ME542 Vehicle Dynamics

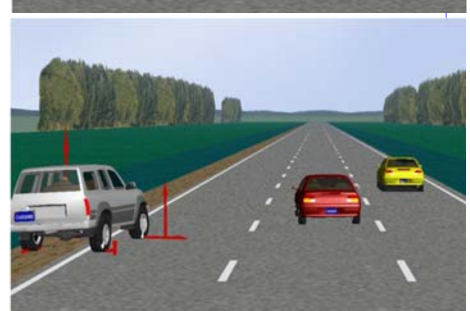
Winter 2014

Tu & Th 1:30-3:00pm

Huei Peng 3012 PML
Department of Mechanical Engineering
hpeng@umich.edu, 1-734-769-6553

Office hours: Mon. 4-5pm (Peng), Wed. 1-2pm (Peng)
GSI office hours to be announced

Vehicle Dynamics and Safety



- Accident occurred in May 2006 on I-96, caught by police dash-cam.
- Induced by an initial light side impact, SUV rolled over multiple times.

Lecture 1--Introduction and Motivation

(and administrative stuff)

- About this course
  - Introduction and administrative information
  - Major course content
  - Grading policy
  - MATLAB/SIMULINK
- Review of rigid body dynamics

Administrative Information

- Textbook (not required)
  - J.Y. Wong, Theory of Ground Vehicles, John Wiley &Sons, Inc, 4th edition, 2008. (http://www.wiley.com/WileyCDA/WileyTitle/productCd-0470170387.html)
- Course material will be made available on the Ctools site (PPT files should be downloaded and printed before the lectures, HW, solutions, example MATLAB programs will be distributed using this site)

Course Requirements

- Prerequisites
  - Knowledge in Newtonian Dynamics (ME240 level) is essential
  - That of Automotive Engineering (ME458) and Intermediate Dynamics (ME440) are helpful but not required.
  - Familiarity with Matlab/Simulink, since Matlab/Simulink is used extensively in the lecture examples and homework assignments.

Major course content

Background

Tire

Handling

Ride

Lec.	Date	Lecture contents
1	1/9	Introduction, motivation
2	1/14	Review of Rigid Body Dynamics
3	1/16	MATLAB-SIMULINK review
4	1/21	Tire Models: Overview, Terminology, Definitions
5	1/23	Brush Tire Model (Lateral)
6	1/28	Brush Tire Model (Lateral)
7	1/30	Brush Tire Model (Longitudinal)
8	2/4	Combined-Slip Tire Model
9	2/6	(Lateral) Taut String Tire Model, Magic Formula Tire Model
10	2/11	Off-road tire model
11	2/13	Steady-State Handling
12	2/18	Steady-State Handling: Understeering and Oversteering
13	2/20	Transient Handling
14	2/25	Lateral-Yaw-Roll model
15	2/27	Lateral-Yaw-Roll model
16	3/11	Four-Wheel Steering
	3/13	**Midterm (in class)**
17	3/18	Guest lecture: chassis control systems
18	3/20	On-Center Handling, Steady-State And Transient Handling of Articulated Vehicles
19	3/25	Driver-vehicle interaction
20	3/30	Cross-wind stability
21	4/1	Ride dynamics—Principle and Vibration Isolation and human perception
22	4/3	Ride dynamics—road excitations
23	4/8	Ride dynamics—Quarter-car suspension Model
24	4/10	Ride dynamics—Quarter-car suspension Model
25	4/15	Ride dynamics—Bounce and Pitch Model
26	4/17	Ride dynamics—Active Suspensions
27	4/22	**Summary**
	4/25	**Final exam (4-6pm)**

Related Courses

- ME458 Automotive Engineering

  Emphasizes the vehicle as an engineering system and review design consideration associated with all major systems including the vehicle structure, powertrain, suspension, steering and braking.

- ME568 Vehicle Control Systems

  Covers control issues for all major vehicle control systems including engine control, cruise control, ABS/traction control, four-wheel steering, active suspension and advanced control systems for Intelligent Transportation Systems.

Grading Policy

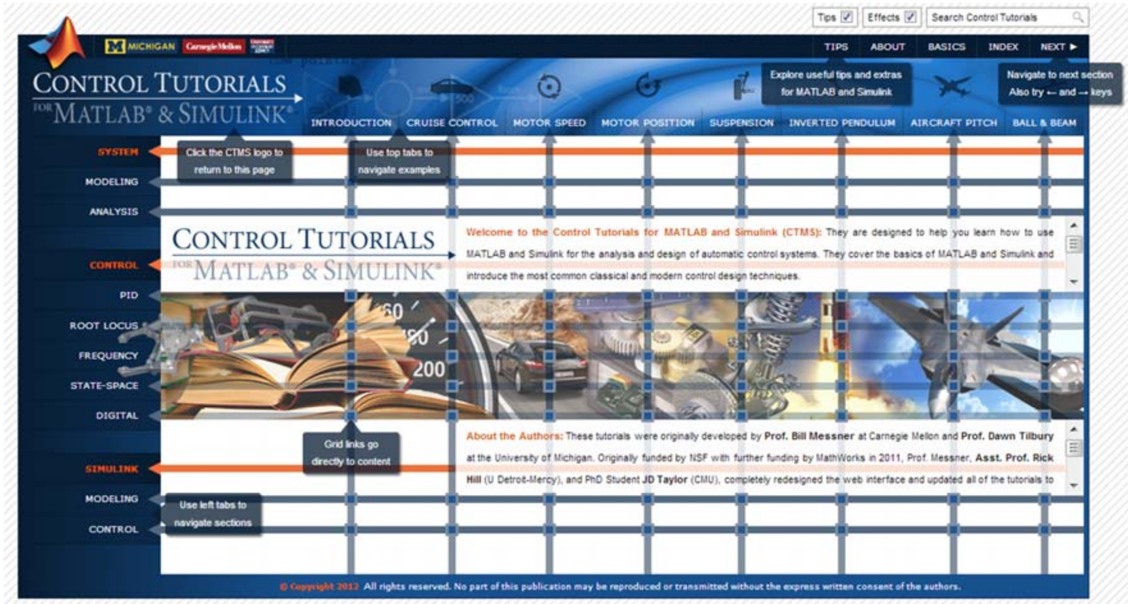
- Grading:

5-6 Homework	55%
Midterm exam	20%
Final exam	25%

- Homework: Must be handed in on the due date in class (on-campus students) or uploaded to Ctools (distance learning students). Late homework will be accepted up to 48 hours late with a 20% penalty for each 24 hours (rounded up, i.e., 0 to 24 hours late = 24 hours late). All problem sets (home work assignments) are to be completed on your own. You may discuss homework assignments with your fellow students at the conceptual level, but must complete all calculations and write-up, from scrap to final form, on your own. Verbatim copying of another student's work is forbidden. If you have any questions about this policy, please do not hesitate to contact the instructor.

- Exams: Midterm: in class, Dynamics, Tire, Handling
  Final exam: 2 hours, Accumulative but more weight on Handling and Ride

MATLAB/SIMULINK Tutorial



http://www.engin.umich.edu/class/ctms/

Getting a Copy of MATLAB/SIMULINK

CAEN labs

Student version:

http://www.mathworks.com/products/studentversion/

Virtual Site

http://virtualsites.umich.edu/

MATLAB/SIMULINK

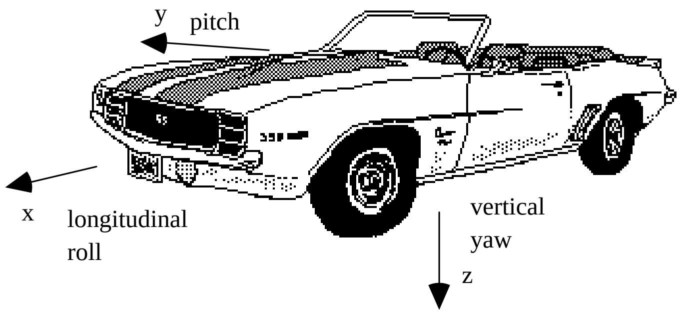
- Example MATLAB/SIMULINK programs will be distributed, which you can freely use/modify.
- MATLAB is used for simple (usually linear) vehicle dynamic simulations and analysis for this course.
- SIMULINK: A GUI based simulation program. Strength:
  - Realistic dynamic phenomenon such as nonlinearity, quantization, noise, switches, look-up tables, delays, etc. can be simulated easily.
  - GUI makes it easy to recycle vehicle simulation modules

Review of Rigid Body Dynamics

- Vehicle Coordinate Systems
- Newton/Euler Formulation
- Lagrange Formulation

Vehicle Coordinate System

- First step in deriving vehicle dynamic equations.
- The Society of Automotive Engineers (SAE) has introduced standard coordinates and notations for describing vehicle dynamics



ISO coordinate: x is the same but y and z are reversed.

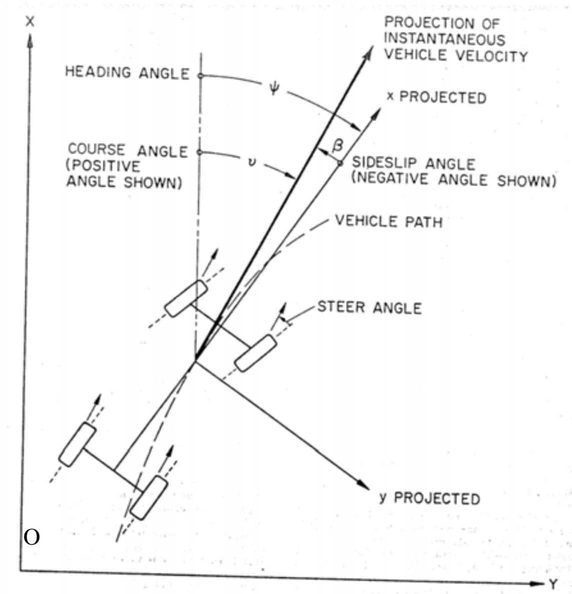
SAE Vehicle-Fixed Coordinate System --Symbols and Definitions

Axis	Translational Velocity	Angular Displacement	Angular Velocity	Force Component	Moment Component
x	u (forward)	ϕ /ˈfaɪ/	p or $\dot{\phi}$ (roll)	F_x	M_x
y	v (lateral)	θ /ˈθiːtə/, US /ˈθeɪtə/	q or $\dot{\theta}$ (pitch)	F_y	M_y
z	w (vertical)	ψ /ˈsaɪ/, /ˈpsaɪ/	r or $\dot{\psi}$ (yaw)	F_z	M_z

Pitch angle: the angle between x-axis and the horizontal plane.
Roll angle: the angle between y-axis and the horizontal plane.
Yaw angle: the angle between x-axis and the X-axis of a inertia frame

Earth Fixed Coordinate and Vehicle Slip

OXYZ fixed on Earth (does not turn with the vehicle)



Course angle $\nu = \psi + \beta$

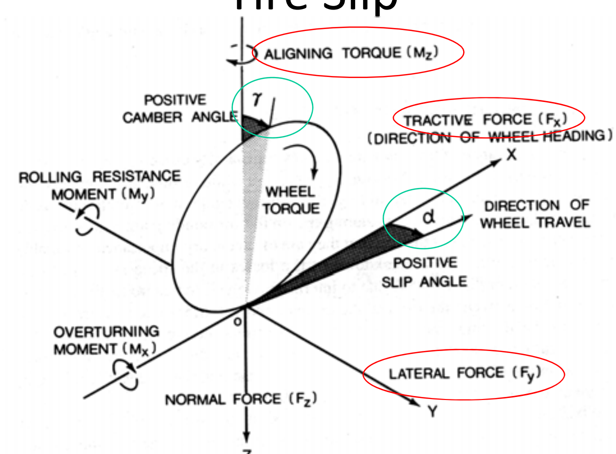
Heading angle

Sideslip angle

In the left figure, sideslip angle is negative)

$$\tan \beta = \frac{v}{u}$$

Tire Slip



Source: Wong page 7

Fig. 1.2 Tire axis system.

o: Center of tire contact patch

Z: Perpendicular to the ground plane.

X and Y axes: On the ground plane.

Tire camber angle: Angle between the XZ plane and the wheel plane.

Ride/Handling Dynamics

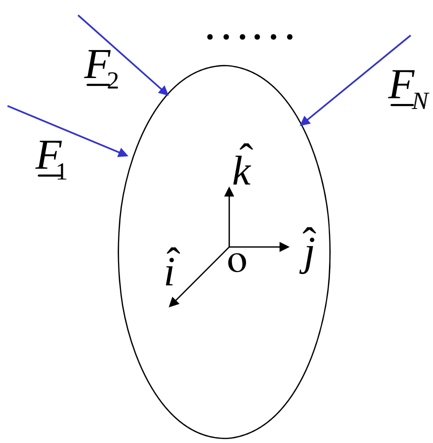
Involves vehicle motions in several directions

SPRUNG MASS	RIDE	HANDLING
longitudinal	(√)	√
lateral		√
vertical	√	
pitch	√	
roll	( )√	(√)
yaw		√
UNSPRUNG MASS		
vertical	√	√
wheel rotation	(√)	√

Newton/Euler Formulation

Consider a rigid body of mass *m*, with c.g. at point o, subject to N external forces.

For unconstrained motion, a rigid body possesses 6 DOF, 3 translational (x, y, z) and 3 rotational (ϕ, θ, ψ).



Newton/Euler Equations

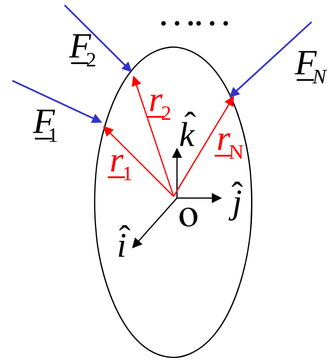
Let $\underline{V}$ and $\underline{\omega}$ to denote the absolute velocity of mass center and angular velocity of the rigid body, ($\hat{i}, \hat{j}, \hat{k}$) be unit vectors of the body-fixed coordinate oxyz. The Newton/Euler equations of motion are then

Newton's 2nd law:

$$\sum_{i=1}^{N} \underline{F}_i = \underline{F} = m\frac{d\underline{V}}{dt} = \frac{d\underline{L}}{dt}$$

Euler

$$\sum_{i=1}^{N} \underline{r}_i \times \underline{F}_i = \underline{M}_o = \frac{d\underline{H}}{dt}$$



Where $\underline{L}$ is the linear momentum and $\underline{H}$ is the angular momentum of the rigid body about the mass center o.

Angular Momentum

Let $\underline{H} = H_x\hat{i} + H_y\hat{j} + H_z\hat{k}$

$$\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} \equiv \underline{I}\cdot\underline{\omega}$$

where $\underline{I} \equiv \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$

$$I_{xx} \equiv \int_V (y^2+z^2)\rho\cdot dV$$

$$I_{xy} \equiv \int_V xy\rho\cdot dV$$

etc.

$$\underline{H} = \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} I_{xx}p - I_{xy}q - I_{xz}r \\ -I_{xy}p + I_{yy}q - I_{yz}r \\ -I_{xz}p - I_{yz}q + I_{zz}r \end{bmatrix}$$

Rigid Body Motion of a Vehicle

(a key concept: rate of change of a vector with respect to a rotating frame)

Translational motion: $\sum F = ma$

$$\sum F = ma = m \cdot \frac{D}{Dt}\left[u\hat{i} + v\hat{j} + w\hat{k}\right]$$

$$= m \cdot \left[\dot{u}\hat{i} + \dot{v}\hat{j} + \dot{w}\hat{k} + u\dot{\hat{i}} + v\dot{\hat{j}} + w\dot{\hat{k}}\right]$$

Change of the unit vectors

$$= m\left\{\frac{d}{dt}\begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \begin{bmatrix} u \\ v \\ w \end{bmatrix}\right\}$$

→

$$\sum F_x = m(\dot{u} + qw - rv)$$
$$\sum F_y = m(\dot{v} + ru - pw)$$
$$\sum F_z = m(\dot{w} + pv - qu)$$

Cross Product

$$\frac{D}{Dt} = \frac{d}{dt} + \underline{\omega}\times$$

Rate of change wrt a fixed frame

Rate of change wrt a rotating frame

Rate of change due to the rotation of the frame

Rigid Body Motion of a Vehicle (cont.)

Rotational motion: $\sum M = \dot{H}$

$$\underline{H} = \begin{bmatrix} I_{xx}p - I_{xy}q - I_{xz}r \\ -I_{xy}p + I_{yy}q - I_{yz}r \\ -I_{xz}p - I_{yz}q + I_{zz}r \end{bmatrix}$$

$$\frac{D}{Dt} = \frac{d}{dt} + \underline{\omega}\times$$

$$\sum M = \left\{\frac{d}{dt}\begin{bmatrix} I_{xx}p - I_{xy}q - I_{xz}r \\ -I_{xy}p + I_{yy}q - I_{yz}r \\ -I_{xz}p - I_{yz}q + I_{zz}r \end{bmatrix} + \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \begin{bmatrix} I_{xx}p - I_{xy}q - I_{xz}r \\ -I_{xy}p + I_{yy}q - I_{yz}r \\ -I_{xz}p - I_{yz}q + I_{zz}r \end{bmatrix}\right\}$$

→

$$\sum M_x = I_{xx}\dot{p} - I_{xy}\dot{q} - I_{xz}\dot{r} - I_{xz}pq - I_{yz}q^2 + I_{zz}rq + I_{xy}pr - I_{yy}qr + I_{yz}r^2$$
$$\sum M_y = -I_{xy}\dot{p} + I_{yy}\dot{q} - I_{yz}\dot{r} + I_{xx}pr - I_{xy}qr - I_{xz}r^2 + I_{xz}p^2 + I_{yz}qp - I_{zz}rp$$
$$\sum M_z = -I_{xz}\dot{p} - I_{yz}\dot{q} + I_{zz}\dot{r} - I_{xy}p^2 + I_{yy}qp - I_{yz}rp - I_{xx}pq + I_{xy}q^2 + I_{xz}rq$$

Equations of Vehicle Rigid Body Motion

$$\sum F_x = m(\dot{u} + qw - rv)$$
$$\sum F_y = m(\dot{v} + ru - pw)$$
$$\sum F_z = m(\dot{w} + pv - qu)$$
$$\sum M_x = I_{xx}\dot{p} - I_{xy}\dot{q} - I_{xz}\dot{r} - I_{xz}pq - I_{yz}q^2 + I_{zz}rq + I_{xy}pr - I_{yy}qr + I_{yz}r^2$$
$$\sum M_y = -I_{xy}\dot{p} + I_{yy}\dot{q} - I_{yz}\dot{r} + I_{xx}pr - I_{xy}qr - I_{xz}r^2 + I_{xz}p^2 + I_{yz}qp - I_{zz}rp$$
$$\sum M_z = -I_{xz}\dot{p} - I_{yz}\dot{q} + I_{zz}\dot{r} - I_{xy}p^2 + I_{yy}qp - I_{yz}rp - I_{xx}pq + I_{xy}q^2 + I_{xz}rq$$

Simplified Equations of Motion

- Assume
  - vehicle is symmetric wrt the xz plane ($I_{xy} = I_{yz} = 0$)
  - p,q,r,v, and w are small, i.e., their higher order terms are negligible.
  - $u = u_o + u'$, u' is small compared with u_o.

$$\sum F_x = m(\dot{u} + qw - rv)$$
$$\sum F_y = m(\dot{v} + ru - pw)$$
$$\sum F_z = m(\dot{w} + pv - qu)$$

.
.
.

$$\sum F_x = m\dot{u}'$$
$$\sum F_y = m(\dot{v} + ru_o)$$
$$\sum F_z = m(\dot{w} - qu_o)$$
$$\sum M_x = I_{xx}\dot{p} - I_{xz}\dot{r}$$
$$\sum M_y = I_{yy}\dot{q}$$
$$\sum M_z = -I_{xz}\dot{p} + I_{zz}\dot{r}$$

Linear!

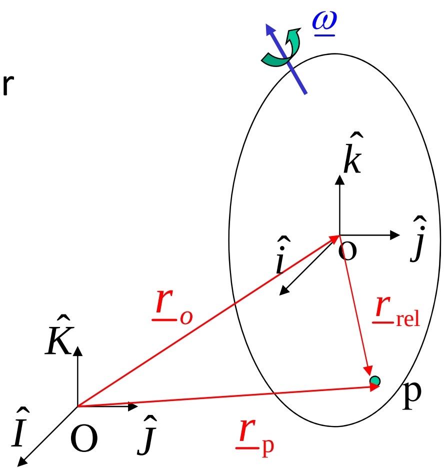
Naturally groups into 3 sets

Review of Rigid Body Kinematics

- Kinematics: study of the geometry of motions.
- Quantities: position, velocity and acceleration.

<u>Given</u>: Kinematics quantities of one point (mass center), and the angular motion of the rigid body.

<u>Find</u>: Those at another point on the same rigid body.



Application: e.g., use motion of c.g. to infer motion of other points on the vehicle

Rigid Body Kinematics

Position:

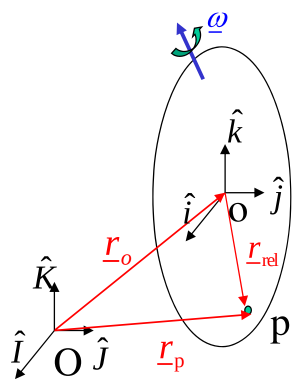
$$\underline{r}_{\text{p}} = \underline{r}_{\text{o}} + \underline{r}_{\text{rel}}$$

$$\Rightarrow \quad \dot{\underline{r}}_{\text{p}} = \dot{\underline{r}}_{\text{o}} + \dot{\underline{r}}_{\text{rel}}$$

Velocity:

$$\underline{V}_{\text{p}} = \underline{V}_{\text{o}} + \underline{\omega} \times \underline{r}_{\text{rel}}$$

$$\Rightarrow \quad \dot{\underline{V}}_{\text{p}} = \dot{\underline{V}}_{\text{o}} + \dot{\underline{\omega}} \times \underline{r}_{\text{rel}} + \underline{\omega} \times \dot{\underline{r}}_{\text{rel}}$$



Acceleration:

$$\underline{a}_{\text{p}} = \underline{a}_o + \dot{\underline{\omega}} \times \underline{r}_{\text{rel}} + \underline{\omega} \times (\underline{\omega} \times \underline{r}_{\text{rel}})$$

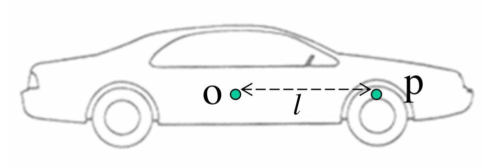
Example 1

A car is traveling in the xz plane, pitching, bouncing

Given: $\underline{V}_o = u\hat{i} + w\hat{k}$

$\underline{\omega} = q\hat{j}$

$\underline{r}_{rel} = l\hat{i}$



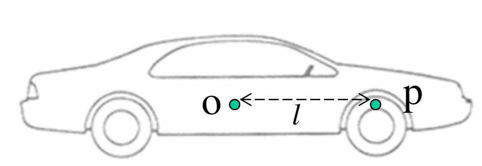
Find: $\underline{V}_p \quad \underline{a}_p$

Example 1 Solution

$\underline{V}_o = u\hat{i} + w\hat{k}$

$\underline{\omega} = q\hat{j}$

$\underline{r}_{rel} = l\hat{i}$



$$\underline{V}_p = \underline{V}_o + \underline{\omega} \times \underline{r}_{rel} = \begin{bmatrix} u \\ 0 \\ w \end{bmatrix} + \begin{bmatrix} 0 \\ q \\ 0 \end{bmatrix} \times \begin{bmatrix} l \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} u \\ 0 \\ w - ql \end{bmatrix}$$

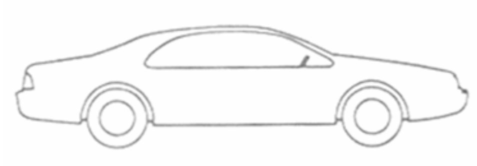
$$= u\hat{i} + (w - ql)\hat{k}$$

Example 1 Solution

$\underline{V}_o = u\hat{i} + w\hat{k}$

$\underline{\omega} = q\hat{j}$

$\underline{r}_{rel} = l\hat{i}$



$$\underline{a}_p = \underline{a}_o + \dot{\underline{\omega}} \times \underline{r}_{rel} + \underline{\omega} \times (\underline{\omega} \times \underline{r}_{rel})$$

$$= \begin{bmatrix} \dot{u} \\ 0 \\ \dot{w} \end{bmatrix} + \begin{bmatrix} 0 \\ q \\ 0 \end{bmatrix} \times \begin{bmatrix} u \\ 0 \\ w \end{bmatrix} + \begin{bmatrix} 0 \\ \dot{q} \\ 0 \end{bmatrix} \times \begin{bmatrix} l \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ q \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ -ql \end{bmatrix}$$

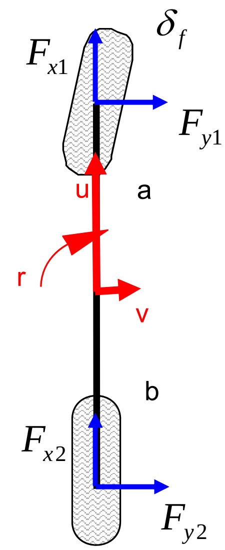
$$= \begin{bmatrix} \dot{u} + qw - q^2 l \\ 0 \\ \dot{w} - qu - \dot{q}l \end{bmatrix}$$

Another Kinematic Example (hw1)

- Vehicle Bicycle Model
- Vehicle is driving in the xy plane at a constant forward speed, and turning. All other motions are ignored
- Find the speed and acceleration at the center of the front axle and the rear left tire

Example 2

- Vehicle Bicycle Model
- Vehicle is driving in the xy plane, and yawing. <u>All other motions are ignored</u>
- Find the dynamic equations



Example 2 Solution

$$\sum F_x = m(\dot{u} + qw - rv)$$
$$\sum F_y = m(\dot{v} + ru - pw)$$
$$\sum F_z = m(\dot{w} + pv - qu)$$
$$\sum M_x = I_{xx}\dot{p} - I_{xy}\dot{q} - I_{xz}\dot{r} - I_{xz}pq - I_{yz}q^2 + I_{zz}rq + I_{xy}pr - I_{yy}qr + I_{yz}r^2$$
$$\sum M_y = -I_{xy}\dot{p} + I_{yy}\dot{q} - I_{yz}\dot{r} + I_{xx}pr - I_{xy}qr - I_{xz}r^2 + I_{xz}p^2 + I_{yz}qp - I_{zz}rp$$
$$\sum M_z = -I_{xz}\dot{p} - I_{yz}\dot{q} + I_{zz}\dot{r} - I_{xy}p^2 + I_{yy}qp - I_{yz}rp - I_{xx}pq + I_{xy}q^2 + I_{xz}rq$$

$\Downarrow$?

Lagrange's Formulation

- An alternative way to formulate the equations of motion.
- Starts from "energy" (scalar!) rather than vectors.
- We first define three terms

Degrees Of Freedom *(DOF)*:

Number of independent coordinates required to uniquely define the motion of a particle, a body or a system of bodies.

Constraints

Kinematic relationship that limit possible motions.

Generalized coordinates

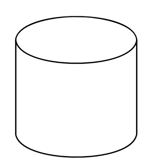
A set of independent coordinates whose number is equal to the DOF and which uniquely define the position and orientation of the rigid body.

Examples for DOF, Constraints, and GC

Position of a particle in space: 3DOF

Position of a particle <u>on a cylindrical surface</u>

constraint



2DOF, position uniquely described by 2 coordinates, which are referred to as generalized coordinates (q_1, q_2).

Rigid wheel rolling on flat surface without slip, DOF=?
Rigid wheel rolling on flat surface with slip, DOF=?

Lagrange Equations

Consider a system of particles or rigid bodies with *n* DOF described by *n* generalized coordinates $\{q_1(t), \ldots, q_n(t)\}$.

The Lagrange's equation is then

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i \qquad i = 1 \ldots n$$

where

$T = T(q_i, \dot{q}_i)$: kinetic energy

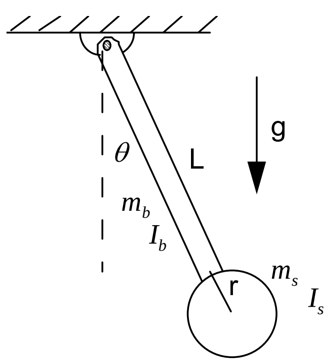
$V = V(q_i)$: potential energy

Q_i : the i[th] generalized non-conservative force.

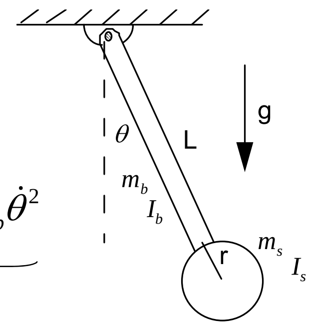
$$Q_i = \sum_{j=1}^{k} \underline{F}_j \cdot \frac{\partial \underline{r}_j}{\partial q_i}$$

Example 3: Compound Pendulum

Derive the equation of motion for the compound pendulum shown below for large angle q.



DOF: only 1 (q)



Kinetic energy

$$T = \underbrace{\frac{1}{2} m_s (L+r)^2 \cdot \dot{\theta}^2 + \frac{1}{2} I_s \dot{\theta}^2}_{\text{disk}} + \underbrace{\frac{1}{2} m_b (\frac{L}{2}\dot{\theta})^2 + \frac{1}{2} I_b \dot{\theta}^2}_{\text{bar}}$$

Potential energy

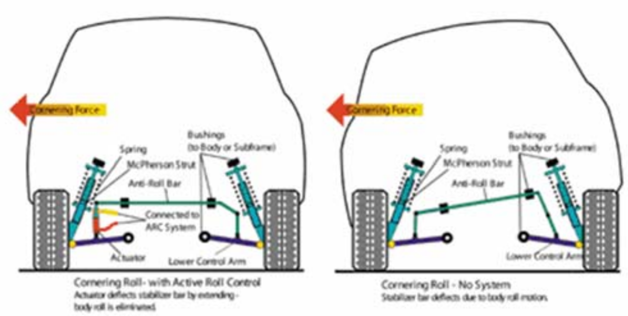
$$V = \underbrace{m_s g (L+r)(1-\cos\theta)}_{\text{disk}} + \underbrace{m_b g \frac{L}{2}(1-\cos\theta)}_{\text{bar}}$$

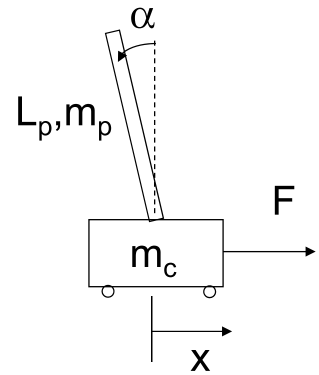
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = \cancel{Q_i} \;\; 0$$

$$\Rightarrow \ddot{\theta}\left[m_s (L+r)^2 + m_b (\frac{L}{2})^2 + I_s + I_b\right] + g\sin\theta\left[m_s (L+r) + m_b \frac{L}{2}\right] = 0$$

Example 4: Inverted Pendulum

- Later in the term, we will apply the Lagrange's method to derive the lateral-yaw-roll model.
- Here we will start from a simpler system, which only involves lateral and roll (yaw is ignored for now)

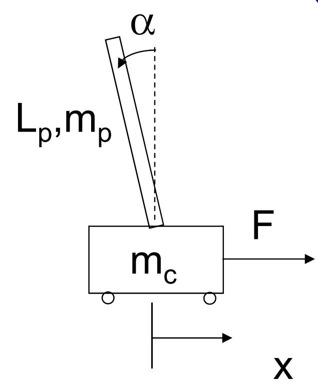




Example 4 Solution

Newtonian Method: for the pendulum

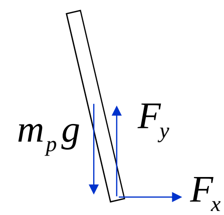
$$x_p = x - \frac{L_p}{2}\sin\alpha \qquad y_p = \frac{L_p}{2}\cos\alpha$$



x-direction

$$F_x = m_p \frac{d^2}{dt^2}(x_p) = m_p \frac{d}{dt}(\dot{x} - \frac{L_p}{2}\dot{\alpha}\cos\alpha)$$

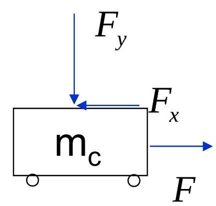
$$= m_p(\ddot{x} - \frac{L_p}{2}\ddot{\alpha}\cos\alpha + \frac{L_p}{2}\dot{\alpha}^2\sin\alpha)$$



y-direction

$$F_y - m_p g = m_p \frac{d^2}{dt^2}(y_p) = m_p \frac{L_p}{2}\frac{d}{dt}(-\dot{\alpha}\sin\alpha)$$

$$= m_p \frac{L_p}{2}(-\ddot{\alpha}\sin\alpha - \dot{\alpha}^2\cos\alpha)$$

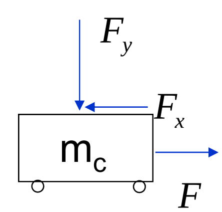


roll-direction

$$F_y \frac{L_p}{2}\sin\alpha + F_x \frac{L_p}{2}\cos\alpha = I_p\ddot{\alpha} \qquad I_p = \frac{1}{12}m_p L_p^2$$

Example 4 Solution (cont.)

Newtonian Method: for the cart



$$F - F_x = m_c\ddot{x}$$

⇨ $$F = (m_c + m_p)\ddot{x} - m_p\frac{L_p}{2}\ddot{\alpha}\cos\alpha + m_p\frac{L_p}{2}\dot{\alpha}^2\sin\alpha$$

$$F_y \frac{L_p}{2}\sin\alpha + F_x \frac{L_p}{2}\cos\alpha = I_p\ddot{\alpha}$$

Plug-in equations from x- and y- direction

⇨ $$m_p g\sin\alpha + m_p\ddot{x}\cos\alpha = \frac{2}{3}m_p L_p\ddot{\alpha}$$

We wrote down four equations, two of them used to cancel the internal forces Fx and Fy.

Example 4 Solution (cont.)

Lagrange's method

$$x_p = x - \frac{L_p}{2}\sin\alpha \quad \Rightarrow \quad \dot{x}_p = \dot{x} - \dot{\alpha}\frac{L_p}{2}\cos\alpha$$

$$y_p = \frac{L_p}{2}\cos\alpha \quad \Rightarrow \quad \dot{y}_p = -\dot{\alpha}\frac{L_p}{2}\sin\alpha$$

The kinetic energy and potential energy are then

$$T = \frac{1}{2}m_c\dot{x}^2 + \frac{1}{2}m_p[\dot{x}_p^2 + \dot{y}_p^2] + \frac{1}{2}I_p\dot{\alpha}^2 = \frac{1}{2}m_c\dot{x}^2 + \frac{1}{2}m_p[\dot{x}^2 - 2\dot{x}\dot{\alpha}\frac{L_p}{2}\cos\alpha + \frac{1}{3}\dot{\alpha}^2L_p^2]$$

$$V = m_p g\frac{L_p}{2}\cos\alpha$$

For x-direction: $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial V}{\partial x} = Q_x = F$

$$F = (m_c + m_p)\ddot{x} - m_p\frac{L_p}{2}\ddot{\alpha}\cos\alpha + m_p\frac{L_p}{2}\dot{\alpha}^2\sin\alpha$$

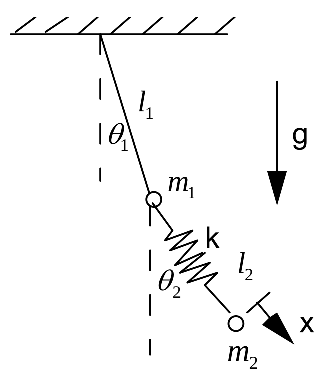
For α-direction: $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\alpha}}\right) - \frac{\partial T}{\partial \alpha} + \frac{\partial V}{\partial \alpha} = Q_\alpha = 0$

$$m_p g\sin\alpha + m_p\ddot{x}\cos\alpha = \frac{2}{3}m_pL_p\ddot{\alpha}$$

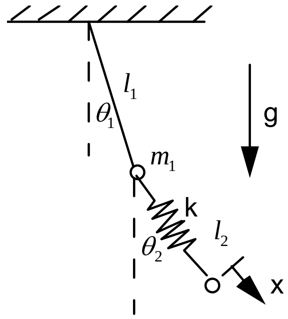
We worked with two equations

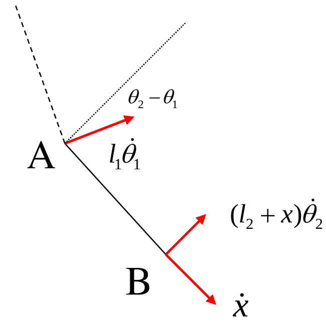
Example 5: Double Pendulum

The double pendulum system consists of a mass-less rigid bar of length l_1, a point-mass m_1, a spring with free length l_2 and spring constant k, and another point-mass m_2. Use Lagrange's equation to derive the equations of motion.



DOF: 3 (θ_1, θ_2, x)





Kinetic energy

$$T = \frac{1}{2} m_1 (l_1 \dot{\theta}_1)^2 +$$

$$\frac{1}{2} m_2 \left\{ \left[\dot{x} + l_1 \dot{\theta}_1 \sin(\theta_2 - \theta_1) \right]^2 + \left[(l_2 + x)\dot{\theta}_2 + l_1 \dot{\theta}_1 \cos(\theta_2 - \theta_1) \right]^2 \right\}$$

Potential energy

$$V = (m_1 + m_2) g l_1 (1 - \cos\theta_1) + m_2 g l_2 (1 - \cos\theta_2) - m_2 g x \cos\theta_2 + \frac{1}{2} k x^2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) - \frac{\partial T}{\partial \theta_1} + \frac{\partial V}{\partial \theta_1} = 0$$

$$m_1 l_1^2 \ddot{\theta}_1 + 2 m_2 l_1 \dot{\theta}_2 \dot{x} \cos(\theta_2 - \theta_1) + m_2 l_1 \ddot{x} \sin(\theta_2 - \theta_1) + m_2 l_1^2 \ddot{\theta}_1$$
$$- m_2 l_1 (l_2 + x) \dot{\theta}_2^2 \sin(\theta_2 - \theta_1) + m_2 l_1 (l_2 + x) \ddot{\theta}_2 \cos(\theta_2 - \theta_1) + (m_1 + m_2) g l_1 \sin(\theta_1) = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) - \frac{\partial T}{\partial \theta_2} + \frac{\partial V}{\partial \theta_2} = 0$$

$$m_2 (l_2 + x)^2 \ddot{\theta}_2 + m_2 l_1 (l_2 + x) \ddot{\theta}_1 \cos(\theta_2 - \theta_1) + m_2 l_1 (l_2 + x) \dot{\theta}_1^2 \sin(\theta_2 - \theta_1)$$
$$+ 2 m_2 \dot{x} \dot{\theta}_2 (l_2 + x) + m_2 g (l_2 + x) \sin(\theta_2) = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial V}{\partial x} = 0$$

$$m_2 \ddot{x} + m_2 l_1 \ddot{\theta}_1 \sin(\theta_2 - \theta_1) - m_2 l_1 \dot{\theta}_1^2 \cos(\theta_2 - \theta_1)$$
$$- m_2 g \cos\theta_2 + kx - m_2 \dot{\theta}_2^2 (l_2 + x) = 0$$

Appendix

Rotation about a fixed axis

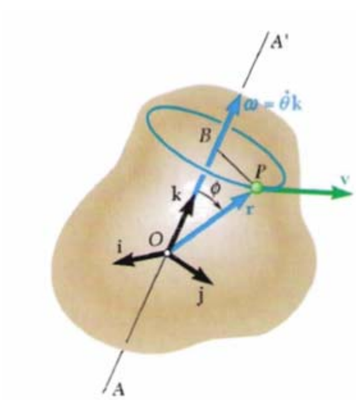


Fig. 15.9

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \times \mathbf{r}$$

$$\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

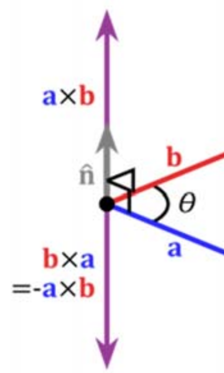
$$\Rightarrow \frac{d\vec{i}}{dt} = \omega \times \vec{i} \quad \frac{d\vec{j}}{dt} = \omega \times \vec{j} \quad \frac{d\vec{k}}{dt} = \omega \times \vec{k}$$

$$\Rightarrow u\dot{\hat{i}} + v\dot{\hat{j}} + w\dot{\hat{k}} = \omega \times \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Beer, Johnson And Clausen, Vector Mechanics for Engineers, Dynamics, 8th edition, page 920

Cross Product

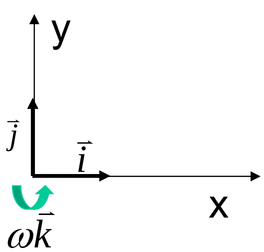
- An operation on two vectors in the three-dimensional space



$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$$

http://en.wikipedia.org/wiki/Cross_product



$$\frac{d\vec{i}}{dt} = \omega\times\vec{j}$$

$$\frac{d\vec{j}}{dt} = -\omega\times\vec{i}$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \boldsymbol{\omega}\times\mathbf{r} \quad \Rightarrow \frac{d}{dt}\hat{i} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \omega \\ 1 & 0 & 0 \end{bmatrix} = \omega\hat{j}$$

$$\frac{d}{dt}\hat{j} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \omega \\ 0 & 1 & 0 \end{bmatrix} = -\omega\hat{i}$$