Finite Element Methods in General Relativity

Harish Narayanan
Computational Mechanics Laboratory

2006 Engineering Graduate Student Symposium

November 3rd, 2006

Some science history

- Newton's *"Principia"* (1687); Maxwell's *"Theory"* (1864)
- Newtonian gravity:
  - Gravitational potential for a point mass m: $\Phi = \frac{-Gm}{r}$
  - Corresponding acceleration: $\boldsymbol{g} = -\boldsymbol{\nabla}\Phi = \frac{-Gm}{r^2}\boldsymbol{e}_r$
- Unable to explain:
  - Bending of light due to stars
  - Magnitude of the precession of the orbit of Mercury
    - $5600 - 5557 = 43$ seconds of arc per *century*
- Action at a distance

Some science history

- Newton's *"Principia"* (1687); Maxwell's *"Theory"* (1864)
- Newtonian gravity:
  - Gravitational potential for a point mass m: $\Phi = \frac{-Gm}{r}$
  - Corresponding acceleration: $\boldsymbol{g} = -\boldsymbol{\nabla}\Phi = \frac{-Gm}{r^2}\boldsymbol{e}_r$
- Unable to explain:
  - Bending of light due to stars
  - Magnitude of the precession of the orbit of Mercury
    - $5600 - 5557 = 43$ seconds of arc per *century*
- Action at a distance

Some science history

- Newton's *"Principia"* (1687); Maxwell's *"Theory"* (1864)
- Newtonian gravity:
  - Gravitational potential for a point mass m: $\Phi = \frac{-Gm}{r}$
  - Corresponding acceleration: $\boldsymbol{g} = -\boldsymbol{\nabla}\Phi = \frac{-Gm}{r^2}\boldsymbol{e}_r$
- Unable to explain:
  - Bending of light due to stars
  - Magnitude of the precession of the orbit of Mercury
    - $5600 - 5557 = 43$ seconds of arc per *century*
- Action at a distance

Warming up with special relativity

- Maxwell's equations ⇒ Speed of EM waves $c \simeq 3 \times 10^8$ m/s
  ... with respect to what?
- *"The laws of physics are the same in all uniformly moving reference frames"*
- Interesting implications:
  - Cosmic speed limit: $1 \leq \gamma = \frac{1}{\sqrt{1-v^2/c^2}} < \infty$
  - Time dilation: $\Delta t' = \gamma \Delta t$
  - Lorentz contraction: $\Delta x' = \frac{\Delta x}{\gamma}$
  - Mass-energy equivalence: $E = Mc^2 = \gamma m c^2$
- Minkowski metric: $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Warming up with special relativity

- Maxwell's equations ⇒ Speed of EM waves $c \simeq 3 \times 10^8$ m/s
  ... with respect to what?
- *"The laws of physics are the same in all uniformly moving reference frames"*
- Interesting implications:
  - Cosmic speed limit: $1 \leq \gamma = \frac{1}{\sqrt{1-v^2/c^2}} < \infty$
  - Time dilation: $\Delta t' = \gamma \Delta t$
  - Lorentz contraction: $\Delta x' = \frac{\Delta x}{\gamma}$
  - Mass-energy equivalence: $E = Mc^2 = \gamma mc^2$
- Minkowski metric: $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Warming up with special relativity

- Maxwell's equations $\Rightarrow$ Speed of EM waves $c \simeq 3 \times 10^8$ m/s
  ... with respect to what?
- *"The laws of physics are the same in all uniformly moving reference frames"*
- Interesting implications:
  - Cosmic speed limit: $1 \leq \gamma = \frac{1}{\sqrt{1-v^2/c^2}} < \infty$
  - Time dilation: $\Delta t' = \gamma \Delta t$
  - Lorentz contraction: $\Delta x' = \frac{\Delta x}{\gamma}$
  - Mass-energy equivalence: $E = Mc^2 = \gamma m c^2$
- Minkowski metric: $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Warming up with special relativity

- Maxwell's equations $\Rightarrow$ Speed of EM waves $c \simeq 3 \times 10^8$ m/s ... with respect to what?
- *"The laws of physics are the same in all uniformly moving reference frames"*
- Interesting implications:
  - Cosmic speed limit: $1 \leq \gamma = \frac{1}{\sqrt{1-v^2/c^2}} < \infty$
  - Time dilation: $\Delta t' = \gamma \Delta t$
  - Lorentz contraction: $\Delta x' = \frac{\Delta x}{\gamma}$
  - Mass-energy equivalence: $E = Mc^2 = \gamma mc^2$
- Minkowski metric: $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

“Do not worry about your difficulties in mathematics, I assure you that mine are greater.” — Einstein

A crash course on modern geometry and topology

- **Spacetime:** Curved, pseudo-Riemannian manifold with a metric of signature $(-+++)$ $\Rightarrow$ Charts and atlases allow us to relate them to Euclidean spaces, $\mathbb{R}^n$
- **Tensor:** Multi-index object which transforms according to $\hat{A}^{i_1\ldots i_q}_{j_1\ldots j_p} = X^{i_1}_{k_1}\cdots X^{i_q}_{k_q} Y^{l_1}_{j_1}\cdots Y^{l_1}_{j_p} A^{k_1\ldots k_q}_{l_1\ldots l_p}$
- **Metric**: Evolving, non-flat, symmetric, 2-index tensor, $g_{\mu\nu}$
- **Christoffel symbols:** $\Gamma_{jks} = \frac{1}{2}\left(\frac{\partial g_{js}}{\partial w^k} + \frac{\partial g_{ks}}{\partial w^j} - \frac{\partial g_{jk}}{\partial w^s}\right)$
- **Covariant derivative:** $Y^i_{;j} = Y^i_{,j} + \Gamma_{jk}{}^i Y^k$
- **Riemann curvature tensor:** $R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$
- **Ricci tensor:** $R_{ij} = R^k{}_{ikj}$
- **Scalar curvature:** $R = R^i{}_i$

A final prelude to general relativity

- What about action at a distance?
- What is so special about special relativity?
  ... Physics is the same for all observers *in uniform motion*
- Do you know if you are in inertial reference frame?

Impossible to tell! $\Rightarrow$ Principle of equivalence

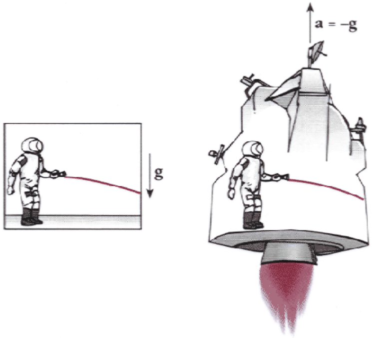
A final prelude to general relativity

- What about action at a distance?
- What is so special about special relativity?
  ... Physics is the same for all observers *in uniform motion*
- Do you know if you are in inertial reference frame?

Impossible to tell! $\Rightarrow$ Principle of equivalence

A final prelude to general relativity

- What about action at a distance?
- What is so special about special relativity?
  . . . Physics is the same for all observers *in uniform motion*
- Do you know if you are in inertial reference frame?



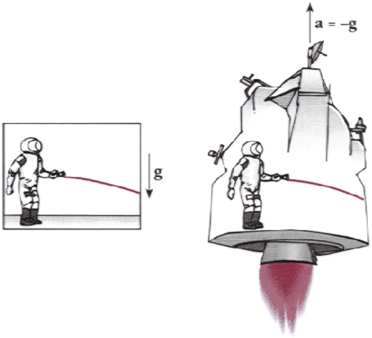
Laboratory in gravitational field

Accelerated laboratory

Impossible to tell! ⇒ Principle of equivalence

A final prelude to general relativity

- What about action at a distance?
- What is so special about special relativity?
  ... Physics is the same for all observers *in uniform motion*
- Do you know if you are in inertial reference frame?

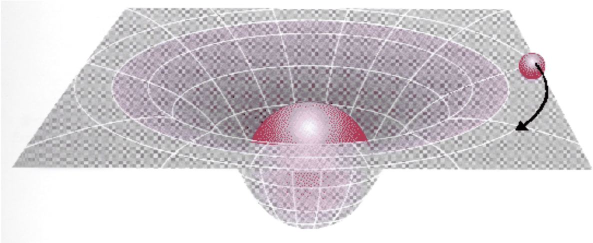


Laboratory in gravitational field

Accelerated laboratory

Impossible to tell! $\Rightarrow$ Principle of equivalence

A new basis for gravity



Gravity is the geometry of spacetime!

A look at the field equations

- System of second order, coupled, nonlinear PDEs:

$$G_{\mu\upsilon} = \frac{8\pi G}{c^4} T_{\mu\upsilon}$$

  G — Gravitational constant
  c — Velocity of light

- **Einstein Tensor:** $G_{\mu\upsilon} = R_{\mu\upsilon} - \frac{1}{2} g_{\mu\upsilon} R$

  $R_{\mu\upsilon}$ — Ricci tensor
  R — Scalar curvature

- **Stress energy tensor:** $T_{\mu\upsilon} = (\rho + p) U_\mu U_\upsilon + \rho g_{\mu\upsilon}$

  Assuming a perfect fluid with 4-velocity U^μ, for e.g.

- Covariant divergence of $\boldsymbol{G}$ and $\boldsymbol{T} = 0 \Rightarrow$ Conservation laws

A famous analytical solution

- Working in a coordinate chart with (r, θ, ϕ, t)
- Spherically symmetric, static spacetime
- General form of such a metric:
  $ds^2 = A(r)\,dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 + B(r)\,dt^2$
- Vacuum field equations: $R_{ab} = 0 \Rightarrow$

$$4\dot{A}B^2 - 2r\ddot{B}AB + r\dot{A}\dot{B}B + r\dot{B}^2A = 0$$
$$r\dot{A}B + 2A^2B - 2AB - r\dot{B}A = 0$$
$$-2r\ddot{B}AB + r\dot{A}\dot{B}B + r\dot{B}^2A - 4\dot{B}AB = 0$$

Unique solution:

$ds^2 = \left(1 - \frac{2Gm}{c^2r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) - c^2\left(1 - \frac{2Gm}{c^2r}\right) dt^2$

using the weak field approximation: $g_{00} = -c^2 + \frac{2Gm}{r}$

Some ado about numerics

- Formulations in *weak form* exist for "$3 + 1$" space×time decomposition (for FEM)

  (First we develop an elegant covariant theory and then turn it back into a $3 + 1$ form!)

- A typical numerical scheme
  - Slice spacetime into spacelike 3D hyperspaces; Successive slices are like "instants" of time
  - Use the constraint equations and solve for the conditions on the initial hypersurface
  - Evolve these solutions forward
  - Peridically check if constraints are propagated correctly
- FeTK: Open source finite element software libraries for solving coupled PDEs on manifolds

Some ado about numerics

- Formulations in *weak form* exist for "$3+1$" space$\times$time decomposition (for FEM)

  (First we develop an elegant covariant theory and then turn it back into a $3+1$ form!)
- A typical numerical scheme
  - Slice spacetime into spacelike 3D hyperspaces; Successive slices are like "instants" of time
  - Use the constraint equations and solve for the conditions on the initial hypersurface
  - Evolve these solutions forward
  - Peridically check if constraints are propagated correctly
- FeTK: Open source finite element software libraries for solving coupled PDEs on manifolds

Some ado about numerics

- Formulations in *weak form* exist for "$3+1$" space$\times$time decomposition (for FEM)

  (First we develop an elegant covariant theory and then turn it back into a $3+1$ form!)
- A typical numerical scheme
  - Slice spacetime into spacelike 3D hyperspaces; Successive slices are like "instants" of time
  - Use the constraint equations and solve for the conditions on the initial hypersurface
  - Evolve these solutions forward
  - Peridically check if constraints are propagated correctly
- FeTK: Open source finite element software libraries for solving coupled PDEs on manifolds

The usefulness of it all

- Better understanding of the physics of our universe
  - Calculates precession of mercury's orbit correctly!
- Simulations for gravitational wave detectors
  - Recall this is a field theory, no action at a distance
- Physics of black holes
  - Accretion disk evolution around black holes
  - Jet formation near black holes
- Relativistic flows: Jets, Shocks

…

An incomplete bibliography

- Hawking, *"On the shoulders of giants"* (Running Press, 2003)
- Misner et al., *"Gravitation"* (Freeman, 1973)
- Smoller, *"Introduction to General Relativity"* (Math 710: *"Topics in Moden Analysis"* lecture notes, 2006)
- Holst, "Adaptive numerical treatment of elliptic systems on manifolds" (Adv. in Comp. Math, 2001)
- Dubrovin et al., *"Modern Geometry – 1"* (Springer, 1991)
- Wolfson, *"Simply Einstein"* (W. W. Norton & Co., 2003)
- Einstein, "Relativity: The special and general theory" (Three Rivers Press, 1995)
- Lehner, "Numerical relativity: a review" (Class. and Quant. Grav., 2001)
- Meier, "Trends in Numerical Relativistic Astrophysics" (Caltech)
- Caroll, "Lecture notes on general relativity" (UCSB)
- Images from:
  - http://archive.ncsa.uiuc.edu/Cyberia/NumRel/NumRelHome.html
  - http://psi.phys.wits.ac.za/teaching/Connell/