Math 116 - 099 Name: Solutions
Quiz 2 §§6.4 - 7.2

This quiz has a total of 22 points. Please show all work to receive full credit.

1. [8 points] Evaluate the following integrals by hand. Answers without work accompanying them will receive zero credit.

(a) $\displaystyle\int_0^{\pi/3} \frac{\sin z}{\cos z} dz$

Method: substitution

$$= \int_{\cos(0)}^{\cos(\pi/3)} -\frac{1}{y}\, dy = -\ln|y|\Big|_{\cos(0)=1}^{\cos(\pi/3)=1/2} = -\ln(1/2) + \underbrace{\ln(1)}_{0} = \boxed{\ln(2)}$$

$y = \cos z$

$dy = -\sin z\, dz$

(b) $\displaystyle\int_0^{\pi} e^{2t}\sin(t)dt$

Method: by parts

$$(*) = -e^{2t}\cos t\Big|_0^{\pi} + \int_0^{\pi} 2e^{2t}\cos t\, dt = \left(-e^{2\pi}\cos\pi + e^0\cos 0\right) + \left(2e^{2t}\sin t\Big|_0^{\pi} - \int_0^{\pi} 4e^{2t}\sin t\, dt\right)$$

$\left[\begin{matrix} u = e^{2t} & v' = \sin t \\ u' = 2e^{2t} & v = -\cos t \end{matrix}\right]$ $\left[\begin{matrix} u = 2e^{2t} & v' = \cos t \\ u' = 4e^{2t} & v = \sin t \end{matrix}\right]$

$$= \left(e^{2\pi} + 1\right) + (0 - 0) - 4\underbrace{\int_0^{\pi} e^{2t}\sin t\, dt}_{= (*)}$$

$$\Rightarrow 5(*) = e^{2\pi} + 1, \quad (*) = \boxed{\frac{e^{2\pi}+1}{5}}$$

(c) $\displaystyle\int \frac{2^p}{2^p + 3} dp$

Method: substitution

$u = 2^p + 3$

$du = \ln(2)\, 2^p\, dp$

$$(*) = \int \frac{1}{u}\cdot\frac{du}{\ln(2)} = \frac{1}{\ln 2}\cdot\ln|u| + C = \boxed{\frac{\ln|2^p+3|}{\ln 2} + C}$$

2. [3 points] If F is the function defined for $x > 0$ by $F(x) = \int_1^x \frac{1}{t} e^t dt$, show that

$$\int F(x)dx = xF(x) - e^x + C.$$

Method 1: integrate by parts

$$\left[\begin{array}{ll} u = F(x) & v' = 1 \\ u' = \underbrace{F'(x) = \frac{1}{x}e^x}_{\text{by 2nd Fund Thm.}} & v = x \end{array}\right]$$

$$\int F(x)\cdot 1\,dx = xF(x) - \int \left(\tfrac{1}{x}e^x\right)(x)\,dx$$
$$= xF(x) - \int e^x\,dx$$
$$= xF(x) - e^x + C$$

Method 2: take derivatives – we must show that RHS is antiderivative of $F(x)$

$$\frac{d}{dx}\left(x\cdot F(x) - e^x + C\right) = \underbrace{F(x) + xF'(x)}_{\text{prod. rule}} - e^x = F(x) + x\underbrace{\left(\tfrac{1}{x}e^x\right)}_{\text{2nd Fund. Thm.}} - e^x = F(x).$$

3. [4 points] Let

$$T(p) = \int_{\sin p}^{p^4} \ln(e^t + 1)dt$$

Calculate $\frac{dT}{dp}$.

Let $F(p) = \int_0^p \ln(e^t + 1)\,dt$, so $F'(p) = \ln(e^p + 1)$ $(\star)$

by the 2nd Fundamental Theorem.

Then $T(p) = F(p^4) - F(\sin p)$ $\left(= \int_0^{p^4} \ln(e^t+1)dt - \int_0^{\sin p} \ln(e^t+1)dt \right)$

so

$$\frac{dT}{dp} = \frac{d}{dp}F(p^4) - \frac{d}{dp}F(\sin p)$$

} Chain rule

$$= 4p^3 \cdot F'(p^4) - \cos p \cdot F'(\sin p)$$

} by $(\star)$

$$= 4p^3 \ln\left(e^{p^4} + 1\right) - \cos p \cdot \ln\left(e^{\sin p} + 1\right)$$

4. [5 points]

(a) Write a formula for the function whose derivative is $\frac{\sin x}{x}$ and whose graph passes through the point $(\pi, -1)$.

By 2nd Fundamental Theorem, want a function like

$$G(x) = \int_a^x \frac{\sin t}{t}\,dt + C$$

where a and C are constants. This satisfies $G(a) = C$,

so we want

$$\int_\pi^x \frac{\sin t}{t}\,dt - 1$$

Note: x = input variable, so it goes in upper limit of integral expression
t = dummy variable, pick anything other than x

(b) Suppose f is a continuous, odd function and define another function F by

$$F(x) = \int_{-12}^{x} f(5t - c)dt.$$

Find a value for the constant c such that the graph of F goes through the origin.

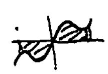
[See Integration Practice Solutions, problem 8.(a)] $c = -30$

Want $F(0) = 0$.

Know $F(0) = \int_{-12}^{0} f(5t-c)\,dt = \int_{5(-12)-c}^{5(0)-c} \frac{1}{5} f(w)\,dw = \frac{1}{5}\int_{-60-c}^{-c} f(w)\,dw$

$w = 5t - c$
$dw = 5\,dt$

$f(x)$ odd $\Rightarrow$ Know $\int_{-a}^{a} f(x)\,dx = 0$



$\Rightarrow$ choose c such that $\underbrace{-c}_{\text{top limit}} = -\underbrace{(-60-c)}_{\text{bottom limit}} = 60 + c$

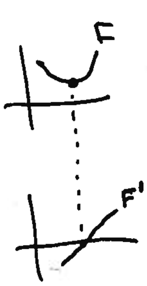
$c = -30$

5. [1 each] Circle True or False. No justification needed.

(a) $\frac{d}{dx}\underbrace{\int_0^5 t^2 dt}_{} = x^2$.

just a constant, $\frac{d}{dx} C = 0$ True $\boxed{\text{False}}$

(b) $F(x) = \int_{-2}^{x} t^2 dt$ has a local minimum at $x = 0$.



(F has local min.) $\iff$ (F' changes from (-) to (+)) True $\boxed{\text{False}}$

but here $F'(x) = x^2$ is always ≥ 0.

6. **[3 points, Extra credit!] Show using a substitution that the following integrals are equal:**

$$\int_1^{10} \frac{1}{x^3+1}\,dx = \int_{\frac{1}{11}}^{\frac{1}{2}} \frac{z}{1-3z+3z^2}\,dz.$$

To figure out the correct substitution, we should look at the new limits & integration. (No easy guess otherwise...)

$$\frac{1}{11} = \frac{1}{\underbrace{10}_{\text{old limit}}+1}, \qquad \frac{1}{2} = \frac{1}{\underbrace{1}_{\text{old limit}}+1}$$

$\Rightarrow$ guess $z = \frac{1}{x+1}$. (z = new variable x = old variable)

$$dz = -\frac{1}{(x+1)^2}\,dx$$

Apply substitution: need to solve for x and dx, in terms of z + dz

$$z = \frac{1}{x+1} \Rightarrow \frac{1}{z} = x+1, \quad x = \frac{1}{z} - 1 = \frac{1-z}{z}$$

$$dz = -\frac{1}{(x+1)^2}\,dx \Rightarrow dx = -(x+1)^2\,dz = -\frac{1}{z^2}\,dz$$

Then

$$\int_1^{10} \frac{1}{x^3+1}\,dx = \int_{\frac{1}{1+1}}^{\frac{1}{10+1}} \frac{1}{\left(\frac{1-z}{z}\right)^3+1}\cdot\left(-\frac{1}{z^2}\,dz\right) = -\int_{1/2}^{1/11} \frac{z^3}{(1-z)^3+z^3}\cdot\frac{1}{z^2}\,dz$$

$$= \int_{1/11}^{1/2} \frac{z}{(1-z)^3+z^3}\,dz = \int_{1/11}^{1/2} \frac{z}{1-3z+3z^2}\,dz.$$