Morning session

1. Let $\mathbb{H} = \{ z \in \mathbb{C} : \text{Re}(z) > 0 \}$, and let $f : \overline{\mathbb{H}} \to \mathbb{C}$ be a bounded continuous function, which is analytic in $\mathbb{H}$. Prove that for any $z = x + iy \in \mathbb{H}$

$$f(z) = \frac{x}{\pi} \int_{-\infty}^{\infty} \frac{f(it)dt}{x^2 + (t - y)^2}.$$

**Proof.** Consider integrating the function

$$g(w) = \frac{f(w)}{(w + \bar{z})(w - z)}$$

over the semicircular contour indicated below:

This function is meromorphic inside the given region, and has a single pole at $w = z$ with residue

$$\text{Res}_h(z) = \lim_{w \to z} (w - z)g(z) = \frac{f(z)}{z + \bar{z}} = \frac{f(z)}{2x}.$$

Along $\gamma_2$, we have

$$\int_{\gamma_2} g(w)dw \leq |\gamma_2| \cdot \sup_{\gamma_2} |g(w)| \leq \pi R \cdot \frac{M}{(R - |z|)^2} \approx \frac{\pi M}{R} \to 0 \quad \text{as} \quad R \to \infty,$$

where $M$ is a uniform bound on $|f(w)|$ in $\bar{\mathbb{H}}$. Along $\gamma_1$, we have

$$\lim_{\epsilon \to 0} \int_{\gamma_1} g(w)dw = \lim_{\epsilon \to 0} \int_{-R}^{R} \frac{f(\epsilon + it)}{(\epsilon + it + \bar{z})(\epsilon + it - z)}idt = -i \int_{-R}^{R} \frac{f(it)dt}{(x + i(t - y))(x + i(t - y))^2} = i \int_{-R}^{R} \frac{f(it)dt}{x^2 + (t - y)^2},$$

where passing the limit inside the integral is justified by dominated convergence under $\frac{M}{(x/2)^2 + (t - y)^2}$ (assuming $\epsilon < x/2$).

1
Thus Cauchy’s integral formula tells us that
\[
\frac{f(z)}{2x} = \sum \text{Res}_g = \lim_{R \to \infty} \frac{1}{2\pi i} \int_{\gamma_1 + \gamma_2} g(w)dw = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{f(it)dt}{x^2 + (t - y)^2},
\]
and multiplying the above equation by \(2x\) gives the desired result.

2. Let \(f: \mathbb{D} \to \mathbb{D}\), \(f(z) = \sum_{n=0}^{\infty} a_n z^n\), be a bounded analytic function.

(a) Prove that for any \(r < 1\)
\[
\sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 dt.
\]

(b) Show that the series \(\sum_{n=0}^{\infty} |a_n|^2\) converges.

**Proof.** (a) Let \(f_N = \sum_{n=0}^{N} a_n z^n\) denote the degree-\(N\) Taylor approximation of \(f\). Note that for any \(r < 1\), the convergence \(f_N \to f\) is uniform on the circle \(|z| = r\) since \(f\) is analytic in \(\mathbb{D}\). We have for any \(z = re^{it} \in \mathbb{D}\)
\[
|f_N(z)|^2 = \left(\sum_{n=0}^{N} a_n z^n\right) \left(\sum_{n=0}^{N} \bar{a}_n \bar{z}^n\right) = \sum_{n=0}^{N} |a_n|^2|z|^{2n} + \sum_{0 \leq n \neq m \leq N} a_n \bar{a}_m z^n \bar{z}^m
\]
\[
= \sum_{n=0}^{N} |a_n|^2 r^{2n} + \sum_{n \neq m} a_n \bar{a}_m r^{n+m} e^{(n-m)it}.
\]

It is straightforward to verify that integrating \(z^k\) or \(\bar{z}^k\) around a circle gives you zero:
\[
\int_0^{2\pi} (re^{it})^k dt = \int_0^{2\pi} (re^{-it})^k dt = 0 \quad \text{for} \quad k \neq 0,
\]
(by orthogonality of \(\cos(kt), \sin(kt)\)) while the other terms are constant. Thus
\[
\frac{1}{2\pi} \int_0^{2\pi} |f_N(re^{it})|^2 dt = \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{n \leq N} |a_n|^2 r^{2n}\right) dt + \sum_{n \neq m} \left(a_n \bar{a}_m r^{n+m} \frac{1}{2\pi} \int_0^{2\pi} e^{(n-m)it} dt\right)
\]
\[
= \sum_{n \leq N} |a_n|^2 r^{2n}
\]

Taking \(N \to \infty\) gives the desired equality.

(b) As \(r\) approaches 1, the uniform bound \(|f(z)| \leq 1\) ensures that
\[
\lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 dt \leq \frac{1}{2\pi} \int_0^{2\pi} dt = 1.
\]
Thus
\[ \sum_{n \geq 0} |a_n|^2 = \lim_{r \to 1} \left( \sum_{n \geq 0} |a_n|^2 r^{2n} \right) \leq 1 \]
is finite by monotone convergence.

3. Let \( f \) be a function which is analytic on the wedge
\[ W = \{ z \in \mathbb{C} : \text{Re } (z) > 0, -\frac{\pi}{6} < \text{Arg } (z) < \frac{\pi}{6} \}, \]
which is bounded on \( W \), and verifies for all \( r > 0 \)
\[ \lim_{\theta \to \pm \frac{\pi}{6}} f(re^{i\theta}) := \varphi(r) \in \mathbb{R}. \]
Show that \( f \) must be real and constant.
\textbf{Hint:} Consider using Schwarz reflection.

\textbf{Proof.} Following the hint, we note that \( f \) may be extended to an entire function by reflecting its values on \( W \) six times around the origin. Since \( f \) is bounded on \( W \), this extension to \( \mathbb{C} \) will also be bounded. Thus \( f \) must be constant, and since it takes real values (in some limit) it must be a real constant.

4. Evaluate
\[ \int_{0}^{\infty} \frac{\ln x}{(x-1)\sqrt{x}} \, dx. \]

\textbf{Proof.} Denote the given integral by \( I \). Substituting \( y = \ln(x) \)
\[ I = \int_{-\infty}^{\infty} \frac{ye^y dy}{(ey-1)e^{y/2}} = \int_{-\infty}^{\infty} \frac{y dy}{e^{y/2} - e^{-y/2}}. \]
If we consider integrating \( f(z) = \frac{z}{e^{z/2} - e^{-z/2}} \) along the rectangular contour that encloses \( \{ z : 0 \leq \text{Im } z \leq \pi, -R \leq \text{Re } z \leq R \} \), which does not contain any poles of \( f \), we see that the integrals over the vertical segments go to 0 as \( R \to \infty \), so the integral along the lower boundary must equal the integral along the upper boundary (taking both integrals from left to right). Thus
\[ I = \int_{-\infty}^{\infty} \frac{(y + \pi i) dy}{e^{(y+\pi i)/2} - e^{-(y+\pi i)/2}} = \int_{-\infty}^{\infty} \frac{y}{ie^{y/2} + ie^{-y/2}} \, dy + \int_{-\infty}^{\infty} \frac{\pi i}{ie^{y/2} + ie^{-y/2}} \, dy \]
\[ = -i \int_{-\infty}^{\infty} \frac{y}{e^{y/2} + e^{-y/2}} \, dy + \pi \int_{-\infty}^{\infty} \frac{1}{e^{y/2} + e^{-y/2}} \, dy \]
\[ = \pi \int_{-\infty}^{\infty} \frac{1}{e^{y/2} + e^{-y/2}} \, dy \]
since the integrand in the first term is an odd function.
Substituting back $w = e^{y/2}$,

$$
\int_{-\infty}^{\infty} \frac{1}{e^{y/2} + e^{-y/2}} dy = \int_{-\infty}^{\infty} e^{y/2} dy = \int_{0}^{\infty} \frac{2dw}{w^2 + 1} = 2 \arctan(w) \bigg|_0^\infty = \pi.
$$

Thus $I = \frac{\pi}{2}$.

5. Let $\Omega \subset \mathbb{C}$ be a bounded, simply connected domain in $\mathbb{C}$. Let $z_0$ and $z_1$ be two distinct points of $\Omega$. If $\varphi_1$ and $\varphi_2$ are two one-to-one and onto analytic maps from $\Omega$ onto itself, and $\varphi_1(z_i) = \varphi_2(z_i)$, $i = 0, 1$, show that $\varphi_1 \equiv \varphi_2$ on $\Omega$.

**Proof.** The given conditions of $\Omega$ imply that it is analytically isomorphic to the unit disk $\mathbb{D}$ by the Riemann mapping theorem. Thus we may take some analytic isomorphism $\Phi: \Omega \rightarrow \mathbb{D}$, which we may assume sends $z_0 \mapsto 0$ without loss of generality. Then $\hat{\varphi}_1 := \Phi \circ \varphi_1 \circ \Phi^{-1}$ defines a bijective analytic map from $\mathbb{D}$ to itself, as does $\hat{\varphi}_2 := \Phi \circ \varphi_2 \circ \Phi^{-1}$.

Now consider $\hat{\varphi}_2^{-1} \circ \hat{\varphi}_1: \mathbb{D} \rightarrow \mathbb{D}$ which is equal to $\Phi \circ (\varphi_2^{-1} \circ \varphi_1) \circ \Phi^{-1}$. This fixes the origin and the point $w_1 := \Phi(z_1)$. In particular, this means $|\hat{\varphi}_1(w_1)| = |w_1|$, so Schwarz’s lemma implies that $\hat{\varphi}_2^{-1} \circ \hat{\varphi}_1$ is the identity (i.e. the only rotation that fixes $w_1$). Then $\varphi_2^{-1} \circ \varphi_1 = \Phi^{-1} \circ (\hat{\varphi}_2^{-1} \circ \hat{\varphi}_1) \circ \Phi = \Phi^{-1} \circ \Phi$ must be the identity on $\Omega$, so $\varphi_1 \equiv \varphi_2$. \qed
Afternoon session

1. Let \( f \in L^1((0,1)) \), and define \( g: (0,1) \to \mathbb{R} \) by
\[
g(x) = \int_x^1 \frac{f(t)}{t} \, dt.
\]
Prove that \( g \in L^1((0,1)) \).

**Proof.** We apply Tonelli’s theorem to exchange the order of integration:
\[
\int_0^1 |g(x)| \, dx \leq \int_0^1 \int_x^1 \frac{|f(t)|}{t} \, dtdx = \int_0^1 |f(t)| \, dxdt = \int_0^1 f(t) \, dt.
\]
Thus \( \|g\|_1 \leq \|f\|_1 < \infty \) by assumption \( f \in L^1 \), so \( g \in L^1 \) as well.

2. Let \( (X, \mathcal{A}, \mu) \) be a finite measure space, and let \( F: \mathbb{R} \to \mathbb{R} \) be a \( C^2 \) function with second derivative \( F'' > 0 \). Let \( f \in L^1(\mu) \) be real-valued. Prove Jensen’s inequality:
\[
F\left( \frac{1}{\mu(X)} \int f \, d\mu \right) \leq \frac{1}{\mu(X)} \int F(f) \, d\mu.
\]

**Proof.** For any \( t_0, t \in \mathbb{R} \),
\[
F(t) = F(t_0) + \int_{t_0}^t F'(s) \, ds = F(t_0) + (t-t_0)F'(t_0) + \int_{t_0}^t \int_{t_0}^s F''(r) \, dr \, ds
\]
\[
\geq F(t_0) + (t-t_0)F'(t_0)
\]
since \( F'' > 0 \) and the two integrals are either both positively oriented, or both negatively oriented. (Moreover, the inequality is strict if \( t \neq t_0 \).)

Now take \( f_0 = \frac{1}{\mu(X)} \int f \, d\mu \in \mathbb{R} \), so that
\[
\int_X f \, d\mu = \mu(X)f_0 = \int_X f_0 \, d\mu \quad \Rightarrow \quad \int_X (f - f_0) \, d\mu = 0.
\]
We have \( F(f) \geq F(f_0) + (f - f_0)F'(f_0) \) for any \( f \in \mathbb{R} \), so integrating this inequality over \( X \) we have
\[
\int_X F(f) \, d\mu \geq F(f_0) \int_X d\mu + F'(f_0) \int_X (f - f_0) \, d\mu = \mu(X)F(f_0).
\]
Dividing by \( \mu(X) \) gives the desired result.
3. Let $f, g_1, g_2, \ldots \in L^1(\mathbb{R})$ be non-negative functions. Assume that $g_n \to f$ a.e. and

$$\int_{\mathbb{R}} g_n \, dm = \int_{\mathbb{R}} f \, dm.$$ 

Prove that for any measurable set $A \subseteq \mathbb{R}$

$$\int_{A} g_n \, dm \to \int_{A} f \, dm.$$ 

Proof. For any measurable $A$, Fatou’s lemma says that

$$\int_{A} f \, dm = \int_{A} \liminf_{n \to \infty} g_n \, dm \leq \liminf_{n \to \infty} \int_{A} g_n \, dm.$$ 

To prove $\lim_{n \to \infty} \int_{A} g_n = \int_{A} f$ it suffices to show that $\int_{A} f \geq \limsup_{n \to \infty} \int_{A} g_n$. For this, consider integrating these over the complement $A^c$:

$$\int_{A^c} f \, dm = \int_{A^c} \liminf_{n \to \infty} g_n \, dm \leq \liminf_{n \to \infty} \int_{A^c} g_n \, dm.$$ 

Expressing these in terms of the total integral over $\mathbb{R}$ gives

$$\int_{\mathbb{R}} f \, dm - \int_{A} f \, dm \leq \liminf_{n \to \infty} \left( \int_{\mathbb{R}} g_n \, dm - \int_{A} g_n \, dm \right) = \int_{\mathbb{R}} f \, dm - \limsup_{n \to \infty} \int_{A} g_n \, dm$$ 

so we must have

$$\int_{A} f \, dm \geq \limsup_{n \to \infty} \int_{A} g_n \, dm.$$ 

This shows the desired convergence of integrals. \qed

4. Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Let $\{f_n\}_{n=1}^{\infty} \subseteq L^2(\mu)$ be a sequence of functions such that $\|f_n\|_2 \leq 1$.

(a) Prove that if $f_n \to 0$ in measure, then $f_n \to 0$ in $L^1(\mu)$.

(b) If $f_n \to 0$ in measure, does it necessarily follow that $f_n \to 0$ in $L^2(\mu)$?

Proof. (a) Suppose $f_n \to 0$ in measure, meaning that for any $\epsilon > 0$

$$\lim_{n \to \infty} \mu(\{x : |f_n(x)| > \epsilon\}) = 0.$$ 

The Cauchy-Schwarz inequality implies that for any measurable $A \subseteq X$,

$$\int_{A} |f_n| \, d\mu \leq \left( \int_{A} |f_n|^2 \, d\mu \right)^{1/2} \left( \int_{A} d\mu \right)^{1/2} \leq (\mu(A))^{1/2}$$ 

since we are given that $\|f\|_2 \leq 1$.

Now consider the measurable sets $E_{n,\epsilon} = \{x : |f_n(x)| > \epsilon\}$. By convergence in measure

$$\limsup_{n \to \infty} \int_{E_{n,\epsilon}} |f_n| \, d\mu \leq \limsup_{n \to \infty} \mu(E_{n,\epsilon})^{1/2} = 0.$$
for any fixed $\epsilon > 0$, so

$$
\limsup_{n \to \infty} \int_X |f_n| d\mu = \limsup_{n \to \infty} \left( \int_{E_{n,\epsilon}} |f_n| d\mu + \int_{E_{n,\epsilon}} |f_n| d\mu \right)
\leq \limsup_{n \to \infty} \int_{E_{n,\epsilon}} \epsilon d\mu + \limsup_{n \to \infty} \int_{E_{n,\epsilon}} |f_n| d\mu \leq \epsilon \cdot m(X).
$$

As $\epsilon \to 0$, this bound goes to zero so $f_n \to 0$ in $L^1$ as desired.

(b) No; consider the sequence in $L^2([0, 1])$ with the Lebesgue measure defined by

$$
f_n(x) = \begin{cases} 
\sqrt{n} & \text{if } 0 < x < \frac{1}{n}, \\
0 & \text{otherwise}.
\end{cases}
$$

It is straightforward to verify that $f_n \to 0$ in measure, but $\|f_n\|_2 = 1$ for all $n$ so $f_n \not\to 0$ in $L^2$.

5. Let $F \subset \mathbb{R}$ be a closed set, and define the distance from $x \in \mathbb{R}$ to $F$ by

$$
d(x, F) = \inf_{y \in F} |x - y|.
$$

Prove that

$$
\lim_{x \to y} \frac{d(x, F)}{|x - y|} = 0
$$

for a.e. $y \in F$.

*Hint:* Consider Lebesgue points of $F$.

*Proof.* Recall that $y \in F$ is a Lebesgue point of $F$ if

$$
\lim_{m(B) \to 0} \frac{m(B \cap F)}{m(B)} = 1 \iff \lim_{m(B) \to 0} \frac{m(B) - m(B \cap F)}{m(B)} = 0
$$

where the limit is taken over balls (i.e. intervals) $B$ in $\mathbb{R}$ that contain $y$. Since Lebesgue points have full measure in $F$ it suffices to prove the given equality for Lebesgue points.

Suppose $y$ is a Lebesgue point of $F$, and $x \in \mathbb{R}$ arbitrary. Let $B_r(x)$ denote the ball of radius $r$ centered at $x$, and let $B_r := B_r(x, y)$ denote the smallest open interval containing both $B_r(x)$ and $B_r(y)$. It is clear that $m(B_r) = |x - y| + 2r$. If we intersect this ball with $F$, then we must exclude an interval of length at least $d(x, F)$:

$$
m(B_r \cap F) \leq m(B_r) - d(x, F) \Rightarrow d(x, F) \leq m(B_r) - m(B_r \cap F).
$$

This implies

$$
\frac{d(x, F)}{|x - y|} \leq \liminf_{r \to 0} \frac{m(B_r) - m(B_r \cap F)}{|x - y|} = \liminf_{r \to 0} \frac{m(B_r) - m(B_r \cap F)}{m(B_r)}.
$$
Since \( y \) is a Lebesgue point, for any \( \epsilon > 0 \) there is an \( \delta > 0 \) such that

\[
m(B) < \delta, \ y \in B \quad \Rightarrow \quad \left| \frac{m(B) - m(B \cap F)}{m(B)} \right| < \epsilon.
\]

so taking \( B = B_{\epsilon}(x, y) \) for any \( x \) satisfying \( 0 < |x - y| < \delta \), we have

\[
0 \leq \frac{d(x, F)}{|x - y|} \leq \liminf_{r \to 0} \frac{m(B_r) - m(B_r \cap F)}{m(B_r)} < \epsilon.
\]

In the limit as \( x \) approaches \( y \), we get \( \lim_{x \to y} \frac{d(x, F)}{|x - y|} < \epsilon \). But since \( \epsilon \) was arbitrary, this shows that the limit is in fact 0. \( \Box \)