Morning session

1. Prove or disprove: If $E$ is an open subset of $\mathbb{R}$ with $m(E) = 1$ then there is a finite union of intervals $F$ containing $E$ with $m(F) < 1.1$.

Proof. Consider the countable union of open intervals $E = \bigcup_{n \geq 1} (n, n + 1/2^n) \subset \mathbb{R}$. It is clear that $m(E) = \sum_{n \geq 1} 1/2^n = 1$. However, if $F$ is a finite union of intervals with $m(F) < \infty$, then $F$ is bounded as a subset of $\mathbb{R}$, so $F$ cannot contain $E$. \hfill \Box

2. Let $f \in L_1 \cap L_4$ (on some measure space). Prove that the function

$$[1, 4] \to \mathbb{R}$$

$$p \mapsto \|f\|_p$$

is continuous.

Proof. Let $N : [1, 4] \to \mathbb{R}$ denote the given “norm” function. It suffices to prove $N(p) = \|f\|_p^p = \int |f|^p$ is continuous, since then we have $N(p) = \exp(\log(N(p))/p)$ is the composition of continuous functions. Let

$$E = \{ x : |f(x)| \leq 1 \} \quad \text{and} \quad F = \{ x : |f(x)| > 1 \}.$$

We first check that $\|f\|_p^p < \infty$ for all $p \in [1, 4]$, so that $N$ is well-defined. Indeed

$$\int |f|^p = \int_E |f|^p + \int_F |f|^p \leq \int_E |f|^p + \int_F |f|^4 \leq \|f\|_1 + \|f\|_4^4 < \infty.$$

Now to show continuity of $N$ at $p \in [1, 4]$, we must prove $N(p + \epsilon_n) \to N(p)$ for any sequence $\epsilon_n \to 0$ (for which $p + \epsilon_n \in [1, 4]$). Given such a sequence, define

$$g_n = |f|^{p+\epsilon_n} - |f|^p \quad \Rightarrow \quad \|g_n\|_1 \geq |N(p + \epsilon_n) - N(p)|.$$

It is clear that $g_n \to 0$ pointwise, and what wish to prove is that $g_n \to 0$ in $L_1$. As $\int |g_n| = \int_E |g_n| + \int_F |g_n|$, it suffices to check this on $E$ and $F$ separately. On $E$, the sequence $g_n$ is dominated by $2|f|$: 

$$|g_n(x)| \leq |f(x)|^{p+\epsilon_n} + |f(x)|^p - 2|f(x)| \quad \text{for any } x \in E,$$

while of $F$, $g_n$ is dominated by $2|f|^4$. Thus the limit of integrals goes to zero as claimed, by dominated convergence. This shows continuity of $N$ as desired. \hfill \Box

3. Find all $q \geq 1$ such that $f(x^2) \in L_q((0, 1), m)$ for any $f(x) \in L_4((0, 1), m)$.

Proof. If $q > 2$, then $f(x) = x^{-1/2q}$ is in $L_4$ but $f(x^2) = x^{-1/q}$ is not in $L_q$. Thus $q \leq 2$ for the given condition to hold. For $q = 2$, consider the function

$$f(x) = \frac{1}{x^{1/4} \log(x/2)^{1/2}}.$$
We claim that \( f \in L_4 \) but \( f(x^2) \notin L_2 \). Indeed

\[
\int_0^1 f(x)^4 \, dx = \int_0^1 \frac{dx}{x \log(x/2)^2} = -\frac{1}{\log(x/2)} \bigg|_0^1 = \frac{1}{\log 2},
\]

while

\[
\int_0^1 f(x^2)^2 \, dx = \int_0^1 \frac{dx}{x |\log(x^2/2)|} = -\frac{1}{2} \log |\log(x^2/2)| \bigg|_0^1 = -\frac{1}{2} \log \log 2 + \infty.
\]

Finally, consider \( 1 \leq q < 2 \). Observe that by substitution \( u = x^2 \),

\[
\int_0^1 f(x^2)^q \, dx = \int_0^1 f(u)^{q/2} \, du,
\]

and the given bounds on \( q \) imply that \( \frac{2}{q} \leq \frac{2}{\sqrt{q}} < 1 \). By applying Holder with conjugate norms \( p_1 = 4/q \) and \( p_2 = 4/(4-q) \),

\[
\int_0^1 \left| f(u)^q \right| u^{1/2} \, du \leq \| f^q \|_{p_1} \cdot \| u^{-1/2} \|_{p_2} = \left( \int_0^1 f(u)^4 \, du \right)^{q/4} \left( \int_0^1 u^{-r} \, du \right)^{1-q/4}
\]

where \( r = \frac{p_2}{2} = \frac{2}{4-q} \)

and these last two integrals are finite. Thus \( 1 \leq q < 2 \) gives the desired range. \( \square \)

4. Let

\[
E \subset \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq x\}
\]

\[
E_x = \{y : (x, y) \in E\}
\]

\[
E_y = \{x : (x, y) \in E\}
\]

and assume that \( m(E_x) \geq x^3 \) for any \( x \in [0, 1] \).

(a) Prove that there exists \( y \in [0, 1] \) such that \( m(E_y) \geq \frac{1}{4} \).

(b) Prove that there exists \( y \in [0, 1] \) such that \( m(E_y) \geq c \), where \( c > 1/4 \) is a constant independent of \( E \). Give an explicit value of \( c \).

Proof. (a) Suppose for a contradiction that \( m(E_y) < 1/4 \) for all \( y \in [0, 1] \). Then

\[
\int_0^1 m(E_y) \, dy < \int_0^1 \frac{1}{4} \, dy = \frac{1}{4},
\]

while by Fubini this integral is equal to

\[
\int_0^1 m(E_x) \, dx \geq \int_0^1 x^3 \, dx = \frac{1}{4}.
\]

This is a contradiction, so we must have \( m(E_y) \geq 1/4 \) for some \( y \).

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(b) Observe that
\[ \int_0^1 m(E^y) dy \leq \int_0^{3/4} m(E^y)dy + \int_{3/4}^1 (1-y)dy = \int_0^{3/4} m(E^y)dy + \frac{1}{32} \]

since \( E_y \subset \{ x : y \leq x \leq 1 \} \) so \( m(E^y) \leq 1 - y \). As observed in part (a) we must have
\[ \int_0^1 m(E^y)dy \geq 1/4, \]
so
\[ \int_0^{3/4} m(E^y)dy \geq \frac{1}{4} - \frac{1}{32} = \frac{7}{32}. \]

This implies the “average value” of \( m(E^y) \) on this interval is at least \( \frac{7}{32}/\frac{3}{4} = \frac{7}{24} \), so we may take \( c = 7/24 \) using the same argument as in part (a).

5. Let \( E \subset [0,1] \) be a measurable set, \( m(E) \geq \frac{99}{100} \). Prove that there exists \( x \in [0,1] \) such that for any \( r \in (0,1) \)
\[ m(E \cap (x-r,x+r)) \geq \frac{r}{4}. \]

Remark: One approach to this problem involves the Hardy-Littlewood maximal inequality

Proof. Let \( f = \chi_{E^c} \), so the Hardy-Littlewood maximal function \( f^* \) is given by
\[ f^*(x) = \sup_{x \in B} \frac{1}{m(B)} \int_B f(y)dy = \sup_{x \in B} \frac{m(B \cap E^c)}{m(B)} = 1 - \inf_{x \in B} \frac{m(B \cap E)}{m(B)} \]
where the supremum is taken over all balls (i.e. intervals) \( B \) containing \( x \). It suffices to find some \( x \) satisfying \( f^*(x) \leq 7/8 \), since then
\[ \inf_{r \in (0,1)} \frac{m(E \cap (x-r,x+r))}{2r} = \inf_{r \in (0,1)} \frac{m(B_r \cap E)}{m(B_r)} \geq \inf_{x \in B} \frac{m(B \cap E)}{m(B)} \geq \frac{1}{8}, \]
where \( B_r(x) = (x-r,x+r) \) denotes the ball of radius \( r \) around \( x \).

The Hardy-Littlewood maximal inequality states that for any \( \alpha > 0 \),
\[ m(x : f^*(x) > \alpha) \leq \frac{3}{\alpha} \| f \|_1. \]

In our case
\[ \frac{3}{\alpha} \| f \|_1 = \frac{3}{\alpha} (1 - m(E)) \leq \frac{1}{\alpha} \frac{3}{100}. \]

Thus for \( \alpha = 7/8 \),
\[ m(x : f^*(x) > 7/8) \leq \frac{8}{7} \cdot \frac{3}{100} < 1 \]
so the complement \( \{ x : f^*(x) \leq 7/8 \} \) has positive measure. In particular, it must be non-empty.

\[ \square \]
Afternoon session

1. Find all entire functions \( f(z) \) with the property that \( g(z) \overset{\text{def}}{=} f(2z + \bar{z}) \) is also entire.

   \[ g(z) = g(x + iy) = f(2z + \bar{z}) = f(3x + iy), \]

   \[ \partial_x g(z) = 3f'(3x + iy) \quad \text{and} \quad -i\partial_y g(z) = f'(3x + iy), \]

   so \( g \) holomorphic at \( x + iy \) implies \( f' \equiv 0 \) at \( 3x + iy \). Thus if \( g \) is entire, this holds for all \( 3x + iy \in \mathbb{C} \) so \( f' \equiv 0 \). This means \( f \) must be constant.

2. How many zeros does the polynomial

\[
p(z) = z^8 + 10z^3 - 50z + 1
\]

have in the right half-plane?

   \[ |g_2(iy)| = 10|y|^3 \quad \text{while} \quad |f_2(iy)| = |y^8 - 50iy + 1| \geq \max(y^8 + 1, 50|y|). \]

   Since

\[
10|y|^3 < y^8 + 1 \quad \text{on} \quad [0, 1/3] \cup [2, \infty) \quad \text{and} \quad 10|y|^3 < 50|y| \quad \text{on} \quad (0, 2],
\]

we see that \( |g_2(z)| < |f_2(z)| \) along the imaginary axis. It is also straightforward to see this inequality holds on a circle of sufficiently large radius \( R \geq 2 \) by degree considerations. Thus for a sufficiently large semi-circular region in the right half-plane, Rouche’s theorem implies that \( p \) has the same number of zeros in this region as \( f_2 \).

We now consider the location of the zeros of \( f_2(z) = z^8 - 50z + 1 \), using Rouche’s theorem a few more times. Write

\[
f_3(z) = z^8 - 50z, \quad g_3(z) = 1 \quad \text{so} \quad f_2 = f_3 + g_3.
\]
It is clear that $f_3(z) = z(z^7 - 50)$ has one zero at the origin and seven roots spaced evenly about the circle of radius $50^{1/7}$ around the origin, starting at the positive real axis. Thus $f_3$ has three roots in the right half-plane, four in the left half-plane, and one in the middle.

On the circle $\{|z| = 1/2\}$, we have $|f_3(z)| \geq 50|z| - |z|^8 > 20$ while $|g_3| = 1$, so $f_2 = f_3 + g_3$ and $f_3$ both have one zero inside this region.

Along the imaginary axis $\{iy : |y| \geq 1/2\}$ outside this circle, $|f_3(iy)| = |y^8 - 50iy| > 20 > |g_3|$, so both $f_2$ and $f_3$ must have three zeros in the region $\{z : \text{Re } z > 0, 1/2 < |z| < R\}$ for sufficiently large $R$.

It remains to decide whether $f_2$’s zero near the origin is in the right or left half-plane. Considering $f_2$ as a function on the real line, we see that it must have a positive, real zero near the origin because it changes sign between $z = 0$ and $z = 1/2$:

$$f_2(0) = 1 > 0, \quad f_2(1/2) < -20 < 0 \quad \Rightarrow \quad f_2(x) = 0 \text{ for some } x \in (0, 1/2).$$

Thus $f_2$ contains four zeros in the right half-plane, as does $p$.

3. Does there exist an analytic function $f$ with an essential singularity at 0 such that $f(z) + 2f(z^2)$ has a removable singularity?

**Proof.** We claim this is not possible. If $f$ has an isolated singularity at 0, we may write the Laurent expansion of $f$ around 0 as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n = \cdots + a_{-1} z^{-1} + a_0 + a_1 z + \cdots.$$ 

Then $f(z) + 2f(z^2)$ has Laurent expansion

$$f(z) + 2f(z^2) = \sum_{n=-\infty}^{\infty} b_n z^n = \sum_{n=-\infty}^{\infty} a_n z^n + \sum_{n=-\infty}^{\infty} 2a_n z^{2n} = \cdots + (a_{-2} + 2a_{-1}) z^{-2} + a_{-1} z^{-1} + 3a_0 + a_1 z + (a_2 + 2a_1) z^2 + \cdots.$$ 

If this function has a removable singularity at 0, then all negative coefficients $\{b_n : n < 0\}$ in the Laurent expansion must vanish. Since $b_n = a_n$ for odd $n$, this means $a_n = 0$ for all odd $n < 0$. But for even coefficients,

$$b_{2n} = a_{2n} + 2a_n$$

so $b_{2n} = a_n = 0$ implies $a_{2n} = 0$. Thus all negative coefficients $\{a_n : n < 0\}$ must vanish, by induction on the highest power of 2 dividing $n$. This means $f$ has a removable singularity at 0, not an essential singularity.

4. Let $\{f_n : \mathbb{D} \to \mathbb{C}\}_{n=1}^{\infty}$ be a sequence of analytic functions such that $f_n(0) = 0$ for all $n \in \mathbb{N}$, and $\text{Re } f_n(z) \to 0$ uniformly on compact sets. Prove that $\text{Im } f_n(z) \to 0$ uniformly on compact sets.
Proof. Consider the sequence of analytic functions \( g_n = e^{f_n} : \mathbb{D} \to \mathbb{C} - \{0\} \), which satisfy \( g_n(0) = 1 \). Since \( |g_n| = e^{\Re f_n} \to e^0 = 1 \) uniformly on compact sets, the sequence \( g_n \) is uniformly bounded in magnitude. We claim that the sequence \( g_n \) converges uniformly on compact sets to the constant function \( g = 1 \). If this were not true, then for some compact \( K = \{|z| \leq 1 - \delta\} \subset \mathbb{D} \) and some \( \epsilon > 0 \) we could find an infinite subsequence \( g_{n_k} \) such that

\[
\sup_{z \in K} |g_{n_k}(z) - 1| \geq \epsilon.
\]

The family \( g_{n_k} \) is uniformly bounded on \( K \), so by Montel’s theorem there must be a sub(sub)sequence \( g_{n_{k_j}} \) converging uniformly to an analytic function \( \tilde{g} \). But since \( |g_n| \to 1 \) uniformly, \( |\tilde{g}(z)| \equiv 1 \) identically so \( \tilde{g} = e^{i\theta} \) must be a constant on the unit circle (e.g. by the open mapping theorem). But \( g_n(0) = 1 \) for all \( n \) and \( 0 \in K \), so we must have \( e^{i\theta} = 1 \). Then \( g_{n_{k_j}} \) in fact converges uniformly to 1, contrary to our initial assumption. This proves our claim that \( g_n \to 1 \) uniformly on compact subsets.

Now we claim that \( f_n \) must converge to 0, uniformly on compact sets. As before fix a compact set \( K = \{|z| \leq 1 - \delta\} \subset \mathbb{D} \). For any \( \epsilon > 0 \), we can choose sufficiently large \( N \) such that

\[
|g_n(z) - 1| < \epsilon \quad \text{for all } n \geq N, \ z \in K.
\]

Namely, \( g_n \) must send \( K \) into an open \( \epsilon \)-neighborhood of 1. The preimage of such a neighborhood under the map \( z \mapsto e^z \), for \( \epsilon \) sufficiently small, is contained in a disjoint union of small \( \epsilon' \)-neighborhoods around the points \( 2\pi ik, k \in \mathbb{Z} \). Thus \( f_n \) must send \( K \) to such a disjoint union, but since \( K \) is connected, its image must lie in a single connected component. Moreover, it must lie in the component containing 0, since \( f_n(0) = 0 \). Thus \( f_n \) sends \( K \) to a small \( \epsilon' \) neighborhood of 0 for sufficiently large \( n \). As \( \epsilon, \epsilon' \to 0 \) this proves uniform convergence of \( f_n \to 0 \) as claimed. Thus \( \Im f_n \to 0 \) uniformly on compact sets.

5. Use complex integration methods to compute

\[
\int_0^\infty \frac{x^t}{(x+1)(x+2)} \, dx,
\]

where \( t \in (0, 1) \).

Proof. Consider integrating the holomorphic function \( f(z) = \frac{z^t}{(z+1)(z+2)} \) over the “keyhole” contour shown below (the small circle has radius \( r \)):
We have

\[ \lim_{\epsilon \to 0, \quad R \to \infty} \int_{\gamma_1} f(z)dz = \int_0^\infty \frac{x^t}{(x + 1)(x + 2)} \, dx =: I. \]

As we travel once around the origin to \( \gamma_3 \), recalling that \( z^t := e^{t \log z} \), we gain an extra \( 2\pi i \) in the value of \( \log z \) so

\[ z^t = e^{t(\log|z|+2\pi i)} = |z|^{it}e^{2\pi it}. \]

Thus

\[ \lim_{\epsilon \to 0, \quad R \to \infty} \int_{\gamma_3} f(z)dz = \int_0^\infty \frac{e^{2\pi it}x^t}{(x + 1)(x + 2)} \, dx = -e^{2\pi it}I. \]

We claim the integrals along \( \gamma_2 \) and \( \gamma_4 \) go to 0 as \( R \to \infty, \ r \to 0 \) respectively. Indeed

\[ \left| \int_{\gamma_2} f(z)dz \right| \leq |\gamma_2| \cdot \sup_{\gamma_2} |f(z)| \leq 2\pi R \cdot \frac{R^t}{|R - 1||R - 2|} \to 0 \]

as \( R \to \infty \), since \( t < 1 \). Similarly along \( \gamma_4 \)

\[ \left| \int_{\gamma_4} f(z)dz \right| \leq |\gamma_4| \cdot \sup_{\gamma_4} |f(z)| \leq 2\pi r \cdot \frac{r^t}{|1 - r||2 - r|} \to 0 \]

as \( r \to 0 \).

Finally, we compute the residues of \( f \) at its poles, which occur at \( z = -1 \) and \( -2 \). We have

\[ \text{Res}_f(-1) = \lim_{z \to -1} (z + 1)f(z) = \frac{(-1)^t}{-1 + 2} = e^{\pi it}, \quad \text{and} \]

\[ \text{Res}_f(-2) = \lim_{z \to -2} (z + 2)f(z) = \frac{(-2)^t}{-2 + 1} = -2^t e^{\pi it}. \]

By Cauchy’s integral formula

\[ (1 - 2^t)e^{\pi it} = \sum \text{Res}_f = \frac{1}{2\pi i} \lim_{\epsilon \to 0, \quad R \to \infty} \int_{\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4} f(z)dz = \frac{1}{2\pi i} (1 - e^{2\pi it})I, \]

which shows \( I = 2\pi i \frac{(1 - 2^t)e^{\pi it}}{1 - e^{2\pi it}} = \frac{\pi}{\sin(\pi t)}(2^t - 1). \)