Math 116 - 099
Quiz 9 §§10.1–10.3 [in class]

Name: Solutions

This quiz has a total of 28 points. The value of each question is specified. Please show all work.

1. [11 points] Let $f(x) = \frac{1}{1+x^2}$. Recall that $\int \frac{1}{1+x^2}dx = \arctan(x) + C$.

   (a) [3] Using the known Taylor series expansion for $\frac{1}{1-x}$ around 0, find the Taylor series expansion around $x = 0$ for $f(x)$. Write the first three nonzero terms, and also give your answer in sigma notation (i.e. $\sum_{k=0}^{\infty}$).

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

Substitute $-x^2$:

Simplify final answer

$$\frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + \cdots = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

   (b) [4] Using the facts above, find the Taylor series expansion around $x = 0$ for $\arctan(x)$. Write the first three nonzero terms, and also give your answer in sigma notation.

$$\arctan(x) = \arctan(0) + \int_0^x \frac{1}{1+t^2}\,dt$$
$$= 0 + \int_0^x (1 - t^2 + t^4 - t^6 + \cdots)\,dt$$
$$= 0 + \left(t - \tfrac{1}{3}t^3 + \tfrac{1}{5}t^5 - \tfrac{1}{7}t^7 + \cdots\right)\Big|_0^x$$
$$= x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

   (c) [2] Find the exact value of the series

$$(*) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{2k+1} = \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \cdots$$

By part (b), $\arctan(1) = \frac{\pi}{4} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$

$$= 1 - (*) \Rightarrow (*) = 1 - \frac{\pi}{4} \approx 0.2146$$

   (d) [2] Find the third-degree Taylor polynomial of $\cos(x)\arctan(x)$ around 0.

$$\cos(x)\cdot\arctan(x) = \left(1 - \tfrac{1}{2}x^2 + \cdots\right)\left(x - \tfrac{1}{3}x^3 + \cdots\right)$$
$$= x - \tfrac{1}{3}x^3 + \cdots + -\tfrac{1}{2}x^3 + \tfrac{1}{6}x^5 + \cdots$$

ignore since deg > 3

$$(\text{3rd deg. Taylor polyn}) = x + \left(-\tfrac{1}{3} - \tfrac{1}{2}\right)x^3 = x - \tfrac{5}{6}x^3$$

2. [10 points] The function

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{\pi}} e^{-t^2}\, dt$$

is the cumulative distribution function for a random variable.

(a) [3] Using the known Taylor series for e^u around $u = 0$, write the first four nonzero terms of the Taylor series for $\frac{1}{\sqrt{\pi}}e^{-t^2}$ centered at $t = 0$.

$$e^u = 1 + u + \frac{1}{2}u^2 + \frac{1}{6}u^3 + \dots$$

$$\Rightarrow \frac{1}{\sqrt{\pi}} e^{-t^2} = \frac{1}{\sqrt{\pi}}\left(1 + (-t^2) + \frac{1}{2}(-t^2)^2 + \frac{1}{6}(-t^2)^3 + \dots\right)$$

$$= \frac{1}{\sqrt{\pi}} - \frac{1}{\sqrt{\pi}}t^2 + \frac{1}{2\sqrt{\pi}}t^4 - \frac{1}{6\sqrt{\pi}}t^6 + \dots$$

(b) [1] It can be shown that the random variable whose cdf is $F(x)$ has median 0. What does this mean as an equation involving $F(x)$? (Hint: $F(?) = ?$)

$$F(\text{median}) = \frac{1}{2} \quad \Rightarrow \quad F(0) = \frac{1}{2}$$

(c) [4] Use your answer from part (b) to write the first four nonzero terms of the Taylor series for $F(x)$ centered at $x = 0$.

$$F(x) = \int_{-\infty}^{0} \frac{1}{\sqrt{\pi}} e^{-t^2}\,dt + \int_0^x \frac{1}{\sqrt{\pi}} e^{-t^2}\,dt$$

$$= F(0) + \int_0^x \left(\frac{1}{\sqrt{\pi}} - \frac{1}{\sqrt{\pi}}t^2 + \frac{1}{2\sqrt{\pi}}t^4 - \frac{1}{6\sqrt{\pi}}t^6 + \dots\right)dt$$

$$= \frac{1}{2} + \left(\frac{1}{\sqrt{\pi}}t - \frac{1}{3\sqrt{\pi}}t^3 + \frac{1}{10\sqrt{\pi}}t^5 - \frac{1}{42\sqrt{\pi}}t^7 + \dots\right)\Big|_0^x$$

$$(\text{first 4 terms}) = \frac{1}{2} + \frac{1}{\sqrt{\pi}}x - \frac{1}{3\sqrt{\pi}}x^3 + \frac{1}{10\sqrt{\pi}}x^5 - \dots$$

(d) [2] Use part (c) to approximate the definite integral $\int_0^1 \frac{1}{\sqrt{\pi}}e^{-t^2}\,dt$.

$$\int_0^1 \frac{1}{\sqrt{\pi}} e^{-t^2}\,dt = F(1) - F(0)$$

$$\approx \left(\frac{1}{2} + \frac{1}{\sqrt{\pi}} - \frac{1}{3\sqrt{\pi}} + \frac{1}{10\sqrt{\pi}}\right) - \left(\frac{1}{2}\right)$$

$$= \frac{23}{30\cdot\sqrt{\pi}} \approx 0.4325$$

3. [5 points]

(a) Find the first five nonzero terms of the Taylor series of $g(x) = x^2 e^{x^2}$ about $x = 0$.

$$e^u = 1 + u + \frac{1}{2!}u^2 + \frac{1}{3!}u^3 + \frac{1}{4!}u^4 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}u^n$$

$$\Rightarrow x^2 e^{x^2} = x^2\left(1 + x^2 + \frac{1}{2!}(x^2)^2 + \frac{1}{3!}(x^2)^3 + \frac{1}{4!}(x^2)^4 + \cdots\right)$$

$$= x^2 + x^4 + \frac{1}{2!}x^6 + \frac{1}{3!}x^8 + \frac{1}{4!}x^{10} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}x^2(x^2)^n$$

(b) Find the value of $g^{(2020)}(0)$.

$$= \sum_{n=0}^{\infty} \frac{1}{n!}x^{2n+2}$$

Match terms:

general formula $\rightarrow$ $\frac{g^{(2020)}(0)}{2020!}x^{2020} = \frac{1}{1009!}x^{2020}$ $\leftarrow$ part (a) when $n = 1009$

$$\Rightarrow g^{(2020)}(0) = \frac{2020!}{1009!}$$

(c) Find the value of $g^{(2019)}(0)$.

From part (a), Taylor expansion of $g(x) = \sum_{n=0}^{\infty} \frac{1}{n!}x^{2n+2}$

has only <u>even</u> powers of x $\Rightarrow$ coeff. of x^{2019} is zero

$$\Rightarrow g^{(2019)}(0) = 0$$

4. [2 points] When θ is close to 0, which of the following is larger?

$h_1(\theta) = 1 + \sin\theta$ $\boxed{h_2(\theta) = e^\theta}$

Justify using Taylor series.

Near $\theta = 0$,

$$h_1(\theta) = 1 + \theta - \frac{1}{3!}\theta^3 + \cdots$$

$$h_2(\theta) = 1 + \theta + \frac{1}{2!}\theta^2 + \cdots$$

When θ is small, $\frac{1}{2!}\theta^2 > -\frac{1}{3!}\theta^3$ (whether θ is positive <u>or</u> negative)

so $h_2(\theta) > h_1(\theta)$