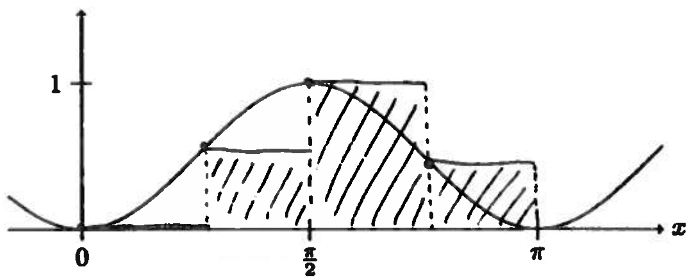
Math 116 - 005 Name: Solutions
Quiz 1 §§5.1 - 6.2

This quiz has a total of 24 points—the value of each question is specified. Please show all work.

1. [4 points] Consider the function $\sin^2(x)$ whose graph is shown below:



(a) [2] Write an expression for the left-hand Riemann sum with $n = 4$ subdivisions which approximates the definite integral

$$\int_0^\pi \sin^2(x)dx.$$

You do not need to simply the expression.

$$\Delta x = \frac{\pi - 0}{4} = \frac{\pi}{4}$$

Riemann sum $= \frac{\pi}{4}\left(\sin^2(0) + \sin^2\left(\frac{\pi}{4}\right) + \sin^2\left(\frac{\pi}{2}\right) + \sin^2\left(\frac{3\pi}{4}\right)\right)$

(b) [1] On the graph above, sketch the area measured by the Riemann sum in part (a).

(c) [1] The sum in part (a) is ____________ the right-hand Riemann sum.

larger than smaller than $\boxed{\text{equal to}}$

2. (a) [3] Find a and b which make the following identity hold:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \underbrace{\frac{10}{n}}_{\Delta x} \underbrace{e^{1+10k/n}}_{f(a+k\Delta x)} = \int_a^b e^x dx.$$

R.H. Riemann Sum

$= \sum_{k=1}^{n} e^{x_k}\Delta x = \sum_{k=1}^{n} e^{(a+k\Delta x)}\cdot\left(\frac{b-a}{n}\right) \Rightarrow$ $b - a = 10$

$a + k\cdot\frac{10}{n} = 1 + k\frac{10}{n}$

$\boxed{a = 1,\ b = 11}$

(b) [1] Evaluate $\int_a^b e^x dx$ in terms of a and b.

$\frac{d}{dx}e^x = e^x$, so $\int_a^b e^x dx = e^x\Big|_a^b = \boxed{e^b - e^a}$ OR $e^{11} - e^1$

2. [5 points] Let $f(x)$ be a function such that $f'(x) > 0$ and $f''(x) < 0$ for all x, and let $F(x)$ be an antiderivative of $f(x)$.
Explain whether each expression below gives the exact value of $f(3.1)$, an underestimate of $f(3.1)$, or an overestimate of $f(3.1)$, or if there is not enough information to decide.

(a) $f(3) + \int_3^{3.1} f'(t)dt$

$\boxed{\text{exact}}$ under over not enough info.

by First Fundamental Theorem,

$$f(3.1) - f(3) = \int_3^{3.1} f'(t)\,dt$$

(b) $\dfrac{F(3.11) - F(3.1)}{0.01}$

exact under $\boxed{\text{over}}$ not enough info.

by First Fund. Theorem,

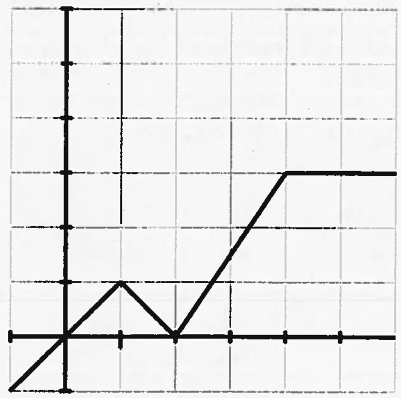
$$\frac{F(3.11) - F(3.1)}{0.01} = \frac{1}{0.01}\int_{3.1}^{3.11} f(x)\,dx$$

$= ($average value of $f(x)$ from 3.1 to 3.11$)$

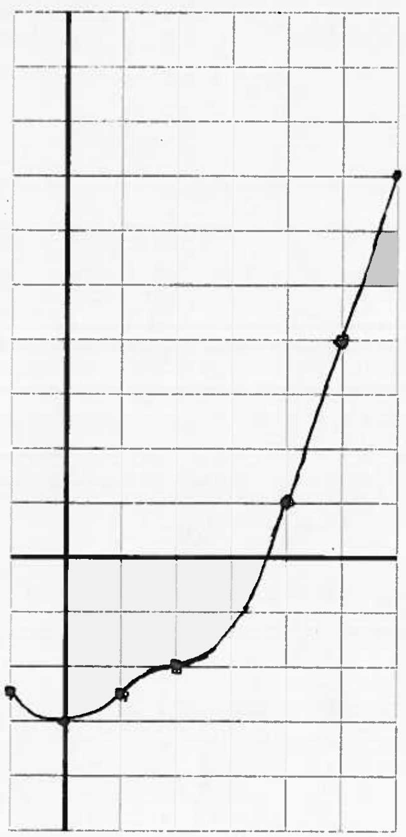
Since $f'(x) > 0$, f is an increasing function so

(avg. value of f on $[3.1, 3.11]$) $\geq$ (value of f at 3.1) $= f(3.1)$

3. [5 points] The graph of $y = g(x)$ is given below.



Suppose $G(x)$ is an antiderivative of $g(x)$ and that $G(0) = -3$. Write down the values for $G(-1)$ and $G(5)$ and carefully draw the graph of $G(x)$ below.



$G(-1) =$ -2.5

$G(5) =$ $\cancel{}$ 4

$G(0) - G(-1) = \int_{-1}^{0} g(x)\,dx = -0.5$

$G(5) - G(0) = \int_{0}^{5} g(x)\,dx = 1 + 3 + 3$

$= 7$

$G(5) = G(0) + 7 = 4$

graph should indicate:

- increasing / decreasing
- concavity
- y-values from area under $g(x)$
- no corners $\Leftrightarrow$ no jumps in $g(x)$

4. [6 points] For each statement below, circle True or False and include a brief justification. Let Avg(f) denote the average value of the function f on the interval $[1, 4]$.

   (a) For any functions f and g, $\text{Avg}(f+g) = \text{Avg}(f) + \text{Avg}(g)$.

   $$\frac{1}{3}\int_1^4 (f+g)\,dx = \frac{1}{3}\int_1^4 f\,dx + \frac{1}{3}\int_1^4 g\,dx$$

   $\boxed{\text{True}}$ False

   (b) For any functions f and g, $\text{Avg}(fg) = \text{Avg}(f)\cdot\text{Avg}(g)$.

   Consider $f(x) = x$ and $g(x) = \frac{1}{x}$, (or many other examples) $\text{Avg}(fg) = 1$, $\text{Avg}(f) = \frac{5}{2}$, $\text{Avg}(g) = \frac{1}{3}\ln(4)$

   True $\boxed{\text{False}}$

   (c) If $f(x)$ is a continuous on the interval $[a, b]$, then $\int_a^b (5 + 3f(x))dx = 5 + 3\int_a^b f(x)dx$.

   $$\int_a^b (5+3f)\,dx = 5\cdot(b-a) + 3\int_a^b f\,dx$$

   True $\boxed{\text{False}}$

5. [Extra Credit!]

   (a) [2] Explain what is wrong with the following argument that $\lim_{n\to\infty}(n\sin(1/n)) = 0$.

   $$\lim_{n\to\infty}(n\sin(1/n)) = n\lim_{n\to\infty}(\sin(1/n)) = n(\sin(0)) = n\cdot 0 = 0.$$

   We cannot move n outside of the limit in the first step, since the value of n depends on the limit.

   (b) [1] What is the actual value of the limit $\lim_{n\to\infty}(n\sin(1/n))$? Justify your answer. (Hint: You can use your calculator.)

   Evaluate when n = (some large numbers):

   $10\cdot\sin(1/10) \approx 0.9983$,

   $100\cdot\sin(1/100) \approx 0.999983$,

   $1000\cdot\sin(1/1000) \approx 0.99999983$ $\Rightarrow \lim_{n\to\infty}(n\sin(1/n)) = 1$