Math 116 - 099
Quiz 5 §§4.7, 7.6–7.7

Name: Solutions

This quiz has a total of ~~25~~ 24 points. The value of each question is specified. Please show all work.

1. [10 points] Evaluate the following improper integrals. Find the exact value if the integral converges, otherwise indicate that it diverges. Approximations with your calculator will not receive credit.

(a) $\displaystyle\int_{-\infty}^{0} \frac{e^{5w}}{2+e^{5w}}\,dw$

$$= \lim_{a\to-\infty} \int_a^0 \frac{e^{5w}}{2+e^{5w}}\,dw \overset{\text{sub}}{=} \lim_{a\to-\infty} \int_{2+e^{5a}}^{2+e^{0}} \frac{1}{5}\cdot\frac{1}{u}\,du = \lim_{a\to-\infty} \left(\frac{1}{5}\ln|u|\right)\Big|_{2+e^{5a}}^{2+1}$$

sub: $u = 2+e^{5w}$, $du = 5e^{5w}\,dw$

$$= \lim_{a\to-\infty} \left(\frac{1}{5}\ln|3| - \frac{1}{5}\ln|\underbrace{2+e^{5a}}_{\to 2 \text{ as } a\to-\infty}|\right) = \frac{1}{5}\ln 3 - \frac{1}{5}\ln 2$$

$$= \frac{1}{5}\ln(3/2)$$

(b) $\displaystyle\int_1^{e^3} \frac{\ln(\ln x)}{x}\,dx$

$$= \lim_{b\to1^+} \int_b^{e^3} \frac{\ln(\ln x)}{x}\,dx \overset{\text{sub}}{=} \lim_{b\to1^+} \int_{\ln b}^{\ln(e^3)} \ln w\,dw$$

sub: $w = \ln x$, $dw = \frac{1}{x}\,dx$

$$= \lim_{b\to1^+} \left(w\ln w - w\right)\Big|_{\ln b}^{3}$$

$$= \lim_{b\to1^+} \left(3\ln 3 - 3 - (\ln b)\ln(\ln b) + \ln b\right)$$

$B = \ln b$, as $b\to1^+$, $B = \ln b \to 0^+$

$$= 3\ln 3 - 3 - \lim_{B\to0^+} \underbrace{\left(B\ln B - B\right)}_{(*)}$$

$$= 3\ln 3 - 3$$

(*) By L'Hopital,

$$\lim_{B\to0^+} \frac{\ln B}{1/B} = \lim_{B\to0^+} \frac{1/B}{-1/B^2}$$

$$= \lim_{B\to0^+} -B = 0$$

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(c) $\displaystyle\int_4^\infty e^{-\sqrt{x}}dx$, given that $\dfrac{d}{dx}\Big(-2e^{-\sqrt{x}}(\sqrt{x}+1)\Big) = e^{-\sqrt{x}}$

$= \lim_{c\to\infty} \int_4^c e^{-\sqrt{x}}\,dx = \lim_{c\to\infty}\left(-2e^{-\sqrt{c}}\left(\sqrt{c}+1\right) + 2e^{-\sqrt{4}}\left(\sqrt{4}+1\right)\right)$

$= \lim_{c\to\infty}\left(-2\dfrac{\sqrt{c}+1}{e^{\sqrt{c}}}\right) + 2e^{-2}(2+1)$

$\overset{L'H}{=} \lim_{c\to\infty}\left(-2\dfrac{1/2\sqrt{c}}{e^{\sqrt{c}}/2\sqrt{c}}\right) + 6e^{-2}$

$= \lim_{c\to\infty}\left(\dfrac{-2}{e^{\sqrt{c}}}\right) + 6e^{-2} = 6e^{-2}$

2. [4 points] Suppose $a > 0$ is a positive constant. Find the exact value of the following improper integral if it converges, otherwise indicate that it diverges.

$$\int_0^\infty \frac{2}{p^2+4a^2}dp$$

Recall: $\int \frac{1}{z^2+1}dz = \arctan z + C$

$= \lim_{b\to\infty} \int_0^b \dfrac{2}{4a^2\left(p^2/4a^2+1\right)}\,dp$

$= \lim_{b\to\infty} \dfrac{1}{2a^2}\int_0^b \dfrac{1}{(p/2a)^2+1}\,dp$

sub: $z = \frac{p}{2a}$, $dz = \frac{1}{2a}dp$

$= \lim_{b\to\infty} \dfrac{1}{a}\int_0^{b/2a} \dfrac{1}{z^2+1}\,dz$

$= \lim_{b\to\infty} \dfrac{1}{a}\left(\arctan\left(\dfrac{b}{2a}\right) - \arctan 0\right)$

$= \dfrac{1}{a}\left(\dfrac{\pi}{2} - 0\right) = \dfrac{\pi}{2a}$

3. [10 points] Use a comparison test to determine whether the following integrals converge or diverge. Show all your work, indicating any theorems you use, and using proper syntax and notation.

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{2}{(1+x^4)^{1/6}}dx$

Proof. This integral is equal to

$$\int_{-\infty}^{-1} \frac{2}{(1+x^4)^{1/6}}dx + \int_{-1}^{1} \frac{2}{(1+x^4)^{1/6}}dx + \int_{1}^{\infty} \frac{2}{(1+x^4)^{1/6}}dx.$$

The integral $\displaystyle\int_{-1}^{1} \frac{2}{(1+x^4)^{1/6}}dx$ is proper so it converges, and $\displaystyle\int_{-\infty}^{-1} \frac{2}{(1+x^4)^{1/6}}dx = \int_{1}^{\infty} \frac{2}{(1+x^4)^{1/6}}dx$ since the integrand is an even function. Thus it suffices to decide whether or not $\displaystyle\int_{1}^{\infty} \frac{2}{(1+x^4)^{1/6}}dx$ converges.

Note that

$$(1+x^4)^{1/6} \leq (x^4+x^4)^{1/6} = 2^{1/6}x^{2/3} \quad \text{for } x \geq 1$$

Thus for $x \geq 1$ we have

$$\frac{2}{(1+x^4)^{1/6}} \geq \frac{2}{2^{1/6}x^{2/3}} = \frac{2^{5/6}}{x^{2/3}}.$$

Let $g(x) = \dfrac{2^{5/6}}{x^{2/3}}$. By the p-test with $p = 4/6$, the integral $\int_1^\infty g(x)dx$ diverges. Thus $\displaystyle\int_{1}^{\infty} \frac{2}{(1+x^4)^{1/6}}dx$ diverges by direct comparison. This means that the original integral from $-\infty$ to ∞ also diverges. □

(b) $\displaystyle\int_{2}^{\infty} \frac{1+e^{-2x}}{\sqrt{x^5 \ln x + x^7}}dx$

Proof. We use the Direct Comparison Test. Note that

$$1 + e^{-2x} \leq 1 + e^0 = 2 \quad \text{for } x \geq 0,$$

and

$$\sqrt{x^5 \ln x + x^7} \geq \sqrt{0 + x^7} \quad \text{for } x \geq 1.$$

Thus for $x \geq 4$ we have

$$0 \leq \frac{1+e^{-2x}}{\sqrt{x^5 \ln x + x^7}} \leq \frac{2}{\sqrt{x^7}} = \frac{2}{x^{3.5}}.$$

Let $g(x) = \dfrac{2}{x^{3.5}}$. By the p-test with $p = 3.5$, the integral $\int_4^\infty g(x)dx$ converges. Thus $\int_4^\infty f(x)dx$ converges by direct comparison, where $f(x)$ is the original integrand. □

4. [5 points, Extra credit!]

(a) Suppose $g(x)$ is a differentiable function which satisfies $-1 \le g(x) \le 1$ for all x. Prove that

$$\int_{-\infty}^{\infty} g'(x)e^{-x^2/2}\,dx = \int_{-\infty}^{\infty} xg(x)e^{-x^2/2}\,dx.$$

Proof. We use integration by parts, with

$$u = e^{-x^2/2} \qquad v' = g'(x)$$
$$u' = -xe^{-x^2/2} \qquad v = g(x).$$

Thus

$$\begin{aligned}
\int_{-\infty}^{\infty} g'(x)e^{-x^2/2}dx &= \lim_{\substack{b\to\infty \\ a\to-\infty}} \int_a^b g'(x)e^{-x^2/2}dx \\
&= \lim_{\substack{b\to\infty \\ a\to-\infty}} \left(g(x)e^{-x^2/2}\Big|_a^b - \int_a^b g(x)\cdot(-e^{-x^2/2})dx \right) \\
&= \lim_{\substack{b\to\infty \\ a\to-\infty}} \left(g(b)e^{-b^2/2} - g(a)e^{-a^2/2} \right) + \lim_{\substack{b\to\infty \\ a\to-\infty}} \int_a^b g(x)e^{-x^2/2}dx \\
&= \lim_{b\to\infty} \left(g(b)e^{-b^2/2} \right) - \lim_{a\to-\infty} \left(g(a)e^{-a^2/2} \right) + \int_{-\infty}^{\infty} xg(x)e^{-x^2/2}dx
\end{aligned}$$

Since $-1 \le g(b) \le 1$ for any b, and since $e^{-b^2/2} > 0$, we have

$$\lim_{b\to\infty} \left(-e^{-b^2/2} \right) \le \lim_{b\to\infty} \left(g(b)e^{-b^2/2} \right) \le \lim_{b\to\infty} \left(e^{-b^2/2} \right)$$

The limits on both sides are equal to 0, so $\lim_{b\to\infty} \left(g(b)e^{-b^2/2} \right) = 0$. The same reasoning shows that $\lim_{a\to-\infty} (g(a)e^{-a^2/2}) = 0$. This proves the desired identity. □

(b) Given that $\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$, find $\int_{-\infty}^{\infty} x^2 e^{-x^2/2}\,dx$.

Proof. Using the above identity with $g(x) = x$, we have

$$\int_{-\infty}^{\infty} x^2 e^{-x^2/2}\,dx = \int_{-\infty}^{\infty} xg(x)e^{-x^2/2}\,dx = \int_{-\infty}^{\infty} g'(x)e^{-x^2/2}\,dx = \int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}.$$

HOWEVER, the function $g(x) = x$ does not satisfy the hypothesis $-1 \le g(x) \le 1$, so we cannot use part (a) directly.
To show that the same identity holds for $g(x) = x$, the key step is to show that $\lim_{b\to\infty} \left(g(b)e^{-b^2/2} \right) = 0$. When $g(x) = x$, this follows from L'Hopital's rule:

$$\lim_{b\to\infty} \left(be^{-b^2/2} \right) = \lim_{b\to\infty} \frac{b}{e^{b^2/2}} \overset{\text{L'Hop}}{=} \lim_{b\to\infty} \frac{1}{b\,e^{b^2/2}} = 0.$$

□