3. [14 points] Consider functions $f(x)$ and $g(x)$ satisfying:

(i) $g(x)$ is an odd function.

(ii) $\int_2^7 g(x)dx = 3$.

(iii) $\int_2^7 f(x)dx = 17$.

(iv) $f(2) = 1$.

(v) $\int_1^6 f'(x)dx = 12$.

(vi) $\int_2^7 f'(x)dx = 3$.

Compute the value of the following quantities. If it is impossible to determine their value with the information provided above, write "NI" (not enough information).

a. [2 points] $\int_{-2}^7 g(x)\ dx =$ _______

Solution: 3, using i and vi.

b. [2 points] $\int_2^7 (f(x) - 8g(x))\ dx =$ _______

Solution: -7, using ii and iii.

c. [2 points] $f(7) =$ _______

Solution: 4, using the Fundamental Theorem of Calculus with iv and vi.

d. [2 points] $\int_1^6 f'(x+1)\ dx =$ _______

Solution: We use u substitution, $u = x + 1$. Making sure to change the limits of integration, we get $\int_2^7 f'(u)du = 3$.

e. [3 points] $\int_2^7 xf'(x)\ dx =$ ______

Solution: We integrate by parts with $u = x, dv = f'$.

$$\int_2^7 xf'(x)dx = xf(x)|_2^7 - \int_2^7 f(x)dx = (7f(7) - 2f(2)) - 17 = 28 - 2 - 17 = 9.$$

f. [3 points] $\int_2^3 xf(x^2 - 2)\ dx =$ ______

Solution: We use u substitution $u = x^2-2, du = 2xdx$. We get $\frac{1}{2}\int_2^7 f(u)du = \frac{17}{2} = 8.5$.

7. [6 points] Suppose that g is a continuous function, and define another function G by

$$G(x) = \int_0^x g(t)\,dt.$$

Given that $\int_0^7 g(x)\,dx = 5$, compute

$$\int_0^7 g(x)(G(x))^2\,dx.$$

Show each step of your computation.

Solution: Substitution gives

$$\int_0^7 g(x)(G(x))^2\,dx = \int_{G(0)}^{G(7)} u^2\,du = \left.\frac{u^3}{3}\right|_0^5 = \frac{125}{3}.$$

Alternatively, integrate by parts to obtain

$$\int_0^7 g(x)(G(x))^2\,dx = (G(x))^3\Big|_0^7 - 2\int_0^7 g(x)(G(x))^2\,dx,$$

which after rearranging gives

$$\int_0^7 g(x)(G(x))^2\,dx = \frac{1}{3}\left((G(x))^3\Big|_0^7\right) = \frac{125}{3}.$$

8. [6 points] Suppose that f is a continuous, **odd** function, and define another function F by

$$F(x) = \int_{-12}^{x} f(3t - c)\,dt,$$

where c is some constant. You do not need to show your work for this problem.

a. [3 points] Find a value of c for which the graph of F goes through the origin.

Solution: The correct value is $c = -18$.

$$F(0) = \int_{-12}^{0} f(3t-c)\,dt$$

substitution: $u = 3t - c$

$du = 3\,dt$

$$= \int_{-36-c}^{-c} f(u)\cdot\frac{1}{3}\,du$$

$$= \frac{1}{3}\int_{-36-c}^{-c} f(u)\,du$$

top limit, bottom limit

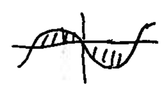
$-c = -(-36-c) = c+36$

$2c = -36$

$c = -18$

f odd $\Rightarrow$ want limits to be opposite

$$\int_{-a}^{a} f(u)\,du = 0$$



b. [3 points] Find a value of c for which the graph of F' goes through the origin.

Solution: The correct value is $c = 0$.

$F'(x) = f(3x - c)$

$F'(0) = f(-c)$

f odd $\Rightarrow$ know $f(0) = 0$, so choose $c = 0$.

8. [6 points] Suppose that $\int_{-3}^{8} f(x)dx = 5$. Use this information to determine the values for the constants $a, b,$ and k that you are certain will satisfy the definite integral $\int_a^b kf(2x)dx = 5$. Write your answers on the spaces provided. You do not need to show your work for this problem.

$$k\int_a^b f(2x)\,dx = k\int_{2a}^{2b} f(u)\cdot \frac{1}{2}du = \frac{k}{2}\int_{2a}^{2b} f(u)du \overset{\text{WANT}}{=} \int_{-3}^{8} f(x)dx$$

$u = 2x$

$du = 2\,dx$

$$\Rightarrow \begin{cases} k/2 = 1, \\ 2a = -3 \\ 2b = 8 \end{cases}$$

$a =$ **-1.5** $b =$ **4** $k =$ **2**

9. [6 points] Suppose $f(x) = f'(x) + 3$. Determine the EXACT value of $\int_0^1 e^x f'(x)dx$ given that $f(0) = 1$ and $f(1) = 4$. Be sure to show enough work to support your answer.

Solution: We use integration by parts, letting $u = e^x$ and $dv = f'(x)dx$ so that $du = e^x dx$ and $v = f(x)$. Then we have

$$\begin{aligned}
\int_0^1 e^x f'(x)dx &= e^x f(x)|_0^1 - \int_0^1 e^x f(x)dx \\
&= ef(1) - f(0) - \int_0^1 e^x(f'(x) + 3)dx \\
&= 4e - 1 - \int_0^1 e^x f'(x)dx - 3\int_0^1 e^x dx \\
2\int_0^1 e^x f'(x) &= 4e - 1 - 3e^x|_0^1 \\
\int_0^1 e^x f'(x) &= \frac{1}{2}((4e-1) - (3e-3)) = \frac{e+2}{2}
\end{aligned}$$

1. [10 points] Indicate if each of the following is true or false by circling the correct answer. No justification is required.

b. [2 points] The function $F(x) = \int_1^{x^2} \sin(e^t)dt$ is an even function.

[True] False

Solution: The function $F(x)$ is even if it satisfies $F(-x) = F(x)$. Since

$$F(-x) = \int_1^{(-x)^2} \sin(e^t)dt = \int_1^{x^2} \sin(e^t)dt = F(x).$$

then $F(x)$ is even.

c. [2 points] Let $h(x)$ be an antiderivative of $g(x)$. If $g(x)$ is measured in kg and x in inches, then the units for $h(x)$ are kg per inch.

True **[False]**

Solution: The second fundamental theorem of calculus says that $h(x) = \int_a^x g(t)dt$ for some contant a. The units of $g(x)$ and x are kg and inches respectively, then the units of $h(x)$ are kg $\cdot$ inches.

d. [2 points] The function $R(t) = \int_t^{1-t} e^{x^3}dx$ is decreasing for all values of t.

[True] False

Solution:

$$R'(t) = -e^{(1-t)^3} - e^{t^3} < 0 \text{ for all values of } t.$$

Hence $R(t)$ is decreasing.