(x)} \, dx = \int_{|f_k| < \epsilon} \frac{f_k(x)}{1 + f_k(x)} \, dx + \int_{|f_k| \geq \epsilon} \frac{f_k(x)}{1 + f_k(x)} \, dx \leq \int_{|f_k| < \epsilon} \frac{\epsilon}{1 + \epsilon} \, dx + \int_{|f_k| \geq \epsilon} 1 \, dx \leq m(E) \frac{\epsilon}{1 + \epsilon} + m(|f_k| \geq \epsilon).$$

Taking $k \to \infty$, we see that $\lim_{k \to \infty} \int_E \frac{f_k(x)}{1 + f_k(x)} \, dx \leq m(E) \frac{\epsilon}{1 + \epsilon}$, and as $\epsilon \to 0$ this bound goes to zero, so the limit is zero as claimed.

($\Leftarrow$): Observe that for any $\epsilon > 0$,

$$\int_E \frac{f_k(x)}{1 + f_k(x)} \, dx = \int_{|f_k| \geq \epsilon} \frac{f_k(x)}{1 + f_k(x)} \, dx + \int_{|f_k| < \epsilon} \frac{f_k(x)}{1 + f_k(x)} \, dx \geq \int_{|f_k| \geq \epsilon} \frac{\epsilon}{1 + \epsilon} \, dx = m(|f_k| \geq \epsilon) \frac{\epsilon}{1 + \epsilon} > 0.$$
3. Let $1 \leq p < \infty$, $f \in L^p(\mathbb{R}^n)$. Let

$$f_*(\lambda) = m(\{x : |f(x)| > \lambda\}), \quad \lambda > 0.$$

Show that

(a) Suppose $\lim_{\lambda \to \infty} \lambda^p f_*(\lambda) = 0$, $\lim_{\lambda \to 0} \lambda^p f_*(\lambda) = 0$.

(b) Given a sequence of integrals converges to zero, then (ignoring the non-zero constant factor $\frac{c}{1+\varepsilon}$)

$$\lim_{k \to \infty} m(x : |f_k(x)| \geq \varepsilon) = 0$$

as well. Thus $f_k \to 0$ in measure. 

Thus the desired equality holds for simple functions. For an arbitrary $f \in L^p(\mathbb{R}^n)$, we may find a monotonically increasing sequence of simple functions $g_n$ that converge pointwise to $f$, so that $\|g_n\|_p \to \|f\|_p$ by monotone convergence. Under these conditions $g_n \leq f$,

$$\{x : |g_n(x)| > \lambda\} \subset \{x : |f(x)| > \lambda\} \Rightarrow (g_n)_*(\lambda) \leq f_*(\lambda)$$

for all $\lambda$, so the sequence $\lambda^{p-1}(g_n)_*$ will also converge monotonically to $\lambda^{p-1}f_*$. By monotone convergence in $L^1(\mathbb{R})$, $\|\lambda^{p-1}(g_n)_*\|_1 \to \|\lambda^{p-1}f_*\|_1$. Thus

$$\int |f(x)|^p dx = \lim_{n \to \infty} \int |g_n(x)|^p dx = \lim_{n \to \infty} p \int_0^\infty \lambda^{p-1}(g_n)_*(\lambda) d\lambda = p \int_0^\infty \lambda^{p-1}f_*(\lambda) d\lambda$$

as desired.

(b) Given a sequence $\lambda_n \to 0$ of positive numbers, define $g_n = \lambda_n \cdot \chi_{|f| > \lambda_n}$. It is clear that $g_n$ is non-negative, bounded above by $|f|$, and converges pointwise (in fact, uniformly) to 0. Thus by dominated convergence

$$\lim_{n \to \infty} \int (g_n)^p dx = 0.$$
But
\[ \lambda_n^p f_*(\lambda_n) = \int \lambda_n^p \cdot \chi_{\{|f| > \lambda_n\}} \, dx = \int (g_n)^p \, dx \]
so the above convergence of integrals implies \( \lim_{n \to \infty} \lambda_n^p f_*(\lambda_n) = 0 \). Since this holds for any \( \lambda_n \to 0 \), we may conclude that \( \lim_{\lambda \to 0} \lambda^p f_*(\lambda) = 0 \).

To see that \( \lim_{\lambda \to \infty} \lambda^p f_*(\lambda) = 0 \), simply repeat the above argument with \( \lambda_n \to \infty \). (The convergence \( g_n \to 0 \) is no longer uniform, but it is still dominated.)

4. Let \( K = \{ f : (0, +\infty) \to \mathbb{R} \mid \int_0^\infty f^4(x) \, dx \leq 1 \} \). Evaluate
\[ \sup_{f \in K} \int_0^\infty f^3(x) e^{-x} \, dx. \]

**Proof.** By Holder’s inequality, for \( p = 4/3 \) and \( q = 4 \),
\[ \int_0^\infty |f^3(x) e^{-x}| \, dx \leq \left( \int_0^\infty |f^3(x)|^{4/3} \, dx \right)^{3/4} \left( \int_0^\infty e^{-4x} \, dx \right)^{1/4} = \left( \int_0^\infty |f^4(x)| \, dx \right)^{3/4} \left( \frac{1}{4} \right)^{1/4}. \]
Thus for any \( f \in K \), the above integral is bounded above by \( 1/\sqrt{2} \). This bound is achieved when \( f(x) = \sqrt{2} e^{-x} \in K \) (i.e. when \( \|f\|_4 = 1 \) and \( f^4 \sim e^{-4x} \) so Holder gives an equality), so
\[ \sup_{f \in K} \int_0^\infty f^3(x) e^{-x} \, dx = \frac{1}{\sqrt{2}}. \]

5. Let \( f : \mathbb{R} \to \mathbb{R} \) be a function such that \( \int_\mathbb{R} |f(x)| \, dx < \infty \). Show that the sequence
\[ h_n(x) = \frac{1}{n} \sum_{k=1}^n f \left( x + \frac{k}{n} \right) \]
converges in \( L^1(\mathbb{R}) \).

**Proof.** Let \( h : \mathbb{R} \to \mathbb{R} \) denote the function \( h(x) = \int_0^1 f(x + \xi) \, d\xi \), which exists by hypothesis that \( f \in L^1(\mathbb{R}) \). By Tonelli’s theorem
\[ \int_\mathbb{R} |h(x)| \, dx = \int_\mathbb{R} \left| \int_0^1 f(x + \xi) \, d\xi \right| \, dx \leq \int_\mathbb{R} \left( \int_0^1 |f(x + \xi)| \, d\xi \right) \, dx = \int_0^1 \left( \int_\mathbb{R} |f(x + \xi)| \, dx \right) \, d\xi = \|f\|_1 < \infty, \]
so \( h \in L^1(\mathbb{R}) \). We claim that \( h_n \to h \) in \( L^1 \). Indeed if \( f \) is Riemann integrable, then we have convergence pointwise by definition of the Riemann integral:

\[
    h(x) = \int_0^1 f(x + \xi) d\xi = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f\left( x + \frac{k}{n} \right) = \lim_{n \to \infty} h_n(x).
\]

To show convergence in \( L^1 \), we will at first assume stronger conditions on \( f \); namely, suppose \( f \) is continuous with bounded support. Then \( f \) is uniformly continuous, so for any \( \epsilon > 0 \) there is some \( \delta > 0 \) such that

\[
    |x - y| < \delta \implies |f(x) - f(y)| < \epsilon.
\]

Then for any \( n > 1/\delta \),

\[
    |h(x) - h_n(x)| = \left| \int_0^1 f(x + \xi) - \frac{1}{n} \sum_{k=1}^{n} f(x + k/n) \right| \leq \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} \left| f(x + \xi) - \frac{1}{n} f(x + k/n) \right| d\xi \leq \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} \epsilon \, d\xi = \epsilon.
\]

Moreover if \( \text{supp}(f) \subset (-R, R) \), then \( h(x) = h_n(x) = 0 \) for any \( x \leq -R - 1 \) or \( x \geq R \), so for \( n \) sufficiently large (depending on \( \epsilon \))

\[
    \|h - h_n\|_1 = \int_{-R}^{R} |h(x) - h_n(x)| \, dx \leq \int_{-R-1}^{R} |h(x) - h_n(x)| \, dx \leq (2R + 1) \epsilon.
\]

Since \( \epsilon \) was arbitrary this shows \( h_n \to h \) in \( L^1 \) under the above assumptions on \( f \).

Now for arbitrary \( f \in L^1 \), there is a sequence \( f^{(m)} \) of continuous, bounded support functions that converge to \( f \) in \( L^1 \). Let \( h_n^{(m)}, h^{(m)} \) denote the corresponding functions for \( f^{(m)} \). It is straightforward to check that

\[
    \|h - h^{(m)}\|_1 \leq \|f - f^{(m)}\|_1 \quad \text{and} \quad \|h_n - h_n^{(m)}\|_1 \leq \|f - f^{(m)}\|_1
\]

under these conditions by exchanging the order of integration, so

\[
    \|h - h_n\|_1 \leq \|h - h^{(m)}\|_1 + \|h^{(m)} - h_n^{(m)}\|_1 + \|h_n^{(m)} - h_n\|_1 \leq \|h^{(m)} - h_n^{(m)}\|_1 + 2\|f - f^{(m)}\|_1.
\]

As \( n \to \infty \) (with \( m \) fixed) this shows

\[
    \lim_{n \to \infty} \|h - h_n\|_1 \leq 2\|f - f^{(m)}\|_1
\]

and by assumption this upper bound goes to zero as \( m \to \infty \). This proves the claim.
Afternoon session

1. Assume that 0 is an isolated singularity of an analytic function $f \neq 0$. Determine the type of the singularity if
\[ \sum_{n=1}^{\infty} |f(1/n)|^{1/n} < +\infty. \]

Proof. If the given series converges, then in particular the terms $|f(1/n)|^{1/n}$ in the summation must approach 0, so for any $1 > \epsilon > 0$ we have
\[ |f(1/n)|^{1/n} < \epsilon \iff |f(1/n)| < \epsilon^n \]
for $n$ sufficiently large. This implies that for any fixed integer $k$,
\[ \lim_{n \to \infty} \frac{|f(1/n)|}{(1/n)^k} \leq \lim_{n \to \infty} \frac{\epsilon^n}{(1/n)^k} = 0. \]
If $f$ has either a removable singularity or a pole at 0, then $f$ has a Taylor series expansion around 0 of the form
\[ f(z) = a_k z^k + a_{k+1} z^{k+1} + \cdots, \]
where $a_k \neq 0$. (If $k \geq 0$ then the singularity is removable; if $k < 0$ it is a pole.) Then for any sequence of points $z_n$ approaching 0, $f(z_n)/z_n^k \to a_k$, so taking magnitudes
\[ \lim_{n \to \infty} \frac{|f(z_n)|}{|z_n|^k} = |a_k| > 0. \]
Considering the sequence $z_n = 1/n \to 0$, this shows the singularity at 0 cannot be removable or a pole if the given series converges, so it must be essential.
To see that it is possible for the given series to converge for some $f$ with an essential singularity at 0, we may take as an example $f(z) = e^{-1/z^2}$.
\[ \left( \sum_{n=1}^{\infty} |e^{-n^2}|^{1/n} = \sum_{n=1}^{\infty} e^{-n} = \frac{1}{e-1}. \right) \]

2. Show that for $x, y \in \mathbb{R}$,
\[ |y| \leq |\sin(x + iy)| \leq e^{|y|}. \]

Proof. Recall that $\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz})$, so
\[ \sin(x + iy) = \frac{1}{2i}(e^{ix}e^{-y} - e^{-ix}e^y). \]
Taking magnitudes and applying the triangle quality in two ways, we have
\[ |\sin(x + iy)| \leq \frac{1}{2} \left( |e^{ix}e^{-y}| + |e^{-ix}e^y| \right) = \frac{1}{2}(e^{-y} + e^y) \leq e^{|y|} \]
since $e^{|y|} = \max\{e^{-y}, e^y\}$, and
\[ |\sin(x + iy)| \geq \frac{1}{2} \left| e^{ix}e^{-y} - e^{-ix}e^y \right| = \frac{1}{2} |e^{-y} - e^y| = \sinh(|y|). \]
so, observing that $|\sinh(y)| = \sinh|y|$, it suffices to prove $|y| \leq \sinh|y|$. Starting from the easier inequality $0 \leq \sinh|y|$, \[
1 = \cosh(0) \leq \cosh(0) + \int_0^{|y|} \sinh(y')dy' = \cosh|y|,
\]
and integrating this inequality again gives us the desired result: \[
|y| = \int_0^{|y|} dy' \leq \int_0^{|y|} \cosh(y')dy' = \sinh(|y|).
\]
(We in fact have the stronger bounds $\sinh|y| \leq |\sin(x + iy)| \leq \cosh(y)$.)

3. Let $a \in (0, 1)$. Find \[
\int_{-\infty}^{\infty} e^{ax} \frac{1}{1 + e^x} dx.
\]

**Proof.** Consider integrating the meromorphic function $f(z) = \frac{e^{az}}{1 + e^z}$ along the rectangular contour with endpoints $-R, R, R + 2\pi i, -R + 2\pi i$, with segments labeled as below:

\[\begin{array}{c}
\gamma_1 & R \rightarrow \infty & R \\
\gamma_4 & -R & \rightarrow -\infty \gamma_3 \rightarrow 0 \rightarrow R \\
& y \uparrow & \gamma_2
\end{array}\]

The poles of $f$ occur where $z = (2k + 1)\pi i$ for $k \in \mathbb{Z}$, so the only pole in our contour occurs at $z = \pi i$. The residue at this pole is \[
\text{Res}_f(\pi i) = \lim_{z \to \pi i} \frac{(z - \pi i)e^{az}}{1 + e^z} = \lim_{z \to \pi i} \frac{a(z - \pi i)e^{az} + e^{az}}{e^z} = -e^{a\pi i}.
\]

The integral along $\gamma_1$ as $R \to \infty$ is the value we are asked to find: \[
I = \int_{-\infty}^{\infty} e^{ax} \frac{1}{1 + e^x} dx = \lim_{R \to \infty} \int_{\gamma_1} f dz.
\]

The integral along $\gamma_3$ we can relate to the integral along $\gamma_1$: \[
\int_{\gamma_3} f dz = \int_R^{-R} \frac{e^{a(z + 2\pi i)}}{1 + e^{z+2\pi i}} dz = -e^{2a\pi i} \int_{-R}^{R} \frac{e^{ax}}{1 + e^x} dx = -e^{2a\pi i} \int_{\gamma_1} f dz,
\]

and the integrals along $\gamma_2$ and $\gamma_4$ go to zero as $R \to \infty$: \[
\left| \int_{\gamma_2} f dz \right| = \left| \int_0^{2\pi} \frac{e^{a(R + iy)}}{1 + e^{R+iy}} dy \right| \leq \int_0^{2\pi} \frac{e^{aR}}{|1 + e^{R+iy}|} dy \leq 2\pi \left( \frac{e^{aR}}{e^R - 1} \right)
\]
and
\[ \lim_{R \to \infty} \frac{e^{aR}}{e^R - 1} = \lim_{R \to \infty} \frac{1}{e^{(1-a)R} - e^{-aR}} = 0 \] since \( 0 < a < 1 \).

Thus
\[ \lim_{R \to \infty} \int_{\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4} f(z)dz = (1 - e^{2a\pi i})I \]

which by Cauchy's theorem is equal to \( 2\pi i \sum \text{Res} f = -2\pi i e^{a\pi i} \). Thus
\[ I = \frac{-2\pi i e^{a\pi i}}{1 - e^{2a\pi i}} = \frac{\pi}{\sin(a\pi)}. \]

4. Find a conformal mapping of domain \( \mathbb{C}\setminus\{x+xi : 1 \leq x \leq 2\} \) to the upper half plane. (Here \( \mathbb{C} = \mathbb{C} \cup \{\infty\} \).) It is enough to represent the mapping as a composition of several conformal mappings.

**Proof.** We claim the following map works:
\[ f(z) = \sqrt{-\frac{z - (1+i)}{z - (2+2i)}}, \]

where we choose the branch of the square root sending the positive real axis to itself. The expression inside the square root is a Möbius transformation sending the segment \( \{x+xi : 1 \leq x \leq 2\} \) to the positive real axis \( \cup \{\infty\} \), so composing this with the square root will give us the desired behavior, since \( \sqrt{z} \) sends \( \mathbb{C}\setminus\{\text{positive reals}\} \) to the upper half plane conformally.

5. Denote by \( \mathbb{D} \) the unit disk: \( \mathbb{D} = \{z : |z| < 1\} \). Let \( \{f_k : \mathbb{D} \to \mathbb{C}\}_{k \in \mathbb{N}} \) be a normal family. Prove that the functions
\[ g_k(z) = f_k(e^{ik}z), \quad k \in \mathbb{N} \]
form a normal family.

**Proof.** For \( n \in \mathbb{N} \) let \( K_n \subset \mathbb{D} \) denote the compact set
\[ K_n = \{z : |z| \leq 1 - 1/n\}. \]

Since \( \{f_k\} \) is a normal family, \( |f_k(z)| \) must be bounded on each \( K_n \) uniformly as \( k \) varies. Let \( M_n \) be such a bound, so
\[ |f_k(z)| < M_n \quad \text{for all } z \in K_n, \ k = 1, 2, \ldots. \]

We claim this is also a uniform bound for \( \{g_k\} \) on \( K_n \). Indeed, for each fixed \( k, n \), the rotation \( z \mapsto e^{ik}\) sends \( K_n \) to itself so
\[ \sup_{z \in K_n} |g_k(z)| = \sup_{z \in K_n} |f_k(e^{ik}z)| = \sup_{w \in K_n} |f_k(w)| < M_n. \]

Thus \( \{g_k\} \) is a normal family by Montel’s theorem, since it is uniformly bounded on compact subsets. (It is clear that any compact subset of \( \mathbb{D} \) is contained in some \( K_n \).)