2. [13 points] Determine if each of the following sequences is increasing, decreasing or neither, and whether it converges or diverges. Circle all the answers that apply. On parts **a-c**, if the sequence converges, find the limit. No justification is required.

a. [3 points] For $n \geq 1$, let $a_n = 3 + \frac{1}{n}$.

Solution:

1. Increasing $\boxed{\textbf{Decreasing}}$ Neither.
2. $\boxed{\textbf{Convergent}}$: $\lim_{n\to\infty} a_n = 3$ Divergent

b. [3 points] For $n \geq 1$, let $a_n = (-\frac{\pi}{e})^n$.

Solution:

1. Increasing Decreasing $\boxed{\textbf{Neither.}}$
2. Convergent: $\lim_{n\to\infty} a_n =$ __________ $\boxed{\textbf{Divergent}}$

c. [3 points] Let $P(x)$ be the cumulative distribution function of a nonzero probability density function $p(x)$. Define $a_n = P(n)$ for $n \geq 1$.

Solution:

1. $\boxed{\textbf{Increasing}}$ Decreasing Neither.
2. $\boxed{\textbf{Convergent}}$: $\lim_{n\to\infty} a_n = 1$ Divergent

d. [2 points] For $n \geq 1$, let $a_n = 1 - \dfrac{1}{1!} + \dfrac{1}{2!} - \dfrac{1}{3!} + \cdots + \dfrac{(-1)^n}{n!}$.

Solution:

1. Increasing Decreasing $\boxed{\textbf{Neither}}$.
2. $\boxed{\textbf{Convergent}}$ (no need to compute the limit) Divergent

e. [2 points] Let $a_n = \displaystyle\int_2^n \frac{1}{\sqrt{x}-1} dx$, for $n \geq 2$.

Solution:

1. $\boxed{\textbf{Increasing}}$ Decreasing Neither.
2. Convergent (no need to compute the limit) $\boxed{\textbf{Divergent}}$

4. [12 points] Consider the following sequences:

$$f_n = \frac{\pi^n}{e^n} \qquad g_n = (-1)^n \sin(n) \qquad h_n = \cos(e^{-n}) \qquad i_n = \int_1^n \frac{1}{(x+3)^2}dx$$

For each sequence, circle **all** that apply. No justification is necessary.

a. [2 points] The sequence (f_n) is :

Bounded $\qquad$ $\boxed{\text{Increasing}}$ $\qquad$ Decreasing

b. [2 points] The sequence (g_n) is :

$\boxed{\text{Bounded}}$ $\qquad$ Increasing $\qquad$ Decreasing

c. [2 points] The sequence (h_n) is :

$\boxed{\text{Bounded}}$ $\qquad$ $\boxed{\text{Increasing}}$ $\qquad$ Decreasing

d. [2 points] The sequence (i_n) is :

$\boxed{\text{Bounded}}$ $\qquad$ $\boxed{\text{Increasing}}$ $\qquad$ Decreasing

e. [4 points] For each given sequence, if it converges, determine its limit and write that limit in the space provided. If the sequence diverges, write "diverges". No justification is necessary.

(f_n): **diverges** $\qquad$ (h_n): **1**

(g_n): **diverges** $\qquad$ (i_n): **1/4**

2. [10 points] Consider an outdoor pool initially filled with 20,000 gallons of water. Each day 4% of the water in the pool evaporates. Each morning at 10:00am, W gallons of water are added back to the pool where W is a constant.

a. [3 points] Let A_n be the number of gallons of water in the pool immediately after water is added back to the pool for the n^{th} time. Given that $A_1 = 19200 + W$, find A_2 and A_3. Put your final answers in the answer blanks.

Solution:
$A_2 = (20,000)(\frac{24}{25})^2 + W(\frac{24}{25}) + W.$
$A_3 = (20,000)(\frac{24}{25})^3 + W(\frac{24}{25})^2 + W(\frac{24}{25}) + W.$

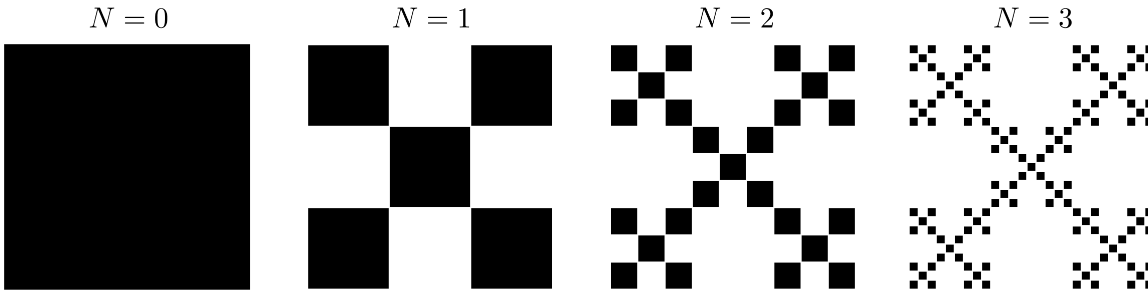
b. [4 points] Find a closed form expression for A_n (i.e. evaluate any sums and solve any recursion). Note your answer may contain the constant W.

Solution: $A_n = \frac{24}{25}A_{n-1} + W$. Expanding this recursion or following the pattern from part a we have $A_n = 20,000(\frac{24}{25})^n + \sum_{k=0}^{n-1} W(\frac{24}{25})^k$. Using the formula for finite geometric series we have $A_n = 20,000(\frac{24}{25})^n + 25W(1 - (\frac{24}{25})^n)$.

c. [3 points] If the pool has a maximum capacity of 25,000 gallons, find the largest value of W so that the pool does not overflow eventually.

Solution: Depending on the value of W, A_n is always increasing or always decreasing. Therefore the amount of water in the pool is the largest either when it is first filled at 20,000 gallons or when n approaches infinity where we have $\lim_{n\to\infty} A_n = 25W$. Therefore our only restriction is $25W \leq 25,000$ thus $W \leq 1,000$. So the largest possible value is $W = 1,000$.

11. [12 points] You construct a snowflake by starting with a square piece of paper of side length 3 inches. You divide the square into a three by three grid of squares of side length one and remove the four squares in the grid that share a side with the center square in the grid. For each remaining square in the grid, subdivide each of them into 9 equally sized squares and remove the four squares in each of these new grids that share a side with the center square in the grid. You continue in this manner for a long time.



a. [3 points] Write a formula that gives the perimeter, P_N, of the black squares that make up the snowflake after N steps.

Solution: $P_N = 12\left(\frac{5}{3}\right)^N$

b. [2 points] Find $\lim_{N\to\infty} P_N$.

Solution: P_N tends to infinity as $N \to \infty$.

c. [3 points] Suppose $N \geq 1$. Write a sum that gives the area, A_N of all the squares you have **removed** after N steps.

Solution: $\sum_{j=0}^{N-1} 4\left(\frac{5}{9}\right)^j$

d. [2 points] Write a closed form expression for A_N.

Solution: $A_N = 4\frac{1-\left(\frac{5}{9}\right)^N}{1-\frac{5}{9}}$

e. [2 points] Find the limit as $N \to \infty$ of your expression in (d).

Solution: $\lim_{N\to\infty} A_N = 9$