ME542 Vehicle Dynamics

Winter 2014

Tu & Th 1:30-3:00pm

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Office hours: Mon. 4-5pm (Peng), Wed. 1-2pm (Peng)
GSI office hours to be announced

Vehicle Dynamics and Safety

- Accident occurred in May 2006 on I-96, caught by police dash-cam.
- Induced by an initial light side impact, SUV rolled over multiple times.
Lecture 1--Introduction and Motivation
(and administrative stuff)

• About this course
  – Introduction and administrative information
  – Major course content
  – Grading policy
  – MATLAB/SIMULINK

• Review of rigid body dynamics

Administrative Information

• Textbook (not required)

• Course material will be made available on the Ctools site
  (PPT files should be downloaded and printed before the lectures, HW, solutions, example MATLAB programs will be distributed using this site)
Course Requirements

• Prerequisites
  – Knowledge in Newtonian Dynamics (ME240 level) is essential
  – That of Automotive Engineering (ME458) and Intermediate Dynamics (ME440) are helpful but not required.
  – Familiarity with Matlab/Simulink, since Matlab/Simulink is used extensively in the lecture examples and homework assignments.

Major course content

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Related Courses

- **ME458 Automotive Engineering**
  Emphasizes the vehicle as an engineering system and review design consideration associated with all major systems including the vehicle structure, powertrain, suspension, steering and braking.

- **ME568 Vehicle Control Systems**
  Covers control issues for all major vehicle control systems including engine control, cruise control, ABS/traction control, four-wheel steering, active suspension and advanced control systems for Intelligent Transportation Systems.

Grading Policy

- **Grading:**
  - 5-6 Homework: 55%
  - Midterm exam: 20%
  - Final exam: 25%

- **Homework:** Must be handed in on the due date in class (on-campus students) or uploaded to Ctools (distance learning students). Late homework will be accepted up to 48 hours late with a 20% penalty for each 24 hours (rounded up, i.e., 0 to 24 hours late = 24 hours late). All problem sets (home work assignments) are to be completed on your own. You may discuss homework assignments with your fellow students at the conceptual level, but must complete all calculations and write-up, from scrap to final form, on your own. Verbatim copying of another student's work is forbidden. If you have any questions about this policy, please do not hesitate to contact the instructor.

- **Exams:**
  - Midterm: in class, Dynamics, Tire, Handling
  - Final exam: 2 hours, Accumulative but more weight on Handling and Ride
MATLAB/SIMULINK Tutorial

Getting a Copy of MATLAB/SIMULINK

CAEN labs

Student version:

http://www.mathworks.com/products/studentversion/

Virtual Site

http://virtualsites.umich.edu/
MATLAB/SIMULINK

• Example MATLAB/SIMULINK programs will be distributed, which you can freely use/modify.

• MATLAB is used for simple (usually linear) vehicle dynamic simulations and analysis for this course.

• SIMULINK: A GUI based simulation program. Strength:
  – Realistic dynamic phenomenon such as nonlinearity, quantization, noise, switches, look-up tables, delays, etc. can be simulated easily.
  – GUI makes it easy to recycle vehicle simulation modules

Review of Rigid Body Dynamics

• Vehicle Coordinate Systems

• Newton/Euler Formulation

• Lagrange Formulation
Vehicle Coordinate System

- First step in deriving vehicle dynamic equations.
- The Society of Automotive Engineers (SAE) has introduced standard coordinates and notations for describing vehicle dynamics.

ISO coordinate: x is the same but y and z are reversed.

SAE Vehicle-Fixed Coordinate System
--Symbols and Definitions

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<th>Translational Velocity</th>
<th>Angular Displacement</th>
<th>Angular Velocity</th>
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<td>u (forward)</td>
<td>φ</td>
<td>p or ( \dot{\phi} ) (roll)</td>
<td>F_x</td>
<td>M_x</td>
</tr>
<tr>
<td>y</td>
<td>v (lateral)</td>
<td>( \theta )</td>
<td>q or ( \dot{\theta} ) (pitch)</td>
<td>F_y</td>
<td>M_y</td>
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<tr>
<td>z</td>
<td>w (vertical)</td>
<td>( \psi )</td>
<td>r or ( \dot{\psi} ) (yaw)</td>
<td>F_z</td>
<td>M_z</td>
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**Pitch** angle: the angle between x-axis and the horizontal plane.

**Roll** angle: the angle between y-axis and the horizontal plane.

**Yaw** angle: the angle between x-axis and the X-axis of a inertia frame.
Earth Fixed Coordinate and Vehicle Slip

OXYZ fixed on Earth (does not turn with the vehicle)

Course angle \( \nu = \psi + \beta \)

Heading angle

Sideslip angle

In the left figure, sideslip angle is negative

\[ \tan \beta = \frac{\nu}{u} \]

Tire Slip

- \( o \): Center of tire contact patch
- \( Z \): Perpendicular to the ground plane.
- \( X \) and \( Y \) axes: On the ground plane.
- Tire camber angle: Angle between the XZ plane and the wheel plane.

Source: Wong page 7
Ride/Handling Dynamics

Involves vehicle motions in several directions

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<th>RIDE</th>
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<td>longitudinal</td>
<td>(√)</td>
<td>(√)</td>
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<tr>
<td>lateral</td>
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<td>vertical</td>
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<td>pitch</td>
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<td>vertical</td>
<td>(√)</td>
<td>(√)</td>
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<tr>
<td>wheel rotation</td>
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<td>(√)</td>
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Newton/Euler Formulation

Consider a rigid body of mass $m$, with c.g. at point $o$, subject to $N$ external forces.

For unconstrained motion, a rigid body possesses 6 DOF, 3 translational $(x, y, z)$ and 3 rotational $(\phi, \theta, \psi)$. 

![Diagram of Newton/Euler Formulation](image)
Newton/Euler Equations

Let $V$ and $\omega$ to denote the absolute velocity of mass center and angular velocity of the rigid body, $(\hat{i}, \hat{j}, \hat{k})$ be unit vectors of the body-fixed coordinate oxyz. The Newton/Euler equations of motion are then

Newton’s 2nd law:

$$\sum_{i=1}^{N} F_i = F = m \frac{dV}{dt} = \frac{dL}{dt}$$

Euler

$$\sum_{i=1}^{N} r_i \times F_i = M_o = \frac{dH}{dt}$$

Where $L$ is the linear momentum and $H$ is the angular momentum of the rigid body about the mass center $o$.

Angular Momentum

Let $H = H_x \hat{i} + H_y \hat{j} + H_z \hat{k}$

$$\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = I \cdot \omega$$

where $I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$

$I_{xx} \equiv \int (y^2 + z^2) \rho \cdot dV$

$I_{xy} \equiv \int xy \rho \cdot dV$

$I_{xz} \equiv \int xz \rho \cdot dV$

$I_{yy} \equiv \int \rho \cdot dV$

$I_{yz} \equiv \int yz \rho \cdot dV$

$I_{zz} \equiv \int \rho \cdot dV$

$$H = \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} I_{xx} p - I_{xy} q - I_{xz} r \\ -I_{xy} p + I_{yy} q - I_{yz} r \\ -I_{xz} p - I_{yz} q + I_{zz} r \end{bmatrix}$$
Rigid Body Motion of a Vehicle
(a key concept: rate of change of a vector with respect to a rotating frame)

Translational motion: \[ \sum F = ma \]

\[ \sum F = ma = m \cdot \frac{D}{Dt} [u\hat{i} + v\hat{j} + w\hat{k}] \]

= \[ m \left[ u\hat{i} + v\hat{j} + w\hat{k} + \frac{d}{dt} [u\hat{i} + v\hat{j} + w\hat{k}] \right] \]

\[ \sum F_x = m(\dot{u} + qw - rv) \]
\[ \sum F_y = m(\dot{v} + ru - pw) \]
\[ \sum F_z = m(\dot{w} + pv - qu) \]

Cross Product
\[ \frac{D}{Dt} = \frac{d}{dt} + \omega \times \]
Rate of change due to the rotation of the frame

Rate of change wrt a fixed frame \hspace{5cm} Rate of change wrt a rotating frame

Rigid Body Motion of a Vehicle (cont.)

Rotational motion: \[ \sum M = \dot{H} \]

\[ \sum M = \left\{ \frac{d}{dt} \begin{bmatrix} I_{xx} p - I_{xy} q - I_{xz} r \\ -I_{xy} p + I_{yy} q - I_{yz} r \\ -I_{xz} p - I_{yz} q + I_{zz} r \end{bmatrix} \right\} + \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \begin{bmatrix} I_{xx} p - I_{xy} q - I_{xz} r \\ -I_{xy} p + I_{yy} q - I_{yz} r \\ -I_{xz} p - I_{yz} q + I_{zz} r \end{bmatrix} \]

\[ \sum M_x = I_{xx} \dot{p} - I_{xy} \dot{q} - I_{xz} \dot{r} - I_{xz} p q - I_{yz} r q + I_{yy} q r + I_{yz} r^2 \]
\[ \sum M_y = -I_{xy} \dot{p} + I_{yy} \dot{q} - I_{yz} \dot{r} + I_{xx} p r - I_{xy} q r - I_{xz} r^2 + I_{xx} p^2 + I_{yy} q p - I_{zz} r p \]
\[ \sum M_z = -I_{xz} \dot{p} + I_{yz} \dot{q} + I_{zz} \dot{r} - I_{xy} p^2 + I_{yy} q p - I_{yz} r p - I_{xx} p q + I_{xx} q^2 + I_{xz} r q \]
Equations of Vehicle Rigid Body Motion

\[
\sum F_x = m(\ddot{u} + qw - rv) \\
\sum F_y = m(\ddot{v} + ru - pw) \\
\sum F_z = m(\ddot{w} + pv - qu) \\
\sum M_x = I_{xx} \dot{p} - I_{xy} \dot{q} - I_{xz} \dot{r} - I_{xz} pq - I_{y} q^2 + I_{zz} \dot{r} q + I_{xy} pr - I_{y} qr + I_{xz} r^2 \\
\sum M_y = -I_{xy} \dot{p} + I_{yy} \dot{q} - I_{yz} \dot{r} + I_{xx} pr - I_{xy} qr - I_{xz} r^2 + I_{xz} p^2 + I_{yz} qp - I_{zz} rp \\
\sum M_z = -I_{xz} \dot{p} - I_{yz} \dot{q} + I_{zz} \dot{r} - I_{xy} p^2 + I_{yy} qp - I_{yz} r p - I_{xz} p q + I_{yy} q^2 + I_{xz} r q 
\]

Simplified Equations of Motion

• Assume
  – vehicle is symmetric wrt the xz plane \((l_{xy} = l_{yz} = 0)\)
  – \(p, q, r, v, \) and \(w\) are small, i.e., their higher order terms are negligible.
  – \(u = u_o + u', \) \(u'\) is small compared with \(u_o.\)

\[
\sum F_x = m(\ddot{u} + qw - rv) \\
\sum F_y = m(\ddot{v} + ru - pw) \\
\sum F_z = m(\ddot{w} + pv - qu) \\
\sum M_x = I_{xx} \dot{p} - I_{xz} \dot{r} \\
\sum M_y = I_{yy} \dot{q} \\
\sum M_z = -I_{xz} \dot{p} + I_{zz} \dot{r}
\]
Review of Rigid Body Kinematics

- Kinematics: study of the geometry of motions.
- Quantities: position, velocity and acceleration.

**Given:** Kinematics quantities of one point (mass center), and the angular motion of the rigid body.

**Find:** Those at another point on the same rigid body.

Application: e.g., use motion of c.g. to infer motion of other points on the vehicle.

---

Rigid Body Kinematics

**Position:**

\[
\mathbf{r}_p = \mathbf{r}_o + \mathbf{r}_{rel}
\]

\[
\mathbf{\dot{r}}_p = \mathbf{\dot{r}}_o + \mathbf{\dot{r}}_{rel}
\]

**Velocity:**

\[
\mathbf{V}_p = \mathbf{V}_o + \mathbf{\omega} \times \mathbf{r}_{rel}
\]

\[
\mathbf{\dot{V}}_p = \mathbf{\dot{V}}_o + \mathbf{\omega} \times \mathbf{r}_{rel} + \mathbf{\omega} \times \mathbf{\dot{r}}_{rel}
\]

**Acceleration:**

\[
\mathbf{a}_p = \mathbf{a}_o + \mathbf{\dot{\omega}} \times \mathbf{r}_{rel} + \mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{r}_{rel})
\]
Example 1

A car is traveling in the xz plane, pitching, bouncing

Given:
\[ V_o = u \hat{t} + w \hat{k} \]
\[ \omega = q \hat{j} \]
\[ r_{rel} = l \hat{t} \]

Find:
\[ V_p \quad a_p \]

Example 1 Solution

\[ V_o = u \hat{t} + w \hat{k} \]
\[ \omega = q \hat{j} \]
\[ r_{rel} = l \hat{t} \]

\[ V_p = V_o + \omega \times r_{rel} = \begin{bmatrix} u \\ 0 \\ w \end{bmatrix} + \begin{bmatrix} 0 \\ q \\ 0 \end{bmatrix} \times \begin{bmatrix} l \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} u \\ 0 \\ w - ql \end{bmatrix} \]

\[ = u \hat{t} + (w - ql) \hat{k} \]
Example 1 Solution

\[ V_o = u\hat{i} + w\hat{k} \]
\[ \omega = q\hat{j} \]
\[ r_{rel} = l\hat{i} \]

\[ a_p = a_o + \dot{\omega} \times r_{rel} + \omega \times (\omega \times r_{rel}) \]

\[
\begin{bmatrix}
\dot{u} \\
0 \\
\dot{w}
\end{bmatrix}
\begin{bmatrix}
0 \\
q \times 0 \\
0
\end{bmatrix}
\begin{bmatrix}
u \\
0 \\
w
\end{bmatrix}
\begin{bmatrix}
l \\
0 \\
0
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
q \times 0 \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
l \\
-q l
\end{bmatrix}
\]

Another Kinematic Example (hw1)

- Vehicle Bicycle Model
- Vehicle is driving in the xy plane at a constant forward speed, and turning. All other motions are ignored
- Find the speed and acceleration at the center of the front axle and the rear left tire
Example 2

- Vehicle Bicycle Model
- Vehicle is driving in the xy plane, and yawing. All other motions are ignored
- Find the dynamic equations

\[ \sum F_x = m(\dot{u} + qw - rv) \]
\[ \sum F_y = m(\dot{v} + ru - pw) \]
\[ \sum F_z = m(\dot{w} + pv - qu) \]
\[ \sum M_x = I_{xx} \dot{p} - I_{xy} \dot{q} - I_{xz} \dot{r} - I_{xx} p \dot{q} - I_{yz} q^2 + I_{yy} \dot{r} q + I_{xy} \dot{p} r - I_{yy} q r + I_{yz} r^2 \]
\[ \sum M_y = -I_{xy} \dot{p} + I_{yy} \dot{q} - I_{xz} \dot{r} + I_{xx} \dot{p} r - I_{xy} q r - I_{xx} r^2 + I_{xz} p^2 + I_{yz} q p - I_{zz} r p \]
\[ \sum M_z = -I_{xz} \dot{p} - I_{yz} \dot{q} + I_{zz} \dot{r} - I_{xy} \dot{p} q + I_{yy} \dot{p} r - I_{xx} \dot{p} q + I_{xy} q^2 + I_{xz} r q \]

\[ ? \]
Lagrange’s Formulation

- An alternative way to formulate the equations of motion.
- Starts from “energy” (scalar!) rather than vectors.
- We first define three terms

**Degrees Of Freedom (DOF):**
Number of independent coordinates required to uniquely define the motion of a particle, a body or a system of bodies.

**Constraints**
Kinematic relationship that limit possible motions.

**Generalized coordinates**
A set of independent coordinates whose number is equal to the DOF and which uniquely define the position and orientation of the rigid body.

Examples for DOF, Constraints, and GC

Position of a particle in space: 3DOF

Position of a particle on a cylindrical surface

2DOF, position uniquely described by 2 coordinates, which are referred to as generalized coordinates \((q_1, q_2)\).

Rigid wheel rolling on flat surface without slip, DOF=?
Rigid wheel rolling on flat surface with slip, DOF=?
Lagrange Equations

Consider a system of particles or rigid bodies with \( n \) DOF described by \( n \) generalized coordinates \( \{q_1(t), \ldots, q_n(t)\} \). The Lagrange's equation is then

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i
\]

where

\[
T = T(q_i, \dot{q}_i) : \text{kinetic energy}
\]

\[
V = V(q_i) : \text{potential energy}
\]

\( Q_i : \text{the } i^{th} \text{ generalized non-conservative force}. \)

\[
Q_i = \sum_{j=1}^{k} E_j \cdot \frac{\partial r_j}{\partial q_i}
\]

Example 3: Compound Pendulum

Derive the equation of motion for the compound pendulum shown below for large angle \( \theta \).
DOF: only 1 (q)

Kinetic energy

\[ T = \frac{1}{2} m_s (L+r)^2 \dot{\theta}^2 + \frac{1}{2} I_s \dot{\theta}^2 + \frac{1}{2} m_b \left( \frac{L}{2} \dot{\theta} \right)^2 + \frac{1}{2} I_b \dot{\theta}^2 \]

disk \hspace{2cm} bar

Potential energy

\[ V = m_s g (L+r)(1-\cos \theta) + m_b g \frac{L}{2} (1-\cos \theta) \]

disk \hspace{2cm} bar

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\theta}} \right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = Q_i \]

\[ \dot{\theta} \left[ m_s (L+r)^2 + m_b \left( \frac{L}{2} \right)^2 + I_s + I_b \right] + g \sin \theta \left[ m_s (L+r) + m_b \frac{L}{2} \right] = 0 \]

Example 4: Inverted Pendulum

• Later in the term, we will apply the Lagrange’s method to derive the lateral-yaw-roll model.
• Here we will start from a simpler system, which only involves lateral and roll (yaw is ignored for now)
Example 4 Solution

Newtonian Method: for the pendulum

\[ x_p = x - \frac{L_p}{2} \sin \alpha \quad y_p = \frac{L_p}{2} \cos \alpha \]

x-direction

\[ F_x = m_p \frac{d^2}{dt^2} (x_p) = m_p \frac{d}{dt} \left( \dot{x} - \frac{L_p}{2} \dot{\alpha} \cos \alpha \right) \]
\[ = m_p \left( \ddot{x} - \frac{L_p}{2} \ddot{\alpha} \cos \alpha + \frac{L_p}{2} \dddot{\alpha} \sin \alpha \right) \]

y-direction

\[ F_y - m_p g = m_p \frac{d^2}{dt^2} (y_p) = m_p \frac{d}{dt} \left( -\dot{\alpha} \sin \alpha \right) \]
\[ = m_p \left( -\ddot{\alpha} \sin \alpha - \dddot{\alpha} \cos \alpha \right) \]

roll-direction

\[ F_y \frac{L_p}{2} \sin \alpha + F_x \frac{L_p}{2} \cos \alpha = I_p \ddot{\alpha} \quad I_p = \frac{1}{12} m_p L_p^2 \]

Example 4 Solution (cont.)

Newtonian Method: for the cart

\[ F - F_x = m_c \ddot{x} \]
\[ F = (m_c + m_p) \ddot{x} - m_p \frac{L_p}{2} \dddot{\alpha} \cos \alpha + m_p \frac{L_p}{2} \dddot{\alpha} \sin \alpha \]

\[ F_y \frac{L_p}{2} \sin \alpha + F_x \frac{L_p}{2} \cos \alpha = I_p \ddot{\alpha} \quad \text{Plug-in equations from x- and y- direction} \]

\[ m_p g \sin \alpha + m_p \dot{x} \cos \alpha = \frac{2}{3} m_p L_p \dddot{\alpha} \]

We wrote down four equations, two of them used to cancel the internal forces Fx and Fy.
Example 4 Solution (cont.)

Lagrange's method

\[ x_p = x - \frac{L_p}{2} \sin \alpha \quad \Rightarrow \quad \dot{x}_p = \dot{x} - \dot{\alpha} \frac{L_p}{2} \cos \alpha \]
\[ y_p = \frac{L_p}{2} \cos \alpha \quad \Rightarrow \quad \dot{y}_p = -\dot{\alpha} \frac{L_p}{2} \sin \alpha \]

The kinetic energy and potential energy are then

\[ T = \frac{1}{2} m_v \dot{x}^2 + \frac{1}{2} m_p [\dot{x}_p^2 + \dot{y}_p^2] + \frac{1}{2} I_p \dot{\alpha}^2 = \frac{1}{2} m_v \dot{x}^2 + \frac{1}{2} m_p [\dot{x}^2 - 2\dot{x}\dot{\alpha}\frac{L_p}{2} \cos \alpha + \frac{1}{3} \dot{\alpha}^2 \frac{L_p^2}{2}] \]
\[ V = m_p g \frac{L_p}{2} \cos \alpha \]

For x-direction:

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}} \right) - \frac{\partial T}{\partial x} + \frac{\partial V}{\partial x} = Q_x = F \]

\[ F = (m_v + m_p) \ddot{x} - m_p \frac{L_p}{2} \ddot{x} \cos \alpha + m_p \frac{L_p}{2} \ddot{\alpha} \sin \alpha \]

For \( \alpha \)-direction:

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\alpha}} \right) - \frac{\partial T}{\partial \alpha} + \frac{\partial V}{\partial \alpha} = Q_\alpha = 0 \]

\[ m_p g \sin \alpha + m_p \ddot{x} \cos \alpha = \frac{2}{3} m_p L_p \ddot{\alpha} \]

We worked with two equations.

Example 5: Double Pendulum

The double pendulum system consists of a mass-less rigid bar of length \( l_1 \), a point-mass \( m_1 \), a spring with free length \( l_2 \) and spring constant \( k \), and another point-mass \( m_2 \). Use Lagrange's equation to derive the equations of motion.
DOF: 3 \((\theta_1, \theta_2, x)\)

**Kinetic energy**

\[
T = \frac{1}{2} m_1 (l_1 \dot{\theta}_1)^2 + \frac{1}{2} m_2 \left[ \left( \ddot{x} + l_1 \dot{\theta}_1 \sin(\theta_2 - \theta_1) \right)^2 + \left( (l_2 + x) \dot{\theta}_2 + l_1 \dot{\theta}_1 \cos(\theta_2 - \theta_1) \right)^2 \right]
\]

**Potential energy**

\[
V = (m_1 + m_2) g l_1 (1 - \cos \theta_1) + m_2 g l_2 (1 - \cos \theta_2) - m_2 g x \cos \theta_2 + \frac{1}{2} k x^2
\]

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\theta}_1} \right) - \frac{\partial T}{\partial \theta_1} + \frac{\partial V}{\partial \theta_1} = 0
\]

\[
m_1 l_1^2 \ddot{\theta}_1 + 2 m_2 l_1 \dot{\theta}_1 \dot{x} \cos(\theta_2 - \theta_1) + m_2 l_1 \ddot{x} \sin(\theta_2 - \theta_1) + m_2 l_1^2 \dddot{\theta}_1
\]

\[
- m_2 l_1 (l_2 + x) \dot{\theta}_2 \dot{x} \sin(\theta_2 - \theta_1) + m_2 l_1 (l_2 + x) \dot{\theta}_2 \dot{x} \cos(\theta_2 - \theta_1) + (m_1 + m_2) g l_1 \sin(\theta_1) = 0
\]

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\theta}_2} \right) - \frac{\partial T}{\partial \theta_2} + \frac{\partial V}{\partial \theta_2} = 0
\]

\[
m_2 (l_2 + x)^2 \ddot{\theta}_2 + m_2 l_1 (l_2 + x) \dot{\theta}_1 \cos(\theta_2 - \theta_1) + m_2 l_1 (l_2 + x) \dot{\theta}_2 \cos(\theta_2 - \theta_1) + (m_1 + m_2) g l_1 \sin(\theta_1) = 0
\]

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}} \right) - \frac{\partial T}{\partial x} + \frac{\partial V}{\partial x} = 0
\]

\[
m_2 \ddot{x} + m_2 l_1 \dot{\theta}_1 \sin(\theta_2 - \theta_1) - m_2 l_1 \dot{\theta}_1^2 \cos(\theta_2 - \theta_1) - m_2 g \cos \theta_2 - k x - m_2 \dot{\theta}_2^2 (l_2 + x) = 0
\]
Appendix

Rotation about a fixed axis

\[ \mathbf{v} = \frac{d\mathbf{r}}{dt} = \omega \times \mathbf{r} \]

\[ \mathbf{a} = \alpha \times \mathbf{r} + \omega \times (\omega \times \mathbf{r}) \]

\[ \Rightarrow \frac{d\hat{i}}{dt} = \omega \times \hat{i} \quad \frac{d\hat{j}}{dt} = \omega \times \hat{j} \quad \frac{d\hat{k}}{dt} = \omega \times \hat{k} \]

\[ \Rightarrow \dot{\mathbf{u}} + \mathbf{v} = \omega \times \left[ \begin{array}{c} u \\ v \\ w \end{array} \right] = \left[ \begin{array}{c} p \\ q \\ r \end{array} \right] \times \left[ \begin{array}{c} u \\ v \\ w \end{array} \right] \]
Cross Product

- An operation on two vectors in the three-dimensional space

\[ \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = a_2 b_3 \mathbf{i} - a_3 b_2 \mathbf{j} + a_1 b_3 \mathbf{k} \]

\[ \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \]

\[ \frac{d}{dt} \mathbf{\dot{i}} = \omega \times \mathbf{\dot{j}} \]
\[ \frac{d}{dt} \mathbf{\dot{j}} = -\omega \times \mathbf{\dot{i}} \]

\[ \mathbf{v} = \frac{d\mathbf{r}}{dt} = \omega \times \mathbf{r} \]

\[ \Rightarrow \frac{d}{dt} \mathbf{\dot{i}} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ 1 & 0 & 0 \end{bmatrix} = \omega \mathbf{\dot{j}} \]

\[ \frac{d}{dt} \mathbf{\dot{j}} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ 0 & 1 & 0 \end{bmatrix} = -\omega \mathbf{\dot{i}} \]