5. [7 points] The Terrible Telemarketing corporation has realized that people often hang up on their telemarketing calls. After collecting data they found that the probability that someone will hang up the phone at time t seconds after the call begins is given by the probability density function $p(t)$. The formula for $p(t)$ is given below.

$$p(t) = \begin{cases} 0 & t < 0 \\ te^{-ct^2} & t \geq 0 \end{cases}$$

a. [5 points] Find the value of c so that $p(t)$ is a probability density function.

Solution:
We must have $\int_0^\infty te^{-ct^2}\,dt = 1$. Let $u = ct^2$ then $du = 2ctdt$. Thus we get the integral $\frac{1}{2c}\int_0^\infty e^{-u}du = \frac{1}{2c}\lim_{N\to\infty} -e^{-u}|_0^N = \frac{1}{2c}$. Thus we have $1 = \frac{1}{2c}$ so $c = \frac{1}{2}$.

b. [2 points] What is the probability that someone will stay on the phone with a telemarketer for more than 4 seconds?

Solution:
The probability is $\int_4^\infty te^{-\frac{1}{2}t^2}\,dt = e^{-8}$.

6. [6 points] Consider the probability density function $q(t)$ shown below.

$$q(t) = \begin{cases} 0 & t < 0 \\ \frac{t}{2} & 0 \leq t < 2 \\ 0 & t \geq 2 \end{cases}$$

a. [4 points] What is the cumulative distribution function $Q(t)$ of the density given by $q(t)$? Write your final answer in the answer blanks provided.

Solution:
$Q(t) = \int_{-\infty}^t q(s)\,ds = \int_0^t q(s)\,ds$. If $t < 0$ then $Q(t) = 0$ and if $0 \leq t < 2$ then $Q(t) = \int_0^t q(s)\,ds = \int_0^t \frac{s}{2}\,ds = t^2/4$. Then it follows that $Q(t) = 1$ for $t \geq 2$.

If $t < 0$ then $Q(t) = 0$

If $0 \leq t < 2$ then $Q(t) = t^2/4$

If $t \geq 2$ then $Q(t) = 1$

b. [2 points] What is the (mean & median) of the distribution?

Solution: The median is the number T such that $Q(T) = \frac{1}{2}$. Thus we want $T^2/4 = \frac{1}{2}$. Therefore $T = \sqrt{2}$.

The mean is

$$\int_{-\infty}^{\infty} t\,q(t)\,dt = \int_0^2 t\left(\frac{t}{2}\right)dt = \left(\frac{t^3}{6}\right)\Big|_0^2 = \boxed{\frac{8}{6}}$$